Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2023, Volume 29, Number 2, 354–359
DOI: 10.7546/nntdm.2023.29.2.354-359

Intersection of Padovan and Tribonacci sequences

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Received: 13 October 2022 **Revised:** 16 March 2023
Accepted: 10 May 2023 **Online First:** 15 May 2023

Abstract: Assume that T_n is the n-th term of Tribonacci sequence and P_m is the m-th term of Padovan sequence. In this paper we solve the equation $T_n = P_m$ completely.

Keywords: Tribonacci numbers, Padovan numbers, Baker methods, Linear logarithms.
2020 Mathematics Subject Classification: 11J86, 11D72, 11B39.

1 Introduction

There are many valuable studies about finding common terms of number sequences in the literature. By the motivation of these papers, we solve the equation

$$P_m = T_n. \tag{1}$$

Here, the terms P_m and T_n are the m-th Padovan number and n-th Tribonacci number, respectively. Now, we present the definitions of these sequences.

For $m \geq 0$, the Padovan sequence is defined by the recurrence relation $P_{m+3} = P_{m+1} + P_m$ with the initials $P_0 = P_1 = P_2 = 1$. The first few terms of the sequence of Padovan are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, \ldots.$$

Its characteristic equation, $x^3 - x - 1 = 0$, has one real root $a \cong 1.324\ldots$ and two conjugate complex roots b and $\overline{b} = c$.

The Binet-type formula of Padovan sequence is

$$P_m = S_a a^m + S_b b^m + S_c c^m \tag{2}$$

where $S_a = \dfrac{(b-1)(c-1)}{(a-c)(a-b)}, S_b = \dfrac{(a-1)(c-1)}{(b-c)(b-a)}, S_c = \dfrac{(a-1)(b-1)}{(c-a)(c-b)}$ for all $m \geq 0$.

The Padovan sequence is named after Richard Padovan [7]. Shannon, Atanassov, Dimitrov and Kritsana [1, 8, 9] studied the combinatorial identities of Padovan sequence and Pell–Padovan sequence.

For all $n \geq 0$, Tribonacci sequence $\{T_n\}$ is given by the recurrence $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ with the initial values $T_0 = 0$, $T_1 = T_2 = 1$. Its characteristic equation, $x^3 - x^2 - x - 1 = 0$ has one real root α and two conjugate complex roots β and γ such that $\alpha \in (1.83, 1.84)$ and $|\beta| = |\gamma| < 1$. In 1982, Spickerman [10] defined the Binet Formula of Tribonacci numbers as

$$T_n = \frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

for all $n \geq 0$.

Recently, Dresden and Du [3] gave the another form the term of Tribonacci sequence as

$$T_n = \frac{(\alpha-1)\alpha^{n-1}}{4\alpha-6} + \frac{(\beta-1)\beta^{n-1}}{4\beta-6} + \frac{(\gamma-1)\gamma^{n-1}}{4\gamma-6}$$

for all $n \geq 0$.

Put $c_\alpha = \frac{\alpha-1}{4\alpha-6} \cong 0.61$. Then, the member of Tribonacci sequence satisfy

$$T_n = c_\alpha \alpha^{n-1} + e_n, \quad \text{with} \quad |e_n| < \frac{1}{\alpha^{n/2}} \quad \text{for all } n \geq 1. \tag{3}$$

Now, we give our result.

Theorem 1.1. *Assume that T_n is the n-th term of Tribonacci sequence and P_m is the m-th term of Padovan sequence. The solutions of the equation $P_m = T_n$ are $T_1 = P_1 = 1$, $T_1 = P_2 = 1$, $T_2 = P_1 = 1$, $T_2 = P_2 = 1$, $T_3 = P_3 = 2$,, $T_3 = P_4 = 2$, $T_4 = P_6 = 4$, $T_5 = P_8 = 7$.*

2 Preliminary results

In this section, we give several lemmas which will be useful later.

Lemma 2.1. *For $n \geq 2$, and $m \geq 2$, we have*

$$\alpha^{n-2} < T_n < \alpha^{n-1} \quad \textit{and} \quad a^{m-2} < P_m < \alpha^{m-1}.$$

Proof. The proof of the first inequality is in [2] and second one is in [5]. □

Let γ be an algebraic number of degree d over Q with minimal primitive polynomial, where the leading coefficient a_0 is positive.

The logarithmic weight of γ is given by

$$h(\gamma) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log(\max\left\{\left|\gamma^i\right|, 1\right\})\right), \tag{4}$$

where γ^i, $i = 1, \ldots, d$ denote the conjugates of γ. Now, we present the theorem of Matveev [6].

Lemma 2.2. *Let $\mathbb{K}$ be a set of numbers of order D over $\mathbb{Q}$. Let $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_t$ be positive real numbers of $\mathbb{K}$ and $b_1, b_2, b_3, \ldots, b_t$ be nonzero integers. Let*

$$\Lambda := \gamma_1^{b_1}\gamma_2^{b_2}\gamma_3^{b_3}\cdots\gamma_t^{b_t} - 1. \tag{5}$$

The number B is obtained as

$$B \geq \max\left\{|b_1|, |b_2|, |b_3|, \ldots, |b_t|\right\}. \tag{6}$$

$A_1, A_2, A_3, \ldots, A_t$ are real numbers such that$A_i \geq \max\left\{Dh(\gamma_i), |\log\gamma_i|, 0.16\right\}$. If $\Lambda \neq 0$, then the inequality

$$\Lambda > \exp\left\{-1.4 \cdot 30^{t+3}t^{4.5}D^2(1+\log D)(1+\log B)A_1A_2A_3\cdots A_t\right\} \tag{7}$$

holds.

By using the Baker method in Matveev Theorem, an upper limit is obtained. To reduce this bound, we will use the following lemma, which is given by Pethő and Dujella. [4]

Lemma 2.3. *Let γ be a real number, M be a positive integer and p/q be a convergent of the continued fraction of γ such that $q > 6M$. Let A, B, μ be real numbers with the conditions $A > 0$ and $B > 1$. Let ϵ is defined such as $\epsilon := \parallel \mu q \parallel - M \parallel \gamma q \parallel$, where $\parallel . \parallel$ denotes the distance from the nearest integer. Under the assumptions, if $\epsilon > 0$, there is no solution for the inequality*

$$0 < \mid m\gamma - n - \mu \mid < AB^{-m} \tag{8}$$

in positive integers m, n with $m \leq M$ and $m \geq \log(Aq/\epsilon)/\log B$.

3 Proof of the Theorem

Firstly, assume that $m \leq 8$. In this case, we see that $2, 4$ and 7 are the solutions that exceed 1 of the equation $T_n = P_m$. So, we assume $m > 8$. We write the Binet formulas of Tribonacci and Padovan sequences in the equation (1). Then

$$c_\alpha \alpha^{n-1} + e_n = S_a a^m + S_b b^m + S_c c^m \tag{9}$$

follows which gives

$$c_\alpha \alpha^{n-1} - S_a a^m = S_b b^m + S_c c^m - e_n.$$

After dividing both sides by $S_a a^m$, we obtain

$$c_\alpha \alpha^{n-1} S_a^{-1} a^{-m} - 1 = \frac{S_b b^m + S_c c^m - e_n}{S_a a^m}.$$

Then

$$\left|c_\alpha \alpha^{n-1} S_a^{-1} a^{-m} - 1\right| = \left|\frac{S_b b^m + S_c c^m - e_n}{S_a a^m}\right| \tag{10}$$

$$< \left|\frac{S_b b^m}{S_a a^m}\right| + \left|\frac{S_c c^m}{S_a a^m}\right| + \left|\frac{e_n}{S_a a^m}\right|$$

$$< \frac{1/2}{a^m} + \frac{1/2}{a^m} + \frac{1}{a^m} = \frac{2}{a^m} \tag{11}$$

holds. Let

$$\Lambda := c_\alpha \alpha^{n-1} S_a^{-1} a^{-m} - 1 \tag{12}$$

To show $\Lambda \neq 0$, we contrary assume $\Lambda = 0$. It yields that $c_\alpha S_a^{-1} = a^m \alpha^{1-n}$. It is obvious that a and α are algebraic integers. $1/\alpha$ is also algebraic integer since $1/\alpha$ is the root of the monic polynomial $x^3 + x^2 + x - 1$. Since product of algebraic integers is also algebraic integer, then it requires that $a^m(1/\alpha^{n-1}) = a^m(\alpha^{1-n})$ is an algebraic integer. But this is a contradiction. Because the minimal polynomial $-1 + 72x - 2872x^2 + 70500x^3 - 1169872x^4 + 12747152x^5 - 90124672x^6 + 405561024x^7 - 1036433728x^8 + 1036433728x^9$ of $c_\alpha S_a^{-1}$ is not monic, so $c_\alpha S_a^{-1}$ is not an algebraic integer, but an algebraic number. We arrive to the claimed fact $\Lambda \neq 0$.

Let $\mathbb{K} = \mathbb{Q}(\alpha, a, S_a, c_\alpha)$. Since $c_\alpha \in \mathbb{Q}(\alpha)$ and $S_a \in \mathbb{Q}(a)$, then we have $[\mathbb{Q}(\alpha, a, S_a, c_\alpha) : \mathbb{Q}] = [\mathbb{Q}(\alpha, a) : \mathbb{Q}] = [\mathbb{Q}(\alpha, a) : \mathbb{Q}(\alpha)]\,[\mathbb{Q}(\alpha) : \mathbb{Q}] = 9$. This gives that $D = 9$. Take $t = 4$.

Let

$$\begin{aligned}
\gamma_1 &= c_\alpha, & b_1 &= 1,\\
\gamma_2 &= \alpha, & b_2 &= n-1,\\
\gamma_3 &= S_a, & b_3 &= -1\\
\gamma_4 &= a, & b_4 &= -m.
\end{aligned}$$

Since the leading coefficients of the minimal polynomial of c_α and S_a are 44 and 23, then logarithmic weights of c_α, α, S_a and a are obtained as follows

$$\begin{aligned}
h(c_\alpha) &= \frac{1}{3}\left[\log 44 + \log(\max\{c_\alpha, 1\})\right] = \frac{1}{3}\log 44 \cong 1.26\\
h(\alpha) &= \frac{1}{3}\left[\log 1 + \log\alpha + \log 1 + \log 1\right] = \frac{1}{3}\log\alpha \cong 0.201\\
h(S_a) &= \frac{1}{3}\left[\log 23 + \log(\max\{S_a, 1\})\right] = \frac{1}{3}\log 23 \cong 1.044\\
h(a) &= \frac{1}{3}\left[\log 1 + \log a + \log 1 + \log 1\right] = \frac{1}{3}\log a \cong 0.093.
\end{aligned}$$

So, we choose A_1, A_2, A_3 and A_4 as

$$\begin{aligned}
&A_1 \geq \max\{9h(c_\alpha), \log c_\alpha, 0, 16\} \cong 11.34 && A_1 = 11.35\\
&A_2 \geq \max\{9h(\alpha),\ \log\alpha,\ 0, 16\} \cong 1.809 && A_2 = 1.81\\
&A_3 \geq \max\{9h(S_a),\ \log S_a,\ 0, 16\} \cong 9.396 && A_3 = 9.4\\
&A_4 \geq \max\{9h(a),\ \log a,\ 0, 16\} \cong 0.837 && A_4 = 0.84.
\end{aligned}$$

Now, we make a decision about B. If we use the inequalities in (2.1) together, we see that

$$\alpha^{n-2} < T_n = P_m < a^{m-1}.$$

After taking logarithm of both sides, we get the inequality

$$0.6(n-2) < 0.29(m-1),$$

yielding

$$n-1 < m. \tag{13}$$

So, $B = m$.

If we write all the values in the inequality (7), then we get the following inequality by using (7) and (10),

$$\frac{2}{a^m} > \exp\left(-1.4 \cdot 30^7 4^{4,5} 9^2 (1+\log 9)(1+\log m) \cdot 11.35 \cdot 1.81 \cdot 9.4 \cdot 0.84\right).$$

After taking logarithm of both sides, the inequality

$$\log 2 - m \log a > -1.4 \cdot 30^7 4^{4,5} 9^2 (2\log 9)(1+\log m) \cdot 11.35 \cdot 1.81 \cdot 9.4 \cdot 0.84$$

holds. This yields

$$m < 6.78 \cdot 10^{18} \log m,$$

where we use $1+\log 9 < 2\log 9$ and $1+\log m < 2\log m$. So, we obtain

$$m < 3.21 \cdot 10^{20}.$$

From now on, we will apply the Lemma 2.3. Let $M = 3.21 \cdot 10^{20}$. Assume that $\Lambda > 0$. So,

$$\log(c_\alpha \alpha^{n-1} S_a^{-1} a^{-m}) < e^{\log(c_\alpha \alpha^{n-1} S_a^{-1} a^{-m})} - 1 \tag{14}$$

holds. Combining (12) and (14),

$$\log(c_\alpha \alpha^{n-1} S_a^{-1} a^{-m}) < \frac{2}{a^m}$$

follows. If $\Lambda < 0$, the inequality $\left|e^\Lambda - 1\right| < \frac{1}{2}$ holds as $m > 8$. This yields that $e^{|\Lambda|} < 2$ and

$$|\Lambda| < e^{|\Lambda|} - 1 = e^{|\Lambda|}\left|e^\Lambda - 1\right| < \frac{4}{a^m}.$$

In this case,

$$|\Lambda| < \frac{4}{a^m}$$

is obtained. The fact that $n < m$ gives that

$$|(n-1)\log\alpha - m\log a + \log c_\alpha - \log S_a| < \frac{4}{a^m}.$$

Together with $n-1 < m$, we get

$$|(n-1)\log\alpha - m\log a + \log c_\alpha - \log S_a| < 2a^{-(n-1)},$$

After dividing by $\log a$, we obtain

$$\left|(n-1)\frac{\log\alpha}{\log a} - m + \frac{\log(c_\alpha/S_a)}{\log a}\right| < \frac{4}{\log a} \cdot a^{-(n-1)}.$$

If we choose

$$\gamma = \frac{\log\alpha}{\log a}, \quad \mu = \frac{\log(c_\alpha/S_a)}{\log a}, \quad A = \frac{4}{\log a}, \quad B = a$$

on (8), and q_n be the denominator of the n-th convergent of the continued fraction of γ. We choose

$$q_{39} = 216078142307580907755 4 > 6M$$

and $\epsilon := \| \mu q_{39} \| - M \| \gamma q_{39} \| \cong 0.0569 \ldots$. By the result of Dujella and Pethő (Lemma 2.3), we get

$$m \geq \log(Aq/\epsilon)/\log B = 194.328.$$

So $m \leq 194$. A quick calculation by *Mathematica* shows that there is no solution of the equation $T_n = P_m$ for $9 < m \leq 194$. So, the proof is completed. □

Acknowledgements

The authors express their gratitude to the anonymous reviewers for the instructive suggestions and remarks.

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