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A note on the length of some finite continued fractions

Khalil Ayadi[1] and Chiheb Ben Bechir[2]

[1] Sfax University, Higher Institute of Industrial Management of Sfax, Tunisia
e-mail: khalil.ayadi@isgis.usf.tn

[2] Sfax University, Faculty of Sciences, Lab AGTS, Tunisia
e-mail: chihebbenbechir@gmail.com

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Abstract: In this paper, based on a 2008 result of Lasjaunias, we compute the lengths of simple continued fractions for some rational numbers whose numerators and denominators are explicitly given.

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1 Introduction

It is well known that any $\alpha \in \mathbb{Q}$ has a unique simple continued fraction expansion

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} = [a_0, a_1, \ldots, a_n]$$

where $a_1, \ldots, a_n$ are positive integers, $a_0 \in \mathbb{Z}$ and $a_n \geq 2$. The sequence $(a_i)_{0 \leq i \leq n}$ is called the sequence of partial quotients of α. The number n is called the depth of the continued fraction for α, denoted by $\psi(\alpha)$. This function was introduced by Mendès France in [5].

Also, as it is well known, there are close connections between the arithmetic behaviour of algebraic number fields and that of the algebraic function fields in one variable. For instance, in the 19th century the theory of polynomial continued fractions was started. More precisely, the ring of polynomials $\mathbb{K}[T]$ over a given field $\mathbb{K}$ can play the role of the ring of integers $\mathbb{Z}$ and thus its field of fractions $\mathbb{K}(T)$ will correspond to the field of rational numbers $\mathbb{Q}$. Then, the role of $\mathbb{R}$ will be played by the completion of $\mathbb{K}(T)$ with respect to the valuation ord associated to the degree, that is, by $\mathbb{K}((T^{-1}))$, the field of formal Laurent series in T^{-1}. If $\alpha \in \mathbb{Q}(T)$, then α has a unique finite continued fraction $\alpha = [a_0, a_1, \ldots, a_n]$ where $(a_i)_{0\leq i\leq n}$ are polynomials with coefficients in $\mathbb{Q}$ such that $\deg a_i \geq 1$ for $i > 0$. If $\alpha \in \mathbb{Q}((T^{-1}))\backslash\mathbb{Q}(T)$ then α can be represented by an infinite continued fraction expansion. In particular, if the partial quotients in the continued fraction expansion of α belong to $\mathbb{Z}[T]$, we say that it is specializable, see [9]. The reader can consult [1] for several interesting specializable continued fraction expansions describing infinite series.

Almost all of the classical notions and results can be translated into the function fields setting. In 1973, Mendès France [6] gave an upper bound on $\psi(nx)$, when $x \in \mathbb{Q}$ and $n \in \mathbb{N}$. An analogue of this result was given in the function field setting, see [8].

We are interested in the length of continued fractions of rational numbers. Pourchet have proved that for all integer $p > 1, q > 1$ such that $\gcd(p, q) = 1$ we have $\lim_{+\infty} \psi((p/q)^n) = +\infty$. In [5], Mendès France asked the following problem: *Let $(p_n/q_n)_{n\geq 1}$ be a sequence of rational numbers, what can we say about the sequence $\big(\psi(p_n/q_n)\big)_{n\geq 1}$? Is it true that $\big(\psi(p_n/q_n)\big)_{n\geq 1}$ is of the order of* $\log(q_n)$*?* He proved that if F is a rational function with rational coefficients, then the sequence $\big(\psi(F(n)))\big)_{n\geq 1}$ is periodic from a certain point onward. On the other hand, Corvaja and Zannier [3] have proved that for some power sums α and β over $\mathbb{Q}$, the lengths of the continued fractions $\alpha(n)/\beta(n)$ tend to infinity as $n \to \infty$. In this direction, we will construct in this note a sequence of rational numbers with continued fractions of constant length. In parallel, we will expose another sequence of quotients with increasing lengths. Now let us see some examples of sequences of rational numbers and their corresponding lengths:

1) We consider the Fibonacci numbers defined by the recurrence relation $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. We know that $\frac{F_{n+1}}{F_n} = [\overbrace{1, 1, \ldots, 1}^{n\ 1's}]$. So $\big(\psi(F_{n+1}/F_n)\big)_{n\geq 1} = n - 1$. Since F_n is asymptotic to $\varphi^n/\sqrt{5}$, where φ is the golden ratio, then $\big(\psi(F_{n+1}/F_n)\big)_{n\geq 1}$ $\sim \log(F_n)/\log(\varphi)$.

2) Let $(S_n)_{n\geq 1}$ be the sequence of rational numbers defined by the partial sums:

$$S_n = \sum_{i=0}^{n} \frac{1}{u^{2^i}} = \frac{p(n)}{q(n)} = \frac{p(n)}{u^{2^n}}$$

for some integer $u \geq 3$. The continued fraction of this sum was given in [7] and it is of length $2^n + 1$. So that $\big(\psi(p(n)/q(n))\big)_{n\geq 1} \sim \log(q(n))/\log(u)$. Our result is based on the following proposition given by Lasjaunias in his paper [4] for the continued fraction expansion of a rational fraction $P/Q \in \mathbb{Q}(T)$ such that $P = (T^2 - 1)^k$, $k \geq 1$ and

$$Q = \int_0^T (x^2 - 1)^{k-1} dx = \sum_{i=0}^{k-1} (-1)^{k-i-1} C_{k-1}^i (2i + 1)^{-1} T^{2i+1}.$$

Proposition 1.1. *Let $k \geq 1$. Then we have in $\mathbb{Q}(T)$ the following continued fraction expansion*

$$P/Q = [u_1T, u_2T, \ldots, u_{2k}T] \tag{1}$$

where $u_i \in \mathbb{Q}^*$ for all $1 \leq i \leq 2k$. For $1 \leq i \leq 2k$, we have

$$u_i = (2k-2i+1)\Big(\prod_{1\leq j<i/2}(2j)(2k-2j)/\prod_{1\leq j<(i+1)/2}(2j-1)(2k-2j+1)\Big)^{(-1)^i},$$

where as usual the empty product is equal to 1. *Moreover, if we set*

$$w_k = -16^{k-1}(2k-1)^{-2}(C_{k-1}^{2k-2})^{-2}$$

then, we also have

$$u_{2k+1-i} = w_k^{(-1)^{i+1}}u_i \ \ for \ \ 1 \leq i \leq 2k$$

and consequently

$$w_kP/Q = [u_{2k}T, u_{2k-1}T, \ldots, u_1T].$$

The analogue of this continued fraction in the function field case allows us to describe the continued fraction for many algebraic irrational power series, see for example [2, 4]. The great interest of these continued fractions will give us the opportunity to study their analogues in the real number case.

2 Main results

We have the following lemma whose proof is simple.

Lemma 2.1. *For $\beta \in \mathbb{Q}^*$ and $a, b \in \mathbb{N}^*$ we have*

$$[a, -b, -\beta] = [a-1, 1, b-1, \beta]. \tag{2}$$

Proposition 2.1. *Let $k \geq 1$ be an integer. Let $P = (T^2-1)^k$ and $Q = \int_0^T (x^2-1)^{k-1}dx$. Let $X_i = \prod_{j=1}^{i}(2j)(2k-2j)$ and $Y_i = \prod_{j=1}^{i}(2j-1)(2k-2j+1)$ with as usual, in the sequel $X_0 = 1$. Then*

$$\begin{aligned}\frac{P(X_{k-1}Y_kT)}{Q(X_{k-1}Y_kT)} &= \frac{((X_{k-1}Y_kT)^2-1)^k}{\int_0^{X_{k-1}Y_kT}(x^2-1)^{k-1}dx} \\ &= [(2k-1)X_{k-1}Y_kT, (2k-3)X_{k-1}\prod_{j=2}^{k}(2j-1)(2k-2j+1)T, \\ &\quad (2k-5)Y_1Y_k\prod_{j=2}^{k}(2j)(2k-j)T, (2k-5)X_1X_k\prod_{j=3}^{k}(2j-1)(2k-2j+1)T, \\ &\quad \ldots, (-2k+3)Y_{k-1}Y_kT, (-2k+1)(X_{k-1})^2T]\end{aligned}$$

with all partial quotients belonging to $\mathbb{Z}[T]$.

Theorem 2.1. *Let $k \geq 1$ and $n \geq 1$. For $1 \leq i \leq k$ we let $X_i = \prod_{j=1}^{i}(2j)(2k-2j)$ and $Y_i = \prod_{j=1}^{i}(2j-1)(2k-2j+1)$ with $X_0 = 1$. Then the rational number*

$$\frac{p}{q} = \frac{((X_{k-1}Y_k n)^2 - 1)^k}{\sum_{i=0}^{k-1}(-1)^{k-i-1}C_i^{k-1}(2i+1)^{-1}(X_{k-1}Y_k)^{2i+1}n^{2i+1}},$$

has a continued fraction expansion of length $2k$.

Remark 2.1. *Note that the last theorem gives us, for all $k \geq 1$, an infinity of rational numbers of length $2k$. Further we note that for all $k \geq 1$*

$$\begin{aligned} -w_k\frac{p}{q} &= [-u_{2k}X_{k-1}Y_kn, \dots, -u_{k+1}X_{k-1}Y_kn, -u_kX_{k-1}Y_kn, \dots, -u_2X_{k-1}Y_kn, \\ &\quad -u_1X_{k-1}Y_kn] \\ &= [-u_{2k}X_{k-1}Y_kn, \dots, -u_{k+1}X_{k-1}Y_kn - 1, 1, u_kX_{k-1}Y_kn - 1, \dots, u_2X_{k-1}Y_kn, \\ &\quad u_1X_{k-1}Y_kn] \end{aligned}$$

by applying the identity (2).

The following corollary improves Mendès France results [5].

Corollary 2.1. *Let $k \geq 1$ be a fixed integer. For all $n \geq 1$ the sequence of rational numbers*

$$F_k(n) = \frac{p(n)}{q(n)} = \frac{((X_{k-1}Y_k n)^2 - 1)^k}{\sum_{i=0}^{k-1}(-1)^{k-i-1}C_i^{k-1}(2i+1)^{-1}(X_{k-1}Y_k)^{2i+1}n^{2i+1}}$$

satisfies $\psi(F_k(n)) = 2k$. So the sequence $(\psi(F_k(n)))_{n\geq 1}$ is constant and equal to $o(\log(q(n)))$.

The following corollary give us the length of the simple continued fraction of quotients of sequences having special progressions kind.

Corollary 2.2. *Let $n \geq 1$ be a fixed integer. For all $k \geq 1$ the sequence*

$$G_n(k) = \frac{p(k)}{q(k)} = \frac{((X_{k-1}Y_k n)^2 - 1)^k}{\sum_{i=0}^{k-1}(-1)^{k-i-1}C_i^{k-1}(2i+1)^{-1}(X_{k-1}Y_k)^{2i+1}n^{2i+1}}$$

has an unbounded length $\psi(G_n(k)) = 2k$. Further $\psi(G_n(k)) = o(\log(q(k)))$.

3 Proofs of the main results

Proof of Proposition 2.1. We follow the same notation of the partial quotients of (1). We have $u_1 = 2k-1$ and for $1 \leq i \leq k$

$$u_{2i} = (2k-4i+1)X_{i-1}/Y_i \qquad u_{2i-1} = (2k-4i+3)Y_{i-1}/X_{i-1}.$$

Then

$$u_{2k} = (-2k+1)X_{k-1}/Y_k \qquad u_{2k-1} = (-2k+3)Y_{k-1}/X_{k-1}.$$

Now, to obtain partial quotients in the continued fraction (1) with coefficients in $\mathbb{Z}$, we substitute T by $X_{k-1}Y_kT$. We get:

$$\begin{aligned}
u_1X_{k-1}Y_kT &= (2k-1)X_{k-1}Y_kT,\\
u_2X_{k-1}Y_kT &= (2k-3)(X_0/Y_1)X_{k-1}Y_kT = (2k-3)X_{k-1}\prod_{j=2}^{k}(2j-1)(2k-2j+1)T,\\
u_3X_{k-1}Y_kT &= (2k-5)(Y_1/X_1)X_{k-1}Y_kT = (2k-5)Y_1Y_k\prod_{j=2}^{k}(2j)(2k-2j)T,\\
u_4X_{k-1}Y_kT &= (2k-7)(X_1/Y_2)X_{k-1}Y_kT = (2k-5)X_1X_k\prod_{j=3}^{k}(2j-1)(2k-2j+1)T,\\
&\vdots\\
u_{2k-1}X_{k-1}Y_kT &= (-2k+3)Y_{k-1}Y_kT\\
u_{2k}X_{k-1}Y_kT &= (-2k+1)(X_{k-1})^2T.
\end{aligned}$$

□

Proof of Theorem 2.1. According to the result of the last proposition, since $u_1 = 2k-1$ and for $1 \le i \le k$

$$\begin{aligned}
u_{2i} &= (2k-4i+1)X_{i-1}/Y_i\\
u_{2i-1} &= (2k-4i+3)Y_{i-1}/X_{i-1},
\end{aligned}$$

then

$$\begin{aligned}
u_{2i} < 0 \text{ for } i \ge \frac{k+1}{2} &\text{ when } k \text{ is odd}\\
u_{2i-1} < 0 \text{ for } i \ge \frac{k+2}{2} &\text{ when } k \text{ is even.}
\end{aligned}$$

So $u_{k+1}, u_{k+2}, \dots, u_{2k}$ are negative integers. If we specialize by "$T = n$" with $n \ge 1$ and we apply the identity (2) we obtain

$$\begin{aligned}
\frac{p}{q} = \frac{P(X_{k-1}Y_kn)}{Q(X_{k-1}Y_kn)} &= [u_1X_{k-1}Y_kn, u_2X_{k-1}Y_kn, \dots, u_kX_{k-1}Y_kn, u_{k+1}X_{k-1}Y_kn,\\
&\quad u_{k+2}X_{k-1}Y_kn, \dots, u_{2k}X_{k-1}Y_kn]\\
&= [u_1X_{k-1}Y_kn, u_2X_{k-1}Y_kn, \dots, u_kX_{k-1}Y_kn - 1, 1,\\
&\quad -u_{k+1}X_{k-1}Y_kn - 1, -u_{k+2}X_{k-1}Y_kn, \dots, -u_{2k}X_{k-1}Y_kn].
\end{aligned}$$

This continued fraction has a length equal to $2k$. □

Proof of Corollary 2.1. By a simple calculation of the limit, we easily show that

$$(\psi(F_k(n)))_{n\ge1} = o(\log(q(n))).$$

□

Proof of Corollary 2.2. By a simple calculation of the limit, we easily show that

$$\psi(G_n(k)) = o(\log(q(k))).$$

□

Example 3.1. *For $k = 2$. We have $X_1 = 4$, $Y_1 = 3$ and $Y_2 = 9$, then $X_{k-1}Y_k = X_1Y_2 = 36$. So*

$$\frac{(1296T^2-1)^2}{15552T^3-36T} = [108T, 12T, -27T, -48T].$$

So for all $n \geq 1$

$$F_2(n) = \frac{(1296n^2-1)^2}{15552n^3-36n} = [108n, 12n, -27n, -48n] = [108n, 12n-1, 1, 27n-1, 48n].$$

Furthermore, $w_2 = -\frac{4}{9}$, then

$$-w_2F_2(n) = -w_2\frac{(1296n^2-1)^2}{15552n^3-36n} = [48n, 27n-1, 1, 12n-1, 108n].$$

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