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On the multiplicative group generated by $\left\{\frac{[\sqrt{2}n]}{n} \;\middle|\; n \in \mathbb{N}\right\}$. V

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Abstract: Let f, g be completely multiplicative functions, $|f(n)| = |g(n)| = 1$ $(n \in \mathbb{N})$. Assume that

$$\frac{1}{\log x} \sum_{n \le x} \frac{|g([\sqrt{2}n]) - Cf(n)|}{n} \to 0 \quad (x \to \infty).$$

Then

$$f(n) = g(n) = n^{i\tau}, \quad C = (\sqrt{2})^{i\tau}, \tau \in \mathbb{R}.$$

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1 Introduction

This paper is a continuation of [2–5].

A function $f : \mathbb{N} \to \mathbb{C}$ belongs to $\mathcal{M}^*$ if $f(nm) = f(n)f(m)$ holds for all $n, m \in \mathbb{N}$ and let

$$\mathcal{M}_1^* := \left\{ f \in \mathcal{M}^* \middle| |f(n)| = 1 \quad n \in \mathbb{N} \right\}.$$

We shall write $f(\frac{n}{m}) = \frac{f(n)}{f(m)}$ if $f \in \mathcal{M}_1^*$ and $n, m \in \mathbb{N}$. Let $\{x\}$ = fractional part of x and let

$$\|x\| = \min(\{x\}, 1 - \{x\}).$$

In [4] we proved

Theorem A. *Let* $\varepsilon(n) \downarrow 0$, $f, g \in \mathcal{M}^*$, $C \in \mathbb{C}$,

$$|g([\sqrt{2}n]) - Cf(n)| \le \varepsilon(n) \ \ (n \in \mathbb{N}),$$

furthermore

$$\sum_{n=2}^{\infty} \frac{\varepsilon(n) \log\log 2n}{n} < \infty.$$

Then $f(n) = g(n) = n^{i\tau} \ \ (\tau \in \mathbb{R})$, *where* $C = (\sqrt{2})^{i\tau}$.

The proof was based on the following:

Main lemma. ([1]) *If* $f \in \mathcal{M}_1^* \ \ (n \in \mathbb{N})$ *and*

$$\sum_{n=1}^{\infty} \frac{|f(n+1) - f(n)|}{n} < \infty,$$

then $f(n) = n^{i\tau} \ \ (\tau \in \mathbb{R})$.

Recently O. Klurman by using the results of [8] and [9] proved a very important theorem:

Theorem B. ([6] and [7]) *If* $f \in \mathcal{M}_1^*$,

$$\frac{1}{\log x} \sum_{n \le x} \frac{|f(n+1) - f(n)|}{n} \to 0 \ \ (x \to \infty),$$

then $f(n) = n^{i\tau} \ \ (\tau \in \mathbb{R})$.

By using Theorem B we improve Theorem A as follows:

Theorem 1. *Let* $f, g \in \mathcal{M}_1^*$,

$$\frac{1}{\log x} \sum_{n \le x} \frac{|g([\sqrt{2}n]) - Cf(n)|}{n} \to 0 \ \ (x \to \infty),$$

then $f(n) = g(n) = n^{i\tau} \ \ (\tau \in \mathbb{R})$, *where* $C = (\sqrt{2})^{i\tau}$.

2 Auxiliary results

Let

$$J_1 = \left\{n \mid \{\sqrt{2}n\} < \frac{1}{\sqrt{2}}\right\} \quad \text{and} \quad J_2 = \left\{n \mid \{\sqrt{2}n\} > \frac{1}{\sqrt{2}}\right\}. \tag{2.1}$$

Then $\mathbb{N} = J_1 \cup J_2$.

Let

$$a_n := \frac{\left[\sqrt{2}[\sqrt{2}n]\right]}{n}. \tag{2.2}$$

We proved in [2] (Lemma 2) that

$$a_n = \begin{cases} \frac{2n-1}{n} & \text{if} \quad n \in J_1, \\ \frac{2(n-1)}{n} & \text{if} \quad n \in J_2. \end{cases} \tag{2.3}$$

For some $N \in \mathbb{N}$, let

$$\mathcal{B}_0 = N \quad \text{and} \quad \mathcal{B}_j = 2\mathcal{B}_{j-1} - 1 \quad \text{for all} \quad j \in \mathbb{N}.$$

In [2] (Lemma 3) we proved that either $N \in J_2$, or there is a positive k, for which $\mathcal{B}_k \in J_2$. Let $T(N) := k + 1$, where k is the smallest positive integer for which $\mathcal{B}_k \in J_2$. We proved that

$$\frac{2^{k+1}(N-1)}{N} = \frac{2(\mathcal{B}_k - 1)}{N} = \Big(\prod_{j=0}^{k-1} a_{\mathcal{B}_j}\Big) a_{\mathcal{B}_k}, \tag{2.4}$$

furthermore that

$$T(N) \le \frac{1}{\log 2} \cdot \log \frac{1}{\|\sqrt{2}(N-1)\|} + c_1, \tag{2.5}$$

c_1 is an explicit constant.

Let $\delta(n) = g([\sqrt{2}n]) - Cf(n)$, $k \in \mathbb{N}$. Let

$$\mathcal{T}_k = \Big\{ n \in \mathbb{N} \Big| \{\sqrt{2}n\} < \frac{1}{k} \Big\}.$$

It is clear that

$$\lim_{x \to \infty} \frac{1}{\log x} \sum_{\substack{n \in \mathcal{T}_k \\ n \le x}} \frac{1}{n} = A_k \ (> 0). \tag{2.6}$$

If $n \in \mathcal{T}_k$, then

$$\delta(kn) = g(k) g([\sqrt{2}n]) - Cf(k) f(n)$$

and

$$\frac{\delta(kn)}{f(k)} = \frac{g(k)}{f(k)} g([\sqrt{2}n]) - Cf(n), \ \ \delta(n) = g([\sqrt{2}n]) - Cf(n),$$

and so

$$\lim_{x \to \infty} \frac{1}{\log x} \sum_{\substack{n \in \mathcal{T}_k \\ n \le x}} \Big| \frac{g(k)}{f(k)} - 1 \Big| \cdot |g([\sqrt{2}n])| \to 0,$$

which implies that $g(k) = f(k)$.

Let $\Theta_1 = \{\sqrt{2}m\}, \Theta_2 = \{\sqrt{2} \cdot 2m\}, \Theta_3 = \{\sqrt{2}(2m-1)\}$. If

$$\Theta_1 \in \Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} (\frac{1}{\sqrt{2}} + 1) \Big) := U,$$

then

$$\Theta_2 = 2\Theta_1 - 1 < \frac{1}{\sqrt{2}}, \ 0 < \Theta_3 = \Theta_2 - (\sqrt{2} - 1) < \frac{1}{\sqrt{2}}$$

and so $m \in J_2, 2m-1 \in J_1, 2m \in J_1$. Such integers m for which $\Theta_1 \in U$ can be found, since $\{\sqrt{2}m\}$ is dense in $[0, 1)$.

Then there exist positive integers k_1, k_2 such that

$$\begin{cases} 2^\ell(2m-2)+1 \in \begin{cases} J_1 & \text{for } \ell < k_1 \\ J_2 & \text{for } \ell = k_1, \end{cases} \\ \\ 2^\ell(2m-1)+1 \in \begin{cases} J_1 & \text{for } \ell < k_2 \\ J_2 & \text{for } \ell = k_2. \end{cases} \end{cases}$$

Then there exists a suitable $\delta > 0$ such that if $|\{\sqrt{2}N\} - \Theta_1| < \delta$, then

$$\begin{cases} a_N = \frac{2(N-1)}{N}, \quad N \in J_2 \\ 2^\ell(2N-2)+1 \in \begin{cases} J_1 & \text{for } \ell < k_1 \\ J_2 & \text{for } \ell = k_1, \end{cases} \\ \\ 2^\ell(2N-1)+1 \in \begin{cases} J_1 & \text{for } \ell < k_2 \\ J_2 & \text{for } \ell = k_2. \end{cases} \end{cases}$$

Let S_k be the set of the integers n, for which k is the non-negative smallest integer for which $\mathcal{B}_k \in J_2$. Let $d_n = \frac{[\sqrt{2}n]}{n}$. Then $a_n = d_n \cdot d_{[\sqrt{2}n]}$.

Let $D = f(2)C^{-2}$. If (2.4) holds, then

$$D^{k+1} f\Big(\frac{N-1}{N}\Big) = \prod_{j=0}^{k} C^{-2} f(a_{\mathcal{B}_j}),$$

and so it follows from $D := f(2)C^{-2}$ that

$$\begin{aligned} \Big|D^{k+1} f\Big(\frac{N-1}{N}\Big) - 1\Big| &\le B \sum_{j=0}^{k} \Big|C^{-1} f\Big(d_{\mathcal{B}_j}\Big) - 1\Big| + B \sum_{j=0}^{k} \Big|C^{-1} f\Big(d_{\mathcal{B}_{[\sqrt{2}\mathcal{B}_j]}}\Big) - 1\Big| \\ &\le 2\frac{B}{|C|}\varepsilon(N-1). \end{aligned}$$

Here B is an absolute constant.

Thus

$$\frac{1}{\log x} \sum_{\substack{N \in S_k \\ N \le x}} \frac{1}{N}\Big|D^{k+1} f\Big(\frac{N-1}{N}\Big) - 1\Big| \to 0 \ \ (x \to \infty).$$

Let now $N \in S_0, 2N-1 \in S_{k_1}, 2N \in S_{k_2}$. Then

$$\frac{1}{\log x} \sum_{\substack{N \in S_0 \\ N \le x}} \frac{\Big|Df\Big(\frac{N-1}{N}\Big) - 1\Big|}{N} \to 0 \ \ (x \to \infty),$$

$$\frac{1}{\log x} \sum_{\substack{2N-1 \in S_{k_1} \\ N \le x}} \frac{\Big|D^{k_1+1} f\Big(\frac{2N-2}{2N-1}\Big) - 1\Big|}{2N-1} \to 0 \ \ (x \to \infty),$$

$$\frac{1}{\log x}\sum_{\substack{2N\in S_{k_2}\\ N\le x}}\frac{\left|D^{k_2+1}f\left(\frac{2N-2}{2N-1}\right)-1\right|}{2N}\to 0\ \ (x\to\infty),$$

and so

$$\frac{1}{\log x}\sum_{N\le x}^{\star}\frac{\left|D^{k_1+k_2+2}-D\right|\left|f\left(\frac{N-1}{N}\right)\right|}{N}\to 0,$$

the summation is extended over those N for which $N\in S_0, 2N-1\in S_{k_1}, 2N\in S_{k_2}$ simultaneously holds. Since

$$\lim_{x\to\infty}\frac{1}{\log x}\sum\nolimits_{N\le x}^{\star}\frac{1}{N}>0,$$

we obtain that

$$D^{k_1+k_2+1}=1.$$

One can compute easily that if $m=2$, then $k_1=2, k_2=1$ and if $m=14$, then $k_1=k_2=1$.

Thus, $D=1$ and $C^2=f(2)$.

3 Proof of Theorem 1

We shall prove that

$$\frac{1}{\log x}\sum_{\substack{T(N)=k\\ N\le x}}\frac{\left|f\left(\frac{N-1}{N}\right)-1\right|}{N}\to 0\ \ (x\to\infty) \tag{3.1}$$

holds for every fixed k.

We have

$$f\Big(\frac{N-1}{N}\Big)=\prod_{j=0}^{k}f(d_{\mathcal{B}_j})f(d_{\mathcal{B}_{[\sqrt{2}\mathcal{B}_j]}}),$$

and so

$$\begin{aligned}\left|f\Big(\frac{N-1}{N}\Big)-1\right| &\le c\left|\log f\Big(\frac{N-1}{N}\Big)\right|\\ &\le c\sum_{j=0}^{k}\left|f(d_{\mathcal{B}_j})-1\right|+c\sum_{j=0}^{k}\left|f(d_{\mathcal{B}_{[\sqrt{2}\mathcal{B}_j]}})-1\right|.\end{aligned}$$

The assumption of Theorem 1 implies (3.1).

Finally we note that

$$\lim_{x\to\infty}\frac{1}{\log x}\sum_{\substack{T(N)\ge U\\ N\le x}}\frac{1}{N}=d(U),$$

where $d(U)\to 0$ if $U\to\infty$.

We obtain that

$$\frac{1}{\log x}\sum_{N\le x}\frac{1}{N}\left|f\Big(\frac{N-1}{N}\Big)-1\right|\to 0.$$

We can apply Theorem B and the proof of Theorem 1 is complete. □

The following result is a consequence of Theorem 1.

Theorem 2. *Let F, G be completely additive functions, $A \in \mathbb{R}$ and*

$$\frac{1}{\log x}\sum_{n\le x}\frac{\|G([\sqrt{2}n]) - F(n) - A\|}{n} \to 0 \ \ (x\to\infty).$$

Then $F(n) = G(n) = \frac{2A}{\log 2}\log n$ for every $n \in \mathbb{N}$.

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