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On certain arithmetical functions of exponents in the factorization of integers

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Abstract: Some new results for the maximum and minimum exponents in factorizing integers are obtained. Related functions and generalized arithmetical functions are also introduced.
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1 Introduction

Let $n = \prod_{i=1}^{r} p_i^{a_i} > 1$ be the prime factorization of the positive integer n ($p_1, p_2, \ldots, p_r$ being distinct primes; $r, a_1, a_2, \ldots, a_r \geq 1$ being positive integers). The arithmetical functions

$$\omega(n) = r, \omega(1) = 1$$

and

$$\Omega(n)=\sum_{i=1}^{r}a_i, \Omega(1)=1$$

denoting the number of distinct, respective, total numbers of prime factors of n, have been much studied in the literature (see, e.g., [14], Chapter V).

In 1969, I. Niven [11] introduced and studied the arithmetical functions

$$\begin{aligned} H(n)&=\max\{a_1,a_2,\ldots,a_r\},\ H(1)=1,\\ h(n)&=\min\{a_1,a_2,\ldots,a_r\},\ \ h(1)=1. \end{aligned} \tag{1}$$

For properties of these functions, see [15], Chapter 4. Here, we will mention only that for each natural number a:

$$h(n^a)=ah(n),$$
$$H(n^a)=aH(n).$$

In 1947 D. G. Kendall and R. A. Rankin [9] considered the function

$$\beta(n)=\prod_{i=1}^{r}a_i, \beta(1)=1 \tag{2}$$

(see, [15], Chapter 4).

In what follows, we will consider some new properties of the above functions. Generalizations of (1) will be considered.

2 Main results

In what follows, $f(x)\sim g(x)$ means that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=1.$$

For the number n in the Introduction, define

$$\Omega_1(n)=\sum_{i=1}^{r}\frac{1}{a_i}, \Omega_1(1)=1.$$

Theorem 1. *One has*

$$\sum_{n\le x}\frac{\omega(n)}{\Omega_1(n)}\sim x \tag{3}$$

and

$$\sum_{n\le x}(\beta(n))^{\frac{1}{\omega(n)}}\sim x, \tag{4}$$

as $x\to\infty$.

Proof. If $x_1,x_2,\ldots,x_r$ are positive real numbers, then it is well-known that

$$\begin{aligned} \min\{x_1,x_2,\ldots,x_r\}&\le H_r(x_1,x_2,\ldots,x_r)\le G_r(x_1,x_2,\ldots,x_r)\\ &\le A_r(x_1,x_2,\ldots,x_r)\\ &\le \max\{x_1,x_2,\ldots,x_r\}, \end{aligned} \tag{5}$$

where

$$H_r(x_1, x_2, \ldots, x_r) = \frac{r}{\sum\limits_{i=1}^{r} \frac{1}{x_i}},$$

$$G_r(x_1, x_2, \ldots, x_r) = \sqrt[r]{\prod_{i=1}^{r} x_i},$$

$$A_r(x_1, x_2, \ldots, x_r) = \frac{\sum\limits_{i=1}^{r} x_i}{r}$$

are the harmonic, geometric, and arithmetic means of $x_1, x_2, \ldots, x_r$, respectively.

Let $x_1 = a_1, x_2 = a_2, \ldots, x_r = a_r$, with $r = \omega(n)$. Then (5) can be rewritten as

$$h(n) \leq \frac{\omega(n)}{\Omega_1(n)} \leq (\beta(n))^{\frac{1}{\omega(n)}} \leq \frac{\Omega(n)}{\omega(n)} \leq H(n), \tag{6}$$

where $n > 1$.

Now, I. Niven [11] proved that $\sum\limits_{n \leq x} h(n) \sim x$, while R. L. Duncan [7] proved that $\sum\limits_{n \leq x} \frac{\Omega(n)}{\omega(n)} \sim x$ as $x \to \infty$. By using inequalities (6), relations (3) and (4) will follow. □

Now, let $f : \mathcal{N} \to \mathcal{N}$ be an arithmetical function. we will extend the Niven functions (1) as follows:

$$\begin{aligned} H_f(n) &= \max\{f(a_1), f(a_2), \ldots, f(a_r)\},\ H_f(1) = 1 \\ h_f(n) &= \min\{f(a_1), f(a_2), \ldots, f(a_r)\},\ \ h_f(1) = 1. \end{aligned} \tag{7}$$

For example, for the Euler totient function φ, one has

$$\begin{aligned} H_\varphi(n) &= \max\{\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_r)\},\ H_\varphi(1) = 1 \\ h_\varphi(n) &= \min\{\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_r)\},\ \ h_\varphi(1) = 1. \end{aligned} \tag{8}$$

For the sum-of-divisors function σ, we define

$$\begin{aligned} H_\sigma(n) &= \max\{\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_r)\},\ H_\sigma(1) = 1 \\ h_\sigma(n) &= \min\{\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_r)\},\ \ h_\sigma(1) = 1. \end{aligned} \tag{9}$$

These functions have not been studied up to now.

Following [3, 5, 12, 13], let us define for the above n the irrational and the restrictive factors, respectively, by:

$$IF(n) = \prod_{i=1}^{r} p_i^{\frac{1}{a_i}},$$

$$CF(n) = \prod_{i=1}^{r} p_i^{a_i - 1}.$$

Then we see immediately that

$$\Omega(IF(n)) = \Omega_1(n),$$

$$\Omega(CF(n)) = \Omega(n) - r.$$

Let us define

$$\Omega_f(n) = \sum_{i=1}^{r} f(a_i), \Omega_f(1) = 1, \tag{10}$$

as well as

$$\Omega_{f,1}(n) = \sum_{i=1}^{r} \frac{1}{f(a_i)}, \Omega_{f,1}(1) = 1 \tag{11}$$

and *

$$\beta_f(n) = \prod_{i=1}^{r} f(a_i). \tag{12}$$

The inequalities (6) can be extended as follows:

Theorem 2. *One has*

$$h_f(n) \leq \frac{\omega(n)}{\Omega_{f,1}(n)} \leq (\beta_f(n))^{\frac{1}{\omega(n)}} \leq \frac{\Omega_f(n)}{\omega(n)} \leq H_f(n) \tag{13}$$

for $n > 1$.

Proof. Apply inequalities (5) for $x_1 = f(a_1), x_2 = f(a_2), \ldots, x_r = f(a_r)$ and use the definitions (10)–(12). □

Theorem 3. *Suppose that* $f(n) \geq f(1) = 1$ *for any* $n \geq 1$. *Then*

$$H_f(n) \leq \Omega_f(n) - \omega(n) + 1 \tag{14}$$

and

$$h_f(n) \geq \Omega_f(n) - (\omega(n) - 1)H_f(n). \tag{15}$$

Proof. Let $f(\overline{a}) = H_f(n)$. Then

$$\begin{aligned} \Omega_f(n) &= f(a_1) + \cdots + f(\overline{a}) + \cdots + f(a_r) \\ &\geq (r-1)f(1) + H_f(n), \end{aligned}$$

as $f(a_1) \geq f(1), \ldots, f(a_r) \geq f(1)$, and $f(1) = 1$. Therefore the inequality (14) follows.

Now, let $f(\underline{a}) = h_f(n)$. Then

$$\begin{aligned} \Omega_f(n) &= f(a_1) + \cdots + f(\underline{a}) + \ldots f(a_r) \\ &\leq (r-1)H_f(n) + f(\underline{a}) \\ &= (r-1)H_f(n) + h_f(n), \end{aligned}$$

as $f(a_1) \leq H_f(1), \ldots, f(a_r) \leq H_f(n)$, and $f(1) = 1$. Therefore the inequality (15) follows. □

Let

$$\gamma(n) = \prod_{i=1}^{r} p_i, \gamma(1) = 1.$$

For the properties of this function see, e.g., [14], and [2], where it is denoted by $\underline{\mathrm{mult}}(n)$.

* The function $\beta_\varphi(n) = \prod_{i=1}^{r} \varphi(a_i)$ denoted by $\varphi_l(n)$ has been studied by J. Sándor in 1996 (see [15]).

Theorem 4. *Let f be a multiplicative function such that*

$$f(p^a) \geq p^{f(a)} \tag{16}$$

for any prime p and integer $a \geq 1$. Then

$$h_f(n) \leq \frac{\log f(n)}{\log \gamma(n)} \tag{17}$$

for $n > 1$. If one has the converse of (16), i.e.,

$$f(p^a) \leq p^{f(a)} \tag{18}$$

then

$$H_f(n) \geq \frac{\log f(n)}{\log \gamma(n)}. \tag{19}$$

Proof. By (16) and the multiplicativity of f we can write

$$f(n) = \prod_{i=1}^{r} f(p_i^{a_i}) \geq \prod_{i=1}^{r} p_i^{f(a_i)} \geq \left(\prod_{i=1}^{r} p_i \right)^{h_f(n)},$$

i.e.

$$f(n) \geq (\gamma(n))^{h_f(n)},$$

and inequality (17) follows. The proof of (19) is similar. □

Remark 1. *We must mention that if the inequality (16) holds true for a set of integers $S \subset \mathcal{N}$, then clearly (17) will be true for any $n \in S$. The similar assertion is true for inequality (19).*

Remark 2. *Let $f(n) = \varphi(n)$. Then (16) is true for any $n \geq 2$ with $a \geq 2$, i.e., squarefull n and*

$$\varphi(p^a) = p^{a-1}(p-1) \geq p^{\varphi(a)}$$

by $\varphi(a) \leq a - 1$ for $a \geq 2$ and $p \geq 2$. In the same manner, (18) is true for $f(n) = \sigma(n)$, when $a \geq 2$ since $\sigma(p^a) \leq p^{\sigma(a)}$ is valid by $\frac{p^{a+1}-1}{p-1} < p^{a+1} \leq p^{\sigma(a)}$ by $\sigma(a) \geq a+1$ for $a \geq 2$. Also, (18) is true for $f(n) = \psi(n)$, too, where $\psi(n)$ denotes the Dedekind arithmetical function.

Thus, one has

$$h_\varphi(n) \leq \frac{\log \varphi(n)}{\log \gamma(n)}, \tag{20}$$

$$H_\sigma(n) \geq \frac{\log \sigma(n)}{\log \gamma(n)}, \tag{21}$$

$$H_\psi(n) \geq \frac{\log \psi(n)}{\log \gamma(n)}, \tag{22'}$$

when n is squarefull, i.e., when in the prime factorization of n, all $a_i \geq 2$. If n is not squarefull, then clearly $h_\varphi(n) = h_\sigma(n) = h_\psi(n) = 1$.

Remark 3. *If $n = \prod_{i=1}^{r} a_i > 1$, denote $n_f = \prod_{i=1}^{r} p_i^{f(a_i)}$. Then*

$$h_f(n) \leq \frac{\log n_f}{\log \gamma(n)} \leq H_f(n). \tag{22}$$

Indeed,

$$\left(\prod_{i=1}^{r} p_i\right)^{h_f(n)} \leq n_f \leq \left(\prod_{i=1}^{r} p_i\right)^{H_f(n)}$$

and (22) follows.

Let as above, $S \subset \mathcal{N}$. The asymptotic density of the set S is defined by

$$d(S) = \lim_{x\to\infty} \frac{S(x)}{x},$$

where $S(x)$ denotes the number of elements of S that are less than or equal to x. In 1951, H. Fast [8] defined the statistical convergence of the sequences. Let

$$S_\varepsilon = \{n : n \in \mathcal{N} \ \& \ |x_n - x| \geq \varepsilon\}.$$

Then the sequence of real numbers $\{x_n\}$ is convergent to x (in writing $\text{limstat}\ x_n = x$), if for any $\varepsilon > 0$:

$$d(S_\varepsilon) = 0.$$

Theorem 5. *Suppose that $f(n) \geq f(1) = 1$ for $n \geq 1$, and*

$$\text{limstat}\left(\frac{\Omega_f(n) - \omega(n)}{\log n}\right) = 0. \tag{23}$$

Then

$$\text{limstat}\left(\frac{h_f(n)}{\log n}\right) = \text{limstat}\left(\frac{H_f(n)}{\log n}\right) = 0. \tag{24}$$

Proof. By using inequality (14) of Theorem 3, clearly, the right side of (24) follows. The left side follows by

$$0 < h_f(n) \leq H_f(n). \qquad \square$$

Corollary 1. *If $1 = f(1) \leq f(n) \leq n$, then (23) is true.*

Indeed, in this case, one has $\Omega_f(n) \leq \Omega(n)$. In paper [16] it is proved that

$$\text{limstat}\left(\frac{\Omega(n) - \omega(n)}{\log n}\right) = 0. \tag{25}$$

Thus (23) follows.

Remark 4. *One has*

$$\text{limstat}\left(\frac{h_\varphi(n)}{\log n}\right) = \text{limstat}\left(\frac{H_\varphi(n)}{\log n}\right) = 0. \tag{26}$$

Indeed, $1 = \varphi(1) \leq \varphi(n)$ are true, so (26) are consequences of (24) from Theorem 5.

A connection between the arithmetical functions H_f and φ is provided by the following theorem.

Theorem 6. *Let $n_f > 1$ be defined in Remark 3. Then*

$$\varphi(n_f) \leq n_f - H_f(n). \tag{27}$$

Proof. First, we mention that

$$\varphi\left(\prod_{i=1}^{r} p_i\right) = \prod_{i=1}^{r}(p_i - 1) \leq \prod_{i=1}^{r} p_i - 1.$$

If, multiply both sides of inequality

$$\prod_{i=1}^{r}(p_i - 1) \leq \prod_{i=1}^{r} p_i - 1 \tag{28}$$

with $\prod_{i=1}^{r} p_i^{a-i-1}$, then we get the inequality

$$\varphi(n) \leq n - \prod_{i=1}^{r} p_i^{a_i-1} = n - \frac{n}{\gamma(n)}. \tag{29}$$

Now, we prove that $\prod_{i=1}^{r} p_i^{a_i-1} \geq H(n)$. This follows by $pa - 1 \geq a$ for any $p \geq 2$ and $a \geq 1$. Thus, by (29) we get

$$\varphi(n) \leq n - \frac{n}{\gamma(n)} \leq n - H(n). \tag{30}$$

Apply (30) to $n := n_f$, and using $H(n_f) = H_f(n)$, relation (27) follows. □

Remark 5. *By using (30) and the known inequality (see, e.g., [14], Chapter 3)*

$$\varphi(n)\sigma(n) > \frac{6}{\pi^2}n^2,$$

and U. Annapurna's inequality

$$\sigma(n) < (\frac{6}{\pi^2})^{\omega(n)-1}.n.\sqrt{n}$$

(see, [1]) we get

$$\left(\frac{6}{\pi^2}\right)^{\omega(n)-1}\sqrt{n} > \frac{\sigma(n)}{n} > \frac{n}{n - H(n)} \cdot \frac{6}{\pi^2}n^2. \tag{31}$$

Theorem 7. *One has*

$$h_f \leq \left(\prod_{i=1}^{r} f(a_i)^{p_i}\right)^{\frac{1}{\beta^*(n)}} \leq \frac{\sum_{i=1}^{r} p_i f(a_i)}{\beta^*(n)} \leq H_f, \tag{32}$$

where $\beta^*(n) = \sum_{i=1}^{r} p_i$.

Proof. Applying the weighted geometric-arithmetic inequality

$$\sum_{i=1}^{r} \lambda_i f(a_i) \geq \prod_{i=1}^{r} f(a_i)^{\lambda_i}, \tag{33}$$

with $\lambda_i = \frac{p_i}{\beta^*(n)}$ and $\lambda_i > 0, \sum_{i=1}^{r} \lambda_i = 1$, we get the second inequality of (32). Clearly, from $\prod_{i=1}^{r} f(a_i)^{\lambda_i} \geq \min\{f(a_1), f(a_2), \ldots, f(a_r)\} = h_f$, $\max\{f(a_1), f(a_2), \ldots, f(a_r)\} = H_f$ and all inequalities of (32) the result follows. □

Let us define the converse factor (see [2])

$$CF(n) = \prod_{i=1}^{r} a_i^{p_i}.$$

Letting $f(n) = n$ in (32), and using the function †

$$B(n) = \sum_{i=1}^{r} \alpha_i . p_i,$$

we get from (32)

$$h(n) \le CF(n)^{\frac{1}{\beta^*(n)}} \le \frac{B(n)}{\beta^*(n)} \le H(n). \tag{34}$$

Corollary 2. *One has the asymptotic relation for $x \to \infty$:*

$$\sum_{2 \le n \le x} (CF(n))^{\frac{1}{\beta^*(n)}} \sim x. \tag{35}$$

Proof. By a result of J.-M. Koninck, P. Erdős and A. Ivić (see [14], p. 144), one has

$$\sum_{2 \le n \le x} \frac{B(n)}{\beta^*(n)} \sim x.$$

Now, using Niven's result $\sum_{n \le x} h(n) \sim x$, by the first two inequalities of (34) we get (35). □

Remark 6. *It is immediate that n and $CF(n)$ cannot be compared, as e.g., for*

$$n = p^3 \prod_{i=1}^{r} p_i^{p_i},$$

where $p > \max\{p_1, p_2, \ldots, p_r\}$ one has $n \le CR(n)$. On the other hand, if $n = p^\alpha \prod_{i=1}^{r} p_i^{p_i}$ with $\max\{p_1, p_2, \ldots, p_r\} < p < \alpha$ and $p, \alpha \ge 3$, then $n > CR(n)$.

Now, we will consider the arithmetical function

$$\beta_h^*(n) = \sum_{i=1}^{r} p_i^{h(n)}. \tag{36}$$

Clearly,

$$\beta^*(n) = \sum_{i=1}^{r} p_i \le \beta_h^*(n) \le \sum_{i=1}^{r} p_i^{a_i} \tag{37}$$

because $h(n) \le a_i$ for $1 \le i \le r$.

Now, let $B^1(n) = \sum_{i=1}^{r} p_i^{a_i}$ as defined in [14], p. 147. We have the following theorem.

† The properties of this function are discussed independently in [6, 14] and [4], and in the last paper it is denoted by ζ.

Theorem 8. *One has for $x \to \infty$:*

$$\sum_{2 \le n \le x} \beta_h^*(n) \sim \frac{\pi^2}{12} \cdot \frac{x^2}{\log x} \tag{38}$$

Proof. It is known that

$$\sum_{2 \le n \le x} \beta^*(n) \sim \frac{\pi^2}{12} \cdot \frac{x^2}{\log x}$$

(by S. M. Kerawala) and

$$\sum_{2 \le n \le x} B^1(n) \sim \frac{\pi^2}{12} \cdot \frac{x^2}{\log x}$$

(by T. Z. Xuan, see [14], p. 147). Therefore, by inequality (37), relation (38) follows. □

For the function $H(n)$ we have the following asymptotic result.

Theorem 9. *One has*

$$\sum_{2 \le n \le x} \log \frac{n}{H(n)} \sim x \log x \tag{39}$$

for $x \to \infty$.

Proof. As we have seen in relation (30), one has $H(n) \le \frac{n}{\gamma(n)}$. On the other hand, $H(n) \ge 1$. These together give the double inequality

$$\log \gamma(n) \le \log \frac{n}{H(n)} \log n. \tag{40}$$

Now, by the integral test, it is immediate that

$$\sum_{2 \le n \le x} \log n \sim \int_2^x \log t dt = x \log x - x + C \sim x \log x.$$

On the other hand, by a result of L. Panaitopol (see [14], p. 208) for $x \to \infty$,

$$\sum_{n \le x} \log \gamma(n) \sim x \log x. \tag{41}$$

Thus, together with inequality (40), relation (39) holds. □

Theorem 10. *One has*

$$\limsup \log \beta_\sigma(n) \cdot \frac{\log \log n}{\log n} = \frac{\log 3}{2}. \tag{42}$$

Proof. Here, $\beta_\sigma(n) = \prod_{i=1}^{r} \sigma(a_i)$. In what follows, we will use a classical Theorem by Drozdova and Freiman (see, e.g., [10]). Let f be a multiplicative function with the property $f(p^k) = g(k)$, where p is a prime, and $g(k)$ depends only on k. Suppose $g(k) \ge 1$ and there exists k_0 with $g(k_0) > 1$. Assume that for a certain number $a > 0$ one has for $k \to \infty$

$$\log g(k) = O\left(k^{1-a}\right).$$

Then, the maximal order of the magnitude of $\log f(n)$ is given by $\frac{\log g(m)}{m} \cdot \frac{\log n}{\log \log n}$, where m is defined by

$$\frac{\log g(k)}{k} \leq \frac{\log g(m)}{m}$$

for $k \leq m$ and

$$\frac{\log g(k)}{k} < \frac{\log g(m)}{m}$$

for $k > m$.

In our case, clearly β_σ is a multiplicative function and $\beta_\sigma(p^k) = \sigma(k)$. It is known that for $k \geq 3$ (due to C. C. Lindner, see [5], p. 77)

$$\sigma(k) < k\sqrt{k} = k^{\frac{3}{2}}.$$

Thus, for $k \geq 3$:

$$\frac{\log g(k)}{k} < \frac{3}{2}\frac{\log k}{k}.$$

Define the function $U(x) = \frac{\log x}{x}$ for $x \geq 1$. As

$$U'(x) = \frac{1 - \log x}{x^2} \leq 0$$

for $x \geq e$, it follows that $x_0 = e$ is a maximum point of $U(x)$ and

$$U(x) \leq U(e) = \frac{1}{e}.$$

But $2 < e < 3$ and $\frac{\log 2}{2} < \frac{\log 3}{3}$. So, we get that $\frac{\log k}{k} < \frac{\log 3}{3}$ for $k \geq 3$. Thus, we get for $k \geq 3$ that

$$\frac{\log g(k)}{k} < \frac{3}{2} \cdot \frac{\log 3}{3} = \frac{\log 3}{2}.$$

But, as $\frac{\log g(k)}{k} = \frac{3}{2}$, we obtain that in the Theorem by Drozdova and Freiman $m = 2$ can be selected. This proves Theorem 10. $\square$

3 Open problems

Finally, we state some open problems. Determine the remainder terms in the asymptotic expansions of (3), (4), (35), (38), (39). In near future, we will conduct analogous research on other airthmetical functions.

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