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Recurrence relations connecting mock theta functions and restricted partition functions

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Abstract: In this paper, we provide some recurrence relations connecting restricted partition functions and mock theta functions. Elementary manipulations are used including Jacobi triple product identity, Euler's pentagonal number theorem, and Ramanujan's theta functions for finding the recurrence relations.

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1 Introduction

A partition of a positive integer n is a finite, non-increasing sequence of positive integers $a_1, a_2, a_3, \ldots, a_r$ where

$$\sum_{i=1}^{r} a_i = n.$$

The number of partitions of the positive integer n are denoted by $p(n)$. For example, the partitions of 5 are:

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1.$$

Here, $p(5) = 7$. By convention, $p(0) = 1$. If we impose certain restrictions on the number of parts or/and the size of parts on a partition of positive integer n then it is known as Restricted partition function. Let us denote the number of partitions of n into distinct parts by $q(n)$. For example, the partitions of 5 into distinct parts are $5, 4+1, 3+2$. So, here $q(n) = 3$.

Euler [1] has studied the recurrence relation for unrestricted partition function $p(n)$. For $n > 0$,

$$p(n) + \sum_{k=1}^{\infty}(-1)^j(p(n-\xi(k)) + p(n-\xi(-k)) = 0, \text{ where } \xi(k) = \frac{3k^2+k}{2}.$$

In 1921, parity recurrences for $p(n)$ were studied by MacMahon [12]. Later on, Ewell [6] proved several recurrence relations connecting $p(n)$, $q(n)$, and some other restricted partition functions. One of his results is given as:

Theorem 1.1. [6] *For each non-negative integer n,*

$$q(2n) = p(n) + \sum_{k=1}^{\infty} p(n-k(4k-1)) + p(n-k(4k+1)),$$

$$q(2n+1) = \sum_{k=0}^{\infty} p(n-k(4k+3)) + p(n-1-k(4k+5)).$$

Then, Ono et al. [15], in 1996, derived a number of recurrence relations for partitions into distinct odd parts, partitions into an even number of parts, partitions into an odd number of parts and many more, using classical techniques. For instance,

$$q(n) + \sum_{n=1}^{\infty}(-1)^j q(n-2\xi(k)) + q(n-2\xi(-k))) = \begin{cases} 1, & n = \frac{m(m+1)}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

In addition, Merca [13] in 2017, has derived new recurrence relations for Euler's partition function $p(n)$ using bisectional pentagonal number theorem and obtained a logical method to determine the parity of $p(n)$. Similarly, Choliy et al. [4] proved the following recurrences for $qq(n)$, where $qq(n)$ denotes the number of partitions into distinct and odd parts.

$$p(n) - 2p(n-1) + 2p(n-4) - 2p(n-9) + \cdots + (-1)^j 2p(n-k^2) + \cdots = (-1)^n qq(n).$$

Analogously, Nyirenda [14] has also given some parity and recurrence formulas for partition functions. For more information on recurrence relations, we refer to [5, 7]. Recently, the first two authors [10] presented some recurrence relations for the mock theta function $B(q)$ of order 2. Motivated by this work, we prove recurrence relations for partition functions associated with mock theta functions and restricted partition functions.

We will begin by examining the partition function $p_\omega(n)$, which is associated with the third order mock theta function $\omega(q)$, introduced by Watson [18] and due to Andrews et al. [2]. $\omega(q)$ is defined as:

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2},$$

where

$$(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i), \qquad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$$

for a positive integer n. We will assume that a, q are complex numbers with $|q| < 1$. Also, we have:

$$\sum_{n=1}^{\infty} p_\omega(n)q^n = q\omega(q).$$

where $p_\omega(n)$ counts the number of partitions of n in which all the odd parts are less than twice the smallest part.

Next, we consider the second partition function, $spt_\omega(n)$, which is also associated with the third order mock theta function $\omega(q)$, as described in [2]. $spt_\omega(n)$ is the smallest part function that counts the total number of smallest parts in the partition enumerated by $p_\omega(n)$. Its generating function is:

$$\sum_{n=1}^{\infty} spt_\omega(n)q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}.$$

Finally, we examine the third partition function, $v_0(n)$, which is associated with the eighth order mock theta function $V_0(q)$ given by Gordon and McIntosh [8]. $v_0(n)$ is defined as:

$$\sum_{n=0}^{\infty} v_0(n)q^n = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q;q^2)_n} = \frac{V_0(q)+1}{2}.$$

In this paper, we provide recurrence relations for three partition functions: $p_\omega(n)$, $spt_\omega(n)$, and $v_0(n)$. We also establish their connection with restricted partition functions $\overline{p}_l(n)$ and $p_{ld}(n)$, where $\overline{p}_l(n)$ is the number of overpartitions of n with l copies, and $p_{ld}(n)$ is the number of partitions into distinct parts with l copies, where:

- $\overline{p}_l(n)$: Number of overpartitions of n with l copies,
- $p_{ld}(n)$: Number of partitions into distinct parts with l copies.

This paper is organized as follows: Section 2 provides some preliminaries that will be used in the proof section. Section 3 contains the recurrence relations for partition functions $p_\omega(n)$ and $spt_\omega(n)$. In Section 4, the recurrence relations for $v_0(n)$ are derived using classical techniques. Before proceeding to the main results of the paper, let us review the preliminaries in Section 2.

2 Preliminary results

Ramanujan's theta function is defined as:

$$f(a,b)=\sum_{n=-\infty}^{n=\infty} a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}},$$

for $|ab|<1$. Some special cases of $f(a,b)$ are:

$$\psi(x)=f(x,x)=\sum_{n=0}^{\infty} x^{\frac{n(n+1)}{2}}, \tag{1}$$

$$\varphi(x)=f(x,x^3)=1+2\sum_{n=1}^{\infty} x^{n^2}. \tag{2}$$

Jacobi Triple Product Identity [1]. If q,z are complex numbers such that $\mid q\mid<1$ and $z\neq 0$ then

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}z^{-1})(1+q^{2n-1}z)=\sum_{n=-\infty}^{\infty} q^{n^2}z^n=1+\sum_{n=1}^{\infty} q^{n^2}(z^n+z^{-n}),$$

and the classical Jacobi's identity is

$$(q;q)_\infty^3=\sum_{n=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2}.$$

Euler's pentagonal number theorem [1]. For $\mid q\mid<1$, we have

$$\prod_{n=1}^{\infty}(1-q^n)=1+\sum_{m=1}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)}(1+q^m)=\sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)}.$$

To shorten the notation, we use $f_l^k=(q^l;q^l)_\infty^k$, where l and k are positive integers.
The generating function for $\overline{p}_l(n)$ and $p_{ld}(n)$ given as:

$$\sum_{n=0}^{\infty}\overline{p}_l(n)q^n=\left(\frac{f_2}{f_1^2}\right)^l, \tag{3}$$

$$\sum_{n=0}^{\infty}p_{ld}(n)q^n=\left(\frac{f_2}{f_1}\right)^l. \tag{4}$$

3 Recurrence relations for $p_\omega(n)$ and $spt_\omega(n)$

We have the relations from [17] given as:

$$\sum_{n=0}^{\infty}p_\omega(8n+4)q^n=\frac{4f_2^{10}}{f_1^9}, \tag{5}$$

$$\sum_{n=0}^{\infty}spt_\omega(2n+1)q^n=\frac{f_2^8}{f_1^5}. \tag{6}$$

Theorem 3.1. *We have*

$$p_\omega(8n+4)=4p_{9d}(n)-4p_{9d}(n-2)-4p_{9d}(n-4)+4p_{9d}(n-10)+4p_{9d}(n-14)$$
$$+\cdots+4(-1)^k(p_{9d}(n-k(3k-1))+p_{9d}(n-k(3k+1)))+\cdots .$$

Proof. We have (5)

$$
\begin{aligned}
\sum_{n=0}^{\infty} p_\omega(8n+4)q^n &= \frac{4f_2^{10}}{f_1^9} \\
&= 4\left(\frac{f_2}{f_1}\right)^9 f_2 \\
&= 4\left(\sum_{n=0}^{\infty} p_{9d}(n)q^n\right)\left(\sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k+1)}\right) \\
&= 4\left(\sum_{n=0}^{\infty} p_{9d}(n)q^n\right)\left(1+\sum_{k=-\infty}^{-1}(-1)^k q^{k(3k+1)}+\sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)}\right) \\
&= 4\left(\sum_{n=0}^{\infty} p_{9d}(n)q^n\right)\left(1+\sum_{k=1}^{\infty}(-1)^k q^{k(3k-1)}+\sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)}\right) \\
&= 4\sum_{n=0}^{\infty} p_{9d}(n)q^n + 4\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^k p_{9d}(n)q^{k(3k-1)+n} \\
&\qquad + 4\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^k p_{9d}(n)q^{k(3k+1)+n} \\
&= 4\sum_{n=0}^{\infty} p_{9d}(n)q^n + 4\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^k p_{9d}(n-k(3k-1))q^n \\
&\qquad + 4\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^k p_{9d}(n-k(3k+1))q^n.
\end{aligned}
$$

On comparing the coefficients we have

$$
\begin{aligned}
p_\omega(8n+4) &= 4p_{9d}(n) + 4\sum_{k=1}^{\infty}(-1)^k\left(p_{9d}(n-k(3k-1))+p_{9d}(n-k(3k+1))\right) \\
&= 4p_{9d}(n) - 4p_{9d}(n-2) - 4p_{9d}(n-4) + 4p_{9d}(n-10) + 4p_{9d}(n-14) \\
&\qquad + \cdots + 4(-1)^k\left(p_{9d}(n-k(3k-1))+p_{9d}(n-k(3k+1))\right) + \cdots . \quad \square
\end{aligned}
$$

Theorem 3.2. *We have*

$$spt_\omega(2n+1) = p_{5d}(n) - 3p_{5d}(n-2) + 5p_{5d}(n-6) + \cdots + (-1)^k(2k+1)p_{5d}(n-k(k+1)) + \cdots .$$

Proof. We have (6)

$$
\begin{aligned}
\sum_{n=0}^{\infty} spt_\omega(2n+1)q^n &= \frac{f_2^8}{f_1^5} \\
&= \left(\frac{f_2}{f_1}\right)^5 f_2^3 \\
&= \left(\sum_{n=0}^{\infty} p_{5d}(n)q^n\right)\left(\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k (2k+1) p_{5d}(n) q^{k(k+1)+n} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} (-1)^k (2k+1) p_{5d}(n-k(k+1)) \right) q^n.
\end{aligned}
$$

On comparing the coefficients, we get

$$
\begin{aligned}
spt_{\omega}(2n+1) &= \sum_{k=0}^{\infty} (-1)^k (2k+1) p_{5d}(n-k(k+1)) \\
&= p_{5d}(n) - 3p_{5d}(n-2) + 5p_{5d}(n-6) + \cdots + \\
&\qquad (-1)^k (2k+1) p_{5d}(n-k(k+1)) + \cdots .
\end{aligned}
$$
□

4 Recurrence relations for $v_0(n)$

Theorem 4.1. [3] *We have*

$$\sum_{n=0}^{\infty} v_0(4n+1) q^n = \frac{f_2^5}{f_1^4}, \tag{7}$$

$$\sum_{n=0}^{\infty} (4n+2) q^n = 2\frac{f_4^4}{f_1^2 f_2}, \tag{8}$$

$$\sum_{n=0}^{\infty} v_0(6n+4) q^n = 3\frac{f_2^3 f_6^3}{f_1^4 f_4}, \tag{9}$$

$$\sum_{n=0}^{\infty} v_0(8n+5) q^n = 4\frac{f_2^2 f_4^4}{f_1^5}, \tag{10}$$

$$\sum_{n=0}^{\infty} v_0(8n+6) q^n = 4\frac{f_2^6 f_8^2}{f_1^6 f_4}, \tag{11}$$

$$\sum_{n=0}^{\infty} v_0(12n+5) q^n = 4\frac{f_2^{10} f_3^2}{f_1^{10} f_6}, \tag{12}$$

$$\sum_{n=0}^{\infty} v_0(16n+12) = 16\frac{f_2^{11} f_4^2}{f_1^{12}}. \tag{13}$$

Theorem 4.2. *We have*

$$
\begin{aligned}
v_0(4n+1) = p_{4d}(n) &- (p_{4d}(n-2) + p_{4d}(n-4)) + (p_{4d}(n-10) + p_{4d}(n-14)) \\
&+ \cdots + (-1)^k (p_{4d}(n-k(3k-1)) + p_{4d}(n-k(3k+1)) + \cdots .
\end{aligned}
$$

Proof. We have (7)

$$
\begin{aligned}
\sum_{n=0}^{\infty} v_0(4n+1) q^n &= \frac{f_2^5}{f_1^4} \\
&= \left(\frac{f_2}{f_1} \right)^4 f_2
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\sum_{n=0}^{\infty} p_{4d}(n) q^n\right) \left(\sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k+1)}\right)\\
&= \left(\sum_{n=0}^{\infty} p_{4d}(n) q^n\right) \left(1+\sum_{k=1}^{\infty}(-1)^k q^{k(3k-1)} + \sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)}\right)\\
&= \sum_{n=0}^{\infty} p_{4d}(n) q^n + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^k p_{4d}(n) q^{k(3k-1)+n}\\
&\qquad + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^k p_{4d}(n) q^{k(3k+1)+n}\\
&= \sum_{n=0}^{\infty} p_{4d}(n) q^n + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^k p_{4d}(n-k(3k-1)) q^{n}\\
&\qquad + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^k p_{4d}(n-k(3k+1)) q^{n}.
\end{aligned}
$$

On comparing the coefficients, we have

$$
\begin{aligned}
v_0(4n+1) &= p_{4d}(n) + \sum_{k=1}^{\infty}(-1)^k \left(p_{4d}(n-k(3k-1)) + p_{4d}(n-k(3k+1))\right)\\
&= p_{4d}(n) - (p_{4d}(n-2) + p_{4d}(n-4)) + (p_{4d}(n-10) + p_{4d}(n-14))\\
&\qquad + \cdots + (-1)^k (p_{4d}(n-k(3k-1)) + p_{4d}(n-k(3k+1)) + \cdots .
\end{aligned}
$$ □

Theorem 4.3. *We have*

$$
v_0(4n+2) = 2\sum_{m=0}^{\infty} \sum_{c=0}^{\lfloor \frac{n-2m(m+1)}{2} \rfloor} (-1)^m (2m+1) \overline{p}\left(n-2m(m+1)-2c\right) \overline{p}(c).
$$

Proof. We have (8)

$$
\begin{aligned}
\sum_{n=0}^{\infty} v_0(4n+2) q^n &= 2\frac{f_4^4}{f_1^2 f_2}\\
&= 2\frac{f_4^4}{f_2^2}\frac{f_2}{f_1^2}\\
&= 2\left(\frac{f_4}{f_2^2}\right)\left(\frac{f_2}{f_1^2}\right) f_4^3\\
&= 2\left(\sum_{k=0}^{\infty} \overline{p}(k) q^{2k}\right)\left(\sum_{m=0}^{\infty} \overline{p}(m) q^{m}\right)\left(f_4^3\right)\\
&= 2\left(\sum_{k=0}^{\infty} \overline{p}(k) q^{2k}\right)\left(\sum_{m=0}^{\infty} \overline{p}(m) q^{m}\right)\left(\sum_{n=0}^{\infty}(-1)^n(2n+1) q^{2n(n+1)}\right)\\
&= 2\left(\sum_{k=0}^{\infty}\sum_{c=0}^{\lfloor \frac{k}{2} \rfloor} \overline{p}\left(k-2c\right) \overline{p}(c) q^{k}\right)\left(\sum_{n=0}^{\infty}(-1)^n(2n+1) q^{2n(n+1)}\right)
\end{aligned}
$$

$$= 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{c=0}^{\lfloor\frac{n-2m(m+1)}{2}\rfloor}(-1)^m(2m+1)\overline{p}\left(n-2m(m+1)-2c\right)\overline{p}(c)q^n.$$

On comparing the coefficients we get

$$v_0(4n+2) = 2\sum_{m=0}^{\infty}\sum_{c=0}^{\lfloor\frac{n-2m(m+1)}{2}\rfloor}(-1)^m(2m+1)\overline{p}\left(n-2m(m+1)-2c\right)\overline{p}(c). \qquad \square$$

Theorem 4.4. *We have*

$$\sum_{c=0}^{\lfloor\frac{n}{2}\rfloor} v_0(6n-12c+4)p_d(c) = 3\overline{p}_2(n) - 9\overline{p}_2(n-6) + 15\overline{p}_2(n-18) + \cdots$$
$$+ 3(-1)^k(2k+1)\overline{p}_2(n-3k(k+1)) + \cdots .$$

Proof. We have (9)

$$\sum_{n=0}^{\infty} v_0(6n+4)q^n = 3\frac{f_2^3f_6^3}{f_1^4f_4} = 3\left(\frac{f_2}{f_1^2}\right)^2\frac{f_2f_6^3}{f_4},$$

$$\left(\sum_{n=0}^{\infty} v_0(6n+4)q^n\right)\left(\frac{f_4}{f_2}\right) = 3\left(\frac{f_2}{f_1^2}\right)^2 f_6^3,$$

$$\left(\sum_{n=0}^{\infty} v_0(6n+4)q^n\right)\left(\sum_{n=0}^{\infty} p_d(n)q^{2n}\right) = 3\left(\sum_{n=0}^{\infty}\overline{p}_2(n)q^n\right)\left(\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{3k(k+1)}\right),$$

$$\sum_{n=0}^{\infty}\sum_{c=0}^{\lfloor\frac{n}{2}\rfloor} v_0(6(n-2c)+4)p_d(c)q^n = 3\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k(2k+1)\overline{p}_2(n-3k(k+1))q^n.$$

On comparing the coefficients, we get

$$\sum_{c=0}^{\lfloor\frac{n}{2}\rfloor} v_0(6n-12c+4)p_d(c) = 3\overline{p}_2(n) - 9\overline{p}_2(n-6) + 15\overline{p}_2(n-18) + \cdots$$
$$+3(-1)^k(2k+1)\overline{p}_2(n-3k(k+1)) + \cdots . \qquad \square$$

Theorem 4.5. *We have*

$$v_0(8n+5) = 4p_{8,4}(n) + 4p_{8,4}(n-1) + 4p_{8,4}(n-3) + \cdots + 4p_{8,4}\left(n - \frac{k(k+1)}{2}\right) + \cdots .$$

Proof. We have (10)

$$\begin{aligned}\sum_{n=0}^{\infty} v_0(8n+5)q^n &= 4\frac{f_2^2f_4^4}{f_1^5}\\ &= 4\left(\frac{f_4}{f_1}\right)^4\frac{f_2^2}{f_1}\\ &= 4\left(\prod_{i=1}^{\infty}\frac{1-q^{4i}}{1-q^i}\right)^4\psi(q)\end{aligned}$$

$$
\begin{aligned}
&= 4\left(\prod_{i=1}^{\infty}(1+q^i)(1+q^{2i})\right)^4 \psi(q),\\
&= 4\left(\prod_{i=1}^{\infty}(1+q^{2i})^8(1+q^{2i+1})^4\right)\sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}}\\
&= 4\left(\sum_{n=0}^{\infty} p_{8,4}(n)q^n\right)\left(\sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}}\right)\\
&= 4\left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} p_{8,4}\left(n-\frac{k(k+1)}{2}\right)\right) q^n
\end{aligned}
$$

On comparing the coefficients, we get

$$
v_0(8n+5) = 4p_{8,4}(n) + 4p_{8,4}(n-1) + 4p_{8,4}(n-3) + \cdots + 4p_{8,4}\left(n-\frac{k(k+1)}{2}\right) + \cdots . \square
$$

Theorem 4.6. *We have*

$$
v_0(8n+6) = 4p_{6d}(n)+4p_{6d}(n-4)+4p_{6d}(n-12)+4p_{6d}(n-24)+\cdots+4p_{6d}(n-2k(k+1))+\cdots .
$$

Proof. We have (11)

$$
\begin{aligned}
\sum_{n=0}^{\infty} v_0(8n+6)q^n &= 4\frac{f_2^6 f_8^2}{f_1^6 f_4}\\
&= 4\left(\frac{f_2}{f_1}\right)^6 \frac{f_8^2}{f_4}\\
&= 4\left(\sum_{n=0}^{\infty} p_{6d}(n)q^n\right)\left(\sum_{k=0}^{\infty} q^{\frac{4k(k+1)}{2}}\right)\\
&= 4\left(\sum_{n=0}^{\infty} p_{6d}(n)q^n\right)\left(\sum_{k=0}^{\infty} q^{2k(k+1)}\right)\\
&= 4\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} p_{6d}(n)q^{2k(k+1)+n}\\
&= 4\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} p_{6d}(n-2k(k+1))q^n.
\end{aligned}
$$

On comparing the coefficients, we get

$$
\begin{aligned}
v_0(8n+6) &= 4\sum_{k=0}^{\infty} p_{6d}(n-2k(k+1))q^n,\\
v_0(8n+6) &= 4p_{6d}(n) + 4p_{6d}(n-4) + 4p_{6d}(n-12) + 4p_{6d}(n-24) + \cdots\\
&\qquad + 4p_{6d}(n-2k(k+1)) + \cdots . \qquad \square
\end{aligned}
$$

Theorem 4.7. *We have*

$$
\begin{aligned}
v_0(12n+5) &= 4p_{10d}(n) - 8p_{10d}(n-3) + 8p_{10d}(n-12) - 8p_{10d}(n-27)\\
&\qquad + \cdots + 8(-1)^k p_{10d}(n-3k^2) + \cdots .
\end{aligned}
$$

Proof. We have (12)

$$\begin{aligned}
\sum_{n=0}^{\infty} v_0(12n+5)q^n &= 4\frac{f_2^{10}f_3^2}{f_1^{10}f_6} \\
&= 4\left(\frac{f_2}{f_1}\right)^{10}\frac{f_3^2}{f_6} \\
&= 4\left(\sum_{n=0}^{\infty} p_{10d}(n)q^n\right)\left(\sum_{k=-\infty}^{\infty}(-1)^k q^{3k^2}\right) \\
&= 4\left(\sum_{n=0}^{\infty} p_{10d}(n)q^n\right)\left(1+2\sum_{k=1}^{\infty}(-1)^k q^{3k^2}\right) \\
&= 4\sum_{n=0}^{\infty} p_{10d}(n)q^n + 8\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^k p_{10d}(n)q^{n+3k^2} \\
&= 4\sum_{n=0}^{\infty} p_{10d}(n)q^n + 8\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^k p_{10d}(n-3k^2)q^n.
\end{aligned}$$

On comparing the coefficients we have

$$\begin{aligned}
v_0(12n+5) &= 4p_{10d}(n) + 8\sum_{k=1}^{\infty}(-1)^k\left(p_{10d}(n-3k^2)\right) \\
&= 4p_{10d}(n) - 8p_{10d}(n-3) + 8p_{10d}(n-12) + -8p_{10d}(n-27) + \cdots \\
&\quad + 8(-1)^k\left(p_{10d}(n-3k^2)\right) + \cdots .
\end{aligned}$$

□

Theorem 4.8. *We have*

$$v_0(16n+12) = 16p_{12d}(n) + 16p_{12d}(n-2) + 16p_{12d}(n-6) + \cdots + 16p_{12d}(n-k(k+1)) + \cdots .$$

Proof. We have (13)

$$\begin{aligned}
\sum_{n=0}^{\infty} v_0(16n+12)q^n &= 16\frac{f_2^{11}f_4^2}{f_1^{12}} \\
&= 16\left(\frac{f_2}{f_1}\right)^{11}\frac{f_4^2}{f_1} \\
&= 16\prod_{i=1}^{\infty}(1+q^i)^{11}\frac{f_4^2}{f_1} \\
&= 16\prod_{i=1}^{\infty}(1+q^i)^{11}(1+q^i)(1+q^{2i})(1-q^{4i}) \\
&= 16\prod_{i=1}^{\infty}(1+q^i)^{12}(1+q^{2i})(1-q^{4i}) \\
&= 16\prod_{i=1}^{\infty}(1+q^i)^{12}\frac{f_4^2}{f_2}
\end{aligned}$$

$$
\begin{aligned}
&= 16\left(\sum_{n=0}^{\infty} p_{12d}(n)q^n\right)\left(\sum_{k=0}^{\infty} q^{k(k+1)}\right) \\
&= 16\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} p_{12d}(n)q^{k(k+1)+n} \\
&= 16\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} p_{12d}(n-k(k+1))\right)q^n.
\end{aligned}
$$

On comparing the coefficients, we get

$$
\begin{aligned}
v_0(16n+12) &= 16\sum_{k=0}^{\infty} p_{12d}(n-k(k+1)), \\
&= 16p_{12d}(n) + 16p_{12d}(n-2) + 16p_{12d}(n-6) \\
&\qquad + \cdots + 16p_{12d}(n-k(k+1)) + \cdots .
\end{aligned}
$$

□

5 Conclusion

In literature, we can find the enumeration of mock theta functions, including generating functions and various combinatorial tools (refer to [2,11,16]). In this paper, we have established connections between mock theta functions and some restricted partition functions through recurrence relations. For future research, one could explore recurrence relations connecting universal mock theta functions such as $g_2(x,q), g_3(x,q)$, for more details of universal mock theta functions, refer [9], and other specified partition functions. Additionally, one can investigate alternative techniques to study recurrence relations instead of relying solely on generating functions.

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