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p-Analogue of biperiodic Pell and Pell–Lucas polynomials

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Abstract: In this study, a binomial sum, unlike but analogous to the usual binomial sums, is expressed with a different definition and termed the p-integer sum. Based on this definition, p-analogue Pell and Pell–Lucas polynomials are established and the generating functions of these new polynomials are obtained. Some theorems and propositions depending on the generating functions are also expressed. Then, by association with these, the polynomials of so-called 'incomplete' number sequences have been obtained, and elegant summation relations provided. The paper has also been placed in the appropriate historical context for ease of further development.

Keywords: p-Analogue Pell, Pell–Lucas polynomials, Biperiodic polynomials, Incomplete sequences.
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1 Introduction

Different generalized recurrences of Fibonacci and Lucas number sequences have had an important place in scientific studies for many years. The most basic generalization of the Fibonacci numbers is defined as

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \quad n \geq 2$$

with initial condition $F_{k,0} = 0, F_{k,1} = 1$.

Based on this sequence, when $k = 1$ is taken, we have the ordinary Fibonacci numbers. Based on this sequence, when $k = 2$ is taken, the Pell number given below and the Pell–Lucas numbers are obtained by changing the initial values as shown:

$$P_n = 2P_{n-1} + P_{n-2}, n \geq 2$$

with initial conditions $P_0 = 0, P_1 = 1$ and

$$Q_n = 2Q_{n-1} + Q_{n-2}, n \geq 2$$

with initial conditions $Q_0 = Q_1 = 1$.

Incomplete and biperiodic studies have been discussed by many authors [1, 4, 9–14, 19]. Depending on *k* any integer, incomplete recurrences of the *k*-Pell and *k*-Pell–Lucas sequence [3] are defined as follows:

$$P_{k,n}^{l} = \sum_{j=0}^{l} \binom{n-1-j}{j} k^{j} 2^{n-1-2j}, \;\; 0 \leq l \leq \frac{n-1}{2}, \quad n \in \mathbb{N}$$

and

$$Q_{k,n}^{l} = \sum_{j=0}^{l} \frac{n}{n-j} \binom{n-j}{j} k^{j} 2^{n-2j}, \;\; 0 \leq l \leq \frac{n}{2}, n \in \mathbb{N}.$$

Biperiodic Fibonacci numbers were defined by Yayenie [4, 5] with the recurrence and summation formulas

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases} \tag{1.1}$$

and

$$q_n = a^{\xi(n-1)} \sum_{i=0}^{\frac{n-1}{2}} \binom{n-i-1}{i} (ab)^{\frac{n-1}{2}-i}, \tag{1.2}$$

where $\xi(n) = n - \frac{n}{2}$. Many generalizations related to these numbers have had an important place in recent scientific research and development [1, 4, 12]. This definition of Yayenie not only gives a different perspective to the usual recurrence relations, but also many studies have been added to the literature by studying this definition when it is applied to different number sequences. Thus, based on the above equation, Ramirez [12] defined the following incomplete sequences:

$$q_n(l) = a^{\xi(n-1)} \sum_{i=0}^{l} \binom{n-i-1}{i} (ab)^{\frac{n-1}{2}-i}, \; 0 \leq l \leq \frac{n-1}{2}. \tag{1.3}$$

Many studies on incomplete number sequences began with Ramirez [13]. In this study, Fibonacci and Lucas number sequences are discussed. The following studies are the continuation of these studies; for example, Catarino et al. [3] worked with generalized Pell numbers. Similarly, later Kuloğlu et al. [10] studied the polynomials of Vieta–Pell number sequences and brought a more up-to-date version to these studies as a whole.

The q-integer of the number a is defined by

$$[a]_q = \begin{cases} \dfrac{1-q^a}{1-q}, & \text{if } q \neq 1 \\ a, & \text{if } q = 1 \end{cases}$$

and another factorial definition is defined by

$$[a]_q! = \begin{cases} [a]_q [a-1]_q \cdots [1]_q, & \text{if } a = 1, 2, \ldots \\ 1, & \text{if } a = 0 \end{cases}$$

Similarly, the binomial definition is given by

$$\begin{bmatrix} a \\ n \end{bmatrix} = \frac{[a]_q!}{[a-n]_q![n]_q!}, \quad 0 \leq n \leq a$$

or

$$\begin{bmatrix} a \\ n \end{bmatrix} = \frac{(q;q)_a}{(q;q)_{a-n}(q;q)_n}, \quad 0 \leq n \leq a$$

For $a < n$ this value is taken as 0. Here

$$(a;q)_n = \prod_{j=0}^{n-1} (1 - q^j a).$$

we can take the Pascal-like rule here as

$$\begin{bmatrix} a \\ n \end{bmatrix}_q = q^n \begin{bmatrix} a-1 \\ n \end{bmatrix}_q + \begin{bmatrix} a-1 \\ n-1 \end{bmatrix}_q$$

and

$$\begin{bmatrix} a \\ n \end{bmatrix}_q = \begin{bmatrix} a-1 \\ n \end{bmatrix}_q + q^{a-n} \begin{bmatrix} a-1 \\ n-1 \end{bmatrix}_q.$$

Srivastava [18] focused on the historical development of many pertinent polynomials and the properties of such sequences, from the generalization of classical Bernoulli polynomials (specifically, q-generalization) to Euler polynomials and Euler numbers to Genocchi polynomials. The q-integers are related to the q-series of Carlitz [2] and the Fermatians of Shannon [16] (the name going back to Fermat's Little Theorem and Shanks [15]). The binomial definition above is a generalization of the Fibonomial coefficients of Jerbic [8]. Hoggatt [6] developed properties of them for ordinary Fibonacci numbers. Hoggatt and Lind [7] also applied them to combinatorial problems, and Shannon and Horadam [17] used them to obtain generating functions for powers of elements of third order recursive sequences.

Based on the definitions and properties given above, this article reconstructs the biperiodic studies defined on p-analogues and proves several related theorems.

2 p-Analogue of biperiodic Pell and Pell–Lucas polynomials

In this section, the generating functions of the p-analogues of the biperiodic generalizations of Pell and Pell-Lucas sequences and their important theorems and propositions depending on the generating functions are discussed.

Definition 2.1. For $m,n\in\mathbb{R}$ the p-analogue of biperiodic Pell and Pell–Lucas polynomials are defined by

$$P_k(s,p)=2mP_{k-1}(s,p)+sp^{k-2}P_{k-2}(s,p),\quad \text{if } k \text{ is even}$$
$$P_k(s,p)=2nP_{k-1}(s,p)+sp^{k-2}P_{k-2}(s,p),\quad \text{if } k \text{ is odd}$$

for $k\geq 2$ and $P_0(s,p)=u, P_1(s,p)=v$.

If we take $u=0$ and $v=m=n=s=p=1$ here, we get $P_k(p)$.

Theorem 2.2. *Let $P_k(s,p)$, k-th p-analogue of biperiodic Pell and Pell–Lucas polynomials for* $k\geq 0$

$$P_k(s,p)=n^{\delta(k)}u\sum_{j=0}^{\left\lfloor\frac{k-2}{2}\right\rfloor}\begin{bmatrix}k-j-2\\ j\end{bmatrix}(mn)^{\left\lfloor\frac{k-2}{2}\right\rfloor-j}p^{j(j+1)}s^{j+1}2^{k-2-2j}$$
$$+m^{\delta(k-1)}v\sum_{j=0}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\begin{bmatrix}k-j-1\\ j\end{bmatrix}(mn)^{\left\lfloor\frac{k-1}{2}\right\rfloor-j}p^{j^2}s^j2^{k-1-2j}.$$

Proof. By induction over k, if we take k = 1, the equality holds. Suppose that the equality is true for k. We now show that it is true for k + 1.

$$\begin{aligned}P_{k+1}(s,p)&=2m^{\delta(k)}n^{1-\delta(k)}P_k(s,p)+sp^{k-1}P_{k-1}(s,p)\\
&=2m^{\delta(k)}n^{1-\delta(k)}\left(n^{\delta(k)}u\sum_{j=0}^{\left\lfloor\frac{k-2}{2}\right\rfloor}\begin{bmatrix}k-j-2\\ j\end{bmatrix}(mn)^{\left\lfloor\frac{k-2}{2}\right\rfloor-j}p^{j(j+1)}s^{j+1}2^{k-2-2j}\right.\\
&\qquad\left.+m^{\delta(k-1)}v\sum_{j=0}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\begin{bmatrix}k-j-1\\ j\end{bmatrix}(mn)^{\left\lfloor\frac{k-1}{2}\right\rfloor-j}p^{j^2}s^j2^{k-1-2j}\right)\\
&+sp^{k-1}\left(n^{\delta(k-1)}u\sum_{j=0}^{\frac{k-3}{2}}\begin{bmatrix}k-j-3\\ j\end{bmatrix}(mn)^{\frac{k-3}{2}-j}p^{j(j+1)}s^{j+1}2^{k-3-2j}\right.\\
&\qquad\left.+m^{\delta(k-2)}v\sum_{j=0}^{\frac{k-2}{2}}\begin{bmatrix}k-j-2\\ j\end{bmatrix}(mn)^{\frac{k-2}{2}-j}p^{j^2}s^j2^{k-2-2j}\right)\end{aligned}$$

$$= 2(mn)^{\delta(k)} n^{\delta(k+1)} u \sum_{j=0}^{\left\lfloor \frac{k-2}{2} \right\rfloor} \begin{bmatrix} k-j-2 \\ j \end{bmatrix} (mn)^{\left\lfloor \frac{k-2}{2} \right\rfloor - j} p^{j(j+1)} s^{j+1}$$

$$+ n^{\delta(k+1)} u \sum_{j=0}^{\left\lfloor \frac{k-3}{2} \right\rfloor} \begin{bmatrix} k-j-3 \\ j \end{bmatrix} (mn)^{\left\lfloor \frac{k-3}{2} \right\rfloor - j} p^{j(j+1)+k-1} s^{j+2} 2^{k-3-2j}$$

$$+ 2(mn)^{\delta(k+1)} m^{\delta(k)} v \sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \begin{bmatrix} k-j-1 \\ j \end{bmatrix} (mn)^{\left\lfloor \frac{k-1}{2} \right\rfloor - j} p^{j^2} s^j 2^{k-1-2j}$$

$$+ m^{\delta(k)} v \sum_{j=0}^{\left\lfloor \frac{k-2}{2} \right\rfloor} \begin{bmatrix} k-j-2 \\ j \end{bmatrix} (mn)^{\left\lfloor \frac{k-2}{2} \right\rfloor - j} p^{j^2+k-1} s^{j+1} 2^{k-2-2j}$$

From the definition of k, $\delta(k)$ and the recurrence relation of the p-binomial coefficients, where $\lfloor k \rfloor$ is the floor function and $\delta(k) = k - 2\left\lfloor \frac{k}{2} \right\rfloor$ is the parity function,

$$\left\lfloor \frac{k-1}{2} \right\rfloor + \delta(k+1) = \left\lfloor \frac{k}{2} \right\rfloor \quad \text{and} \quad \begin{bmatrix} k \\ j \end{bmatrix} = \begin{bmatrix} k-1 \\ j \end{bmatrix} + p^{k-j} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}.$$

We now obtain

$$P_{k+1}(s,p) = n^{\delta(k+1)} u \sum_{j=0}^{\left\lfloor \frac{k-2}{2} \right\rfloor} \begin{bmatrix} k-j-2 \\ j \end{bmatrix} (mn)^{\left\lfloor \frac{k-1}{2} \right\rfloor - j} p^{j(j+1)} s^{j+1} 2^{k-1-2j}$$

$$+ m^{\delta(k)} v \sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \begin{bmatrix} k-j-1 \\ j \end{bmatrix} (mn)^{\left\lfloor \frac{k}{2} \right\rfloor - j} p^{j^2} s^j 2^{k-2j}$$

$$+ n^{\delta(k+1)} u \sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \begin{bmatrix} k-j-2 \\ j-1 \end{bmatrix} (mn)^{\left\lfloor \frac{k-1}{2} \right\rfloor - j} p^{j(j-1)+k-1} s^{j+1} 2^{k-1-2j}$$

$$+ m^{\delta(k)} v \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \begin{bmatrix} k-j-1 \\ j-1 \end{bmatrix} (mn)^{\left\lfloor \frac{k}{2} \right\rfloor - j} p^{(j-1)^2+k-1} s^j 2^{k-2j}$$

$$= n^{\delta(k+1)} u \sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \begin{bmatrix} k-j-1 \\ j \end{bmatrix} (mn)^{\left\lfloor \frac{k-1}{2} \right\rfloor - j} p^{j(j+1)} s^{j+1} 2^{k-1-2j} +$$

$$+ m^{\delta(k)} v \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \begin{bmatrix} k-j \\ j \end{bmatrix} (mn)^{\left\lfloor \frac{k}{2} \right\rfloor - j} p^{j^2} s^j 2^{k-2j}.$$

This completes the proof. □

We provide the generating function for $P_k(s,p)$ in the following theorem. Here the functions $\eta f(x)$ defined in [9] with η operator and, similarly, $\eta_2 f(x)$ with the same operator defined in [14], will be a guide for the proof.

Theorem 2.3. For the p-biperiodic Pell and Pell–Lucas polynomials the generating function is

$$\xi(x) = \frac{u+x(v-nu)+(2m-n)xf(x)}{1-nx-x^2 s\eta},$$

where

$$f(x) = \frac{vx+s(nu-v)x^3}{1-4mnx^2-s\left(1+\frac{1}{p}\right)x^2\eta+s^2x^2\eta_2}.$$

Proof. The generating function for the p-analogue biperiodic Pell and Pell–Lucas polynomials is

$$\xi(x) = \sum_{j=0}^{\infty} P_j(s,p)x^j = P_0(s,p)+P_1(s,p)x^1+P_2(s,p)x^2+P_3(s,p)x^3+\cdots$$
$$nx\xi(x) = nP_0(s,p)x+nP_1(s,p)x^2+nP_2(s,p)x^3+nP_3(s,p)x^4+\cdots$$
$$x^2 s\eta\xi(x) = P_0(s,p)x^2 s+pP_1(s,p)x^3 s+p^2P_2(s,p)x^4 s+p^3P_3(s,p)x^5 s+\cdots$$

and since

$$P_{2k+1}(s,p) = 2nP_{2k}(s,p)+sp^{2k-1}P_{2k-1}(s,p),$$

we obtain

$$\begin{aligned}\left(1-nx-x^2s\eta\right)\xi(x) &= P_0(s,p)+x\left(P_1(s,p)-nP_0(s,p)\right)\\ &\quad+\sum_{k=1}^{\infty}\left(P_{2k}(s,p)-nP_{2k-1}(s,p)dsp^{2k-2}P_{2k-2}(s,p)\right)x^{2k},\end{aligned}$$

and since

$$P_{2k}(s,p) = 2mP_{2k-1}(s,p)+sp^{2k-2}P_{2k-2}(s,p),$$

we have

$$\begin{aligned}\left(1-nx-x^2s\eta\right)\xi(x) &= u+x(v-nu)+\sum_{k=1}^{\infty}\left(2mP_{2k-1}(s,p)-nP_{2k-1}(s,p)\right)x^{2k}\\ &= u+x(v-nu)+\sum_{k=1}^{\infty}\left(P_{2k-1}(s,p)(2m-n)\right)x^{2k}\\ &= u+x(v-nu)+(2m-n)x\sum_{k=1}^{\infty}\left(P_{2k-1}(s,p)\right)x^{2k-1}\\ &= u+x(v-nu)+(2m-n)xf(x)\end{aligned}$$

On the other hand, we find that

$$\begin{aligned}P_{2k+1}(s,p) &= 2nP_{2k}(s,p)+sp^{2k-1}P_{2k-1}(s,p)\\ &= 2n\left(2mP_{2k-1}(s,p)+sp^{2k-2}P_{2k-2}(s,p)\right)+sp^{2k-1}P_{2k-1}(s,p)\\ &= P_{2k-1}(s,p)\left(4mn+sp^{2k-1}\right)+2nsp^{2k-2}P_{2k-2}(s,p)\\ &= P_{2k-1}(s,p)\left(4mn+sp^{2k-1}\right)+sp^{2k-2}\left(P_{2k-1}(s,p)-sp^{2k-3}P_{2k-3}(s,p)\right)\end{aligned}$$

$$= P_{2k-1}(s,p)\left(4mn + sp^{2k-1}\right) + sp^{2k-2}P_{2k-1}(s,p) - s^2p^{4k-5}P_{2k-3}(s,p)$$
$$= \left(4mn + sp^{2k-1}\left(1+\frac{1}{p}\right)\right)P_{2k-1}(s,p) - s^2p^{4k-5}P_{2k-3}(s,p).$$

Thus, we can write

$$\left(1-4mnx^2 - s\left(1+\frac{1}{p}\right)x^2\eta + s^2x^2\eta_2\right)f(x) = vx + s(nu-v)x^3,$$

from which we get the desired results as follows

$$\xi(x) = \frac{u+x(v-nu)+(2m-n)xf(x)}{1-nx-x^2s\eta}.$$

This completes the proof. □

From [14] we can obtain the following lemma.

Lemma 2.4. $$\left(nx + x^2s\eta\right)^k x^2 = x^{k+1}\sum_{j=0}^{k} n^{k-j}x^j \begin{bmatrix} k \\ j \end{bmatrix} s^j p^{j^2}$$

and

$$\left(nx + x^2s\eta\right)^k x^2 = x^{k+2}\sum_{j=0}^{k} n^{k-j}x^j \begin{bmatrix} k \\ j \end{bmatrix} s^j p^{j(j+1)}.$$

Theorem 2.5. The generating functions for $P_k(s,p)$ can be also expressed as follows

$$\sum_{k=0}^{\infty} P_k(s,p)x^k = \sum_{i=0}^{\infty} x^{2i}s^ip^{i^2}\left\{\left(\frac{u}{(nx;p)_i}\left(1+\frac{s\eta}{n}xp^i\right) + \frac{x}{(nx;p)_{i+1}}\left(v-nu+(2m-n)f(x)\right)\right)\right\}.$$

Proof. From Theorem 2.3. we obtain

$$\sum_{k=0}^{\infty} P_k(s,p)x^k = \frac{u + x(v-nu) + (2m-n)xf(x)}{\left(1-nx-x^2s\eta\right)}$$

and from Lemma 2.4.

$$= \sum_{k=0}^{\infty}\left(nx + x^2s\eta\right)^k \left(u + x(v-nu) + (2m-n)xf(x)\right)$$
$$= u\sum_{k=0}^{\infty}\left(nx + x^2s\eta\right)^{k-1}\left(nx + x^2s\eta\right) + (v-nu)\sum_{k=0}^{\infty}\left(nx + x^2s\eta\right)^n x$$
$$+ (2m-n)\sum_{k=0}^{\infty}\left(nx + x^2s\eta\right)^n xf(x)$$
$$= un\sum_{k=0}^{\infty}\left(nx + x^2s\eta\right)^{k-1}x + us\sum_{k=0}^{\infty}\left(nx + x^2s\eta\right)^{n-1}x^2\eta$$
$$+ (v-nu)\sum_{k=0}^{\infty}\left(nx + x^2s\eta\right)^n x + (2m-n)\sum_{k=0}^{\infty}\left(nx + x^2s\eta\right)^n xf(x)$$

and using the above Lemma, we get

$$\begin{aligned}\sum_{k=0}^{\infty} P_k(s,p)x^k &= un\sum_{k=0}^{\infty} x^k \sum_{i=0}^{k} n^{k-i-1}s^i x^i \begin{bmatrix} k-1 \\ i \end{bmatrix} p^{i^2} + us\sum_{k=0}^{\infty} x^{k+1}\sum_{i=0}^{k} n^{k-i-1}s^i x^i \begin{bmatrix} k-1 \\ i \end{bmatrix} p^{i(i+1)}\eta \\ &\quad + (v-nu)\sum_{k=0}^{\infty} x^{k+1}\sum_{i=0}^{k} n^{k-i}s^i x^i \begin{bmatrix} k \\ i \end{bmatrix} p^{i^2} + (2m-n)\sum_{k=0}^{\infty} x^{k+1}\sum_{i=0}^{k} n^{k-i}s^i x^i \begin{bmatrix} k \\ i \end{bmatrix} p^{i^2} f(x) \\ &= un\sum_{k=0}^{\infty}\sum_{i=0}^{k} n^{k-i-1}s^i x^{k+i} \begin{bmatrix} k-1 \\ i \end{bmatrix} p^{i^2} + us\sum_{k=0}^{\infty}\sum_{i=0}^{k} n^{k-i-1}s^i x^{k+i+1} \begin{bmatrix} k-1 \\ i \end{bmatrix} p^{i(i+1)}\eta \\ &\quad + (v-nu)\sum_{k=0}^{\infty}\sum_{i=0}^{k} n^{k-i}s^i x^{k+1+i} \begin{bmatrix} k \\ i \end{bmatrix} p^{i^2} + (2m-n)\sum_{k=0}^{\infty}\sum_{i=0}^{k} n^{k-i}s^i x^{k+i+1} \begin{bmatrix} k \\ i \end{bmatrix} p^{i^2} f(x) \\ &= u\sum_{k,i\geq 0} n^k s^i x^{k+2i} \begin{bmatrix} k+i-1 \\ i \end{bmatrix} p^{i^2} + \frac{u}{n}\sum_{k,i\geq 0} n^k s^{i+1} x^{k+2i+1} \begin{bmatrix} k+i-1 \\ i \end{bmatrix} p^{i(i+1)}\eta \\ &\quad + (v-nu)\sum_{k,i\geq 0} n^k s^i x^{k+2i+1} \begin{bmatrix} k+i \\ i \end{bmatrix} p^{i^2} + (2m-n)\sum_{k,i\geq 0} n^k s^i x^{k+2i+1} \begin{bmatrix} k+i \\ i \end{bmatrix} p^{i^2} f(x);\end{aligned}$$

from the p-binomial formula

$$\sum_{j=0}^{\infty} \begin{bmatrix} k+j \\ j \end{bmatrix} x^j = \frac{1}{(x;p)_{k+1}},$$

we get

$$\begin{aligned}\sum_{k=0}^{\infty} P_k(s,p)x^k &= u\sum_{i\geq 0} s^i x^{2i} p^{i^2} \frac{1}{(nx;p)_i} + \frac{u}{n}\sum_{i\geq 0} s^{i+1} x^{2i+1} p^{i(i+1)} \frac{1}{(nx;p)_i}\eta \\ &\quad + (v-nu)\sum_{i\geq 0} s^i x^{2i+1} p^{i^2} \frac{1}{(nx;p)_{i+1}} + (2m-n)\sum_{i\geq 0} s^i x^{2i+1} p^{i^2} \frac{1}{(nx;p)_{i+1}} f(x) \\ &= \sum_{i=0}^{\infty} x^{2i} s^i p^{i^2} \left\{ \left(\frac{u}{(nx;p)_i}\left(1+\frac{s\eta}{n} xp^i\right) + \frac{x}{(nx;p)_{i+1}}\left(v-nu+(2m-n)f(x)\right)\right)\right\}.\end{aligned}$$

This completes the proof. □

3 A p-analogue of biperiodic incomplete Pell and Pell–Lucas polynomials

Definition 3.1. The p-analogues of biperiodic incomplete Pell polynomials are defined by

$$\begin{aligned}P_k^t(s,p) = n^{\delta(k)} u \sum_{i=0}^{t} \begin{bmatrix} k-i-2 \\ i \end{bmatrix} (mn)^{\left\lfloor \frac{k-2}{2} \right\rfloor - i} p^{i(i+1)} s^{i+1} 2^{k-2-2i} \\ + m^{\delta(k-1)} v \sum_{i=0}^{t} \begin{bmatrix} k-i-1 \\ i \end{bmatrix} (mn)^{\left\lfloor \frac{k-1}{2} \right\rfloor - i} p^{i^2} s^i 2^{k-1-2i}\end{aligned} \tag{3.1}$$

for $0 \leq t \leq \dfrac{k-1}{2}$.

Theorem 3.2. The p-analogues of biperiodic incomplete Pell polynomials satisfy the following relations:

$$P_{k+2}^{t+1}(s,p)=2mP_{k+1}^{t+1}(s,p)+sp^kP_k^t(s,p),\ \text{ if } k \text{ is even},$$
$$P_{k+2}^{t+1}(s,p)=2nP_{k+1}^{t+1}(s,p)+sp^kP_k^t(s,p),\ \text{ if } k \text{ is odd}, \tag{3.2}$$

for $0\le t\le \frac{k-2}{2}$.

Proof. Let k be odd. Then by using (3.1), we get

$$\begin{aligned}
2nP_{k+1}^{t+1}(s,p)+sp^kP_k^t(s,p) &= n^{\delta(k+1)+1}u\sum_{i=0}^{t+1}\begin{bmatrix}k-i-1\\ i\end{bmatrix}(mn)^{\left\lfloor\frac{k-1}{2}\right\rfloor-i}\ p^{i(i+1)}s^{i+1}2^{k-2i}\\
&\quad+m^{\delta(k)}v\sum_{i=0}^{t+1}\begin{bmatrix}k-i\\ i\end{bmatrix}(mn)^{\left\lfloor\frac{k}{2}\right\rfloor-i}\ p^{i^2}s^i2^{k+1-2i}\\
&\quad+sp^kn^{\delta(k)}u\sum_{i=0}^{t}\begin{bmatrix}k-i-2\\ i\end{bmatrix}(mn)^{\left\lfloor\frac{k-2}{2}\right\rfloor-i}\ p^{i(i+1)}s^{i+1}2^{k-2i-2}\\
&\quad+sp^km^{\delta(k-1)}v\sum_{i=0}^{t}\begin{bmatrix}k-i-1\\ i\end{bmatrix}(mn)^{\left\lfloor\frac{k-1}{2}\right\rfloor-i}\ p^{i^2}s^i2^{k-1-2i}\\
&= n^{\delta(k+2)}u\sum_{i=0}^{t+1}\begin{bmatrix}k-i-1\\ i\end{bmatrix}(mn)^{\left\lfloor\frac{k-1}{2}\right\rfloor-i}\ p^{i(i+1)}s^{i+1}2^{k-2i}\\
&\quad+nm^{\delta(k+1)+1}v\sum_{i=0}^{t+1}\begin{bmatrix}k-i\\ i\end{bmatrix}(mn)^{\left\lfloor\frac{k}{2}\right\rfloor-i}\ p^{i^2}s^i2^{k+1-2i}\\
&\quad+sp^kn^{\delta(k+2)}u\sum_{i=1}^{t+1}\begin{bmatrix}k-i-1\\ i-1\end{bmatrix}(mn)^{\left\lfloor\frac{k-2}{2}\right\rfloor-i+1}\ p^{i(i-1)}s^i2^{k-2i}\\
&\quad+p^km^{\delta(k+1)}v\sum_{i=0}^{t+1}\begin{bmatrix}k-i\\ i-1\end{bmatrix}(mn)^{\left\lfloor\frac{k-1}{2}\right\rfloor-i+1}\ p^{(i-1)^2}s^i2^{k+1-2i}\ ,
\end{aligned}$$

so that

$$\begin{aligned}
2nP_{k+1}^{t+1}(s,p)+sp^kP_k^t(s,p) &= n^{\delta(k+2)}u\sum_{i=0}^{t+1}s^{i+1}p^{i(i+1)}2^{k-2i}\left\{\begin{bmatrix}k-i-1\\ i\end{bmatrix}+p^{k-2i}\begin{bmatrix}k-i-1\\ i-1\end{bmatrix}\right\}(mn)^{\left\lfloor\frac{k}{2}\right\rfloor-i}\\
&\quad+vm^{\delta(k+1)}\sum_{i=0}^{t+1}s^ip^{i^2}2^{k-2i+1}\left\{\begin{bmatrix}k-i\\ i\end{bmatrix}+p^{k-2i+1}\begin{bmatrix}k-i\\ i-1\end{bmatrix}\right\}(mn)^{\left\lfloor\frac{k+1}{2}\right\rfloor-i}
\end{aligned}$$

and from (1.5) and (3.1) we obtain

$$\begin{aligned}
2nP_{k+1}^{t+1}(s,p)+sp^kP_k^t(s,p) &= n^{\delta(k+2)}u\sum_{i=0}^{t+1}\begin{bmatrix}k-i\\ i\end{bmatrix}(mn)^{\left\lfloor\frac{k}{2}\right\rfloor-i}\ p^{i(i+1)}s^{i+1}2^{k-2i}\\
&\quad+m^{\delta(k+1)}v\sum_{i=0}^{t+1}\begin{bmatrix}k-i+1\\ i\end{bmatrix}(mn)^{\left\lfloor\frac{k+1}{2}\right\rfloor-i}\ p^{i^2}s^i2^{k+1-2i}\\
&= P_{k+2}^{t+1}(s,p).
\end{aligned}$$

A similar proof is made if k is even. □

Theorem 3.3. $P_k^t(s,p)$ satisfy the following equation

$$\sum_{t=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} P_k^t(s,p) = \left(\frac{k-1}{2}+1\right)P_k(s,p) - \frac{d}{dt}P_k(s,p) + n^{\delta(k)}u\sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} s^i \begin{bmatrix} k-i-2 \\ i \end{bmatrix} (mn)^{\left\lfloor \frac{k-2}{2} \right\rfloor - i} p^{i(i+1)} s^i 2^{k-2-2i}.$$

Proof. We have

$$\begin{aligned}
\sum_{t=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} P_k^t(s,p) &= P_k^0(s,p) + P_k^1(s,p) + P_k^2(s,p) + \cdots + P_k^{\left\lfloor \frac{k-1}{2} \right\rfloor}(s,p) \\
&= \left(\left\lfloor \frac{k-1}{2} \right\rfloor + 1\right)\left\{ n^{\delta(k)}us\begin{bmatrix} k-2 \\ 0 \end{bmatrix}(mn)^{\left\lfloor \frac{k-2}{2} \right\rfloor} 2^{k-2} + m^{\delta(k-1)}v\begin{bmatrix} k-1 \\ 0 \end{bmatrix}(mn)^{\left\lfloor \frac{k-1}{2} \right\rfloor} 2^{k-1} \right\} \\
&\quad + \left\lfloor \frac{k-1}{2} \right\rfloor\left\{ n^{\delta(k)}us^2p^2\begin{bmatrix} k-3 \\ 1 \end{bmatrix}(mn)^{\left\lfloor \frac{k-2}{2} \right\rfloor - 1} 2^{k-4} + m^{\delta(k-1)}vsp\begin{bmatrix} k-2 \\ 1 \end{bmatrix}(mn)^{\left\lfloor \frac{k-1}{2} \right\rfloor - 1} 2^{k-3} \right\} + \cdots \\
&\quad + \left\{ n^{\delta(k)}us^{\left\lfloor \frac{k-1}{2} \right\rfloor + 1} p^{\left\lfloor \frac{k-1}{2} \right\rfloor\left(\left\lfloor \frac{k-1}{2} \right\rfloor + 1\right)} \begin{bmatrix} k-2-\left\lfloor \frac{k-1}{2} \right\rfloor \\ \left\lfloor \frac{k-1}{2} \right\rfloor \end{bmatrix}(mn)^{\left\lfloor \frac{k-2}{2} \right\rfloor - \left\lfloor \frac{k-1}{2} \right\rfloor} 2^{k-2-2\left\lfloor \frac{k-1}{2} \right\rfloor} \right. \\
&\quad \left. + m^{\delta(k-1)}vs^{\left\lfloor \frac{k-1}{2} \right\rfloor} p^{\left(\left\lfloor \frac{k-1}{2} \right\rfloor\right)^2} \begin{bmatrix} k-1-\left\lfloor \frac{k-1}{2} \right\rfloor \\ \left\lfloor \frac{k-1}{2} \right\rfloor \end{bmatrix}(mn)^{\left\lfloor \frac{k-1}{2} \right\rfloor - \left\lfloor \frac{k-1}{2} \right\rfloor} 2^{k-1-2\left\lfloor \frac{k-1}{2} \right\rfloor} \right\} \\
&= \left(\left\lfloor \frac{k-1}{2} \right\rfloor + 1\right)P_k(s,p) - n^{\delta(k)}u\sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} i\begin{bmatrix} k-i-2 \\ i \end{bmatrix}(mn)^{\left\lfloor \frac{k-2}{2} \right\rfloor - i} p^{i(i+1)} s^{i+1} 2^{k-2-2i} \\
&\qquad - m^{\delta(k-1)}v\sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} ip^{i^2}s^i\begin{bmatrix} k-i-1 \\ i \end{bmatrix}(mn)^{\left\lfloor \frac{k-1}{2} \right\rfloor - i} 2^{k-1-2i}.
\end{aligned}$$

Also, we obtain

$$\begin{aligned}
&\frac{d}{dt}P_k(s,p) - n^{\delta(k)}u\sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} s^i\begin{bmatrix} k-i-2 \\ i \end{bmatrix}(mn)^{\left\lfloor \frac{k-2}{2} \right\rfloor - i} p^{i(i+1)} s^i 2^{k-2-2i} \\
&\quad = n^{\delta(k)}u\sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} is^{i+1}\begin{bmatrix} k-i-2 \\ i \end{bmatrix}(mn)^{\left\lfloor \frac{k-2}{2} \right\rfloor - i} p^{i(i+1)} s^i 2^{k-2-2i} \\
&\qquad + m^{\delta(k-1)}v\sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} ip^{i^2}s^i\begin{bmatrix} k-i-1 \\ i \end{bmatrix}(mn)^{\left\lfloor \frac{k-1}{2} \right\rfloor - i} 2^{k-1-2i},
\end{aligned}$$

from which we get

$$\sum_{t=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} P_k^t(s,p) = \left(\left\lfloor \frac{k-1}{2} \right\rfloor + 1\right)P_k(s,p) - \frac{d}{dt}P_k(s,p) + n^{\delta(k)}u\sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} s^i\begin{bmatrix} k-i-2 \\ i \end{bmatrix}(mn)^{\left\lfloor \frac{k-2}{2} \right\rfloor - i} p^{i(i+1)} s^i 2^{k-2-2i},$$

as desired. □

4 Conclusion

In this study, we have created p-analogue recurrences, similar to, but different from, familiar Pell and Pell Lucas polynomials. We have then obtained the corresponding generating functions of these sequences and many related properties. We also obtained several additive formulas of these new sequences by forming incomplete recurrences, which are the 'incomplete' new recurrences which yield more descriptive theorems and propositions based on a definition that is different from the usual binomial sums. Again, with the help of this definition, a link has been established among 'incomplete' recurrences. This work can be applied to different number sequences, as well as to expanding the p-analogue part to create (p, q)-analogue number sequences. This work has been placed in the historical perspective going back to Leonard Carlitz of Duke University.

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