Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2023, Volume 29, Number 2, 322–335
DOI: 10.7546/nntdm.2023.29.2.322-335

On the k-Fibonacci and k-Lucas spinors

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Received: 15 September 2022 **Revised:** 17 April 2023
Accepted: 8 May 2023 **Online First:** 9 May 2023

Abstract: In this paper, we introduce a new family of sequences called the k-Fibonacci and k-Lucas spinors. Starting with the Binet formulas we present their basic properties, such as Cassini's identity, Catalan's identity, d'Ocagne's identity, Vajda's identity, and Honsberger's identity. In addition, we discuss their generating functions. Finally, we obtain sum formulae and relations between k-Fibonacci and k-Lucas spinors.

Keywords: k-Fibonacci spinor, k-Lucas spinor, Binet form, Catalan's identity, Generating function.
2020 Mathematics Subject Classification: 15A66, 11B39, 11R52.

* Statements and conclusions made in this paper by R. F. are entirely those of the author. They do not necessarily reflect the views of LBBW.

1 Introduction

In 2007, Falcón and Plaza [5] introduced a generalization of Fibonacci sequence called k-Fibonacci sequence $\{F_{k,n}\}_{n\geq 0}$ as

$$F_{k,0} = 0, \quad F_{k,1} = 1, \quad F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \quad n \geq 2, \quad k \in \mathbb{R}. \tag{1.1}$$

Later on, in 2011, Falcón [4] extended this generalization to the Lucas sequence and introduced k-Lucas sequence $\{L_{k,n}\}_{n\geq 0}$ defined as

$$L_{k,0} = 2, \quad L_{k,1} = k, \quad L_{k,n} = kL_{k,n-1} + L_{k,n-2}, \quad n \geq 2, \quad k \in \mathbb{R}. \tag{1.2}$$

The characteristic equation corresponding to the above recurrence relations is

$$\beta^2 - k\beta - 1 = 0 \tag{1.3}$$

with roots $\beta_1 = \dfrac{k+\sqrt{k^2+4}}{2}$ and $\beta_2 = \dfrac{k-\sqrt{k^2+4}}{2}$ having the following relations:

$$\beta_1 + \beta_2 = k, \quad \beta_1\beta_2 = -1, \quad \beta_1 - \beta_2 = \sqrt{k^2+4}. \tag{1.4}$$

The Binet forms of k-Fibonacci and k-Lucas sequences are given by

$$F_{k,n} = \frac{\beta_1^n - \beta_2^n}{\beta_1 - \beta_2} \qquad \text{and} \qquad L_{k,n} = \beta_1^n + \beta_2^n,$$

respectively. The first few terms of k-Fibonacci and k-Lucas sequences are shown in the next Table 1.

Table 1. The first few terms of k-Fibonacci and k-Lucas sequences

n	$F_{k,n}$	$L_{k,n}$
0	0	2
1	1	k
2	k	k^2+2
3	k^2+1	k^3+3k
4	k^3+2k	k^4+4k^2+2
5	k^4+3k^2+1	k^5+5k^3+5k
6	k^5+4k^3+3k	$k^6+6k^4+9k^2+2$

In 2015, Ramírez [7] introduced a new sequence of quaternions with coefficients being the k-Fibonacci and k-Lucas numbers and studied their properties. The k-Fibonacci quaternions $D_{k,n}$ and the k-Lucas quaternions $P_{k,n}$ are defined by the equations

$$D_{k,n} = F_{k,n} + F_{k,n+1}e_1 + F_{k,n+2}e_2 + F_{k,n+3}e_3, \quad n \geq 0, \tag{1.5}$$

and

$$P_{k,n} = L_{k,n} + L_{k,n+1}e_1 + L_{k,n+2}e_2 + L_{k,n+3}e_3, \quad n \geq 0, \tag{1.6}$$

where the basis e_1, e_2, e_3 satisfies the properties

$$e_1^2 = e_2^2 = e_3^2 = e_1e_2e_3 = -1.$$

In general, a quaternion with real coefficients is of the form $q = a + be_1 + ce_2 + de_3$, where $\{1, e_1, e_2, e_3\}$ is the quaternion basis satisfying

$$e_1^2 = e_2^2 = e_3^2 = -1, \quad e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad e_3e_1 = -e_1e_3 = e_2. \tag{1.7}$$

Ramírez derived many properties of k-Fibonacci and k-Lucas quaternions, some of which are restated here:

- The Binet formulae are

$$D_{k,n} = \frac{\hat{\alpha}\beta_1^n - \hat{\beta}\beta_2^n}{\beta_1 - \beta_2} \quad \text{and} \quad P_{k,n} = \hat{\alpha}\beta_1^n + \hat{\beta}\beta_2^n, \tag{1.8}$$

  where $\hat{\alpha} = 1 + \beta_1 e_1 + \beta_1^2 e_2 + \beta_1^3 e_3$ and $\hat{\beta} = 1 + \beta_2 e_1 + \beta_2^2 e_2 + \beta_2^3 e_3$.

- The Catalan's identity for the k-Fibonacci quaternions is

$$D_{k,n-r}D_{k,n+r} - D_{k,n}^2 = (-1)^{n-r+1}(2F_{k,r}D_{k,r} - L_{k,2}F_{k,2r}e_3). \tag{1.9}$$

- The Cassini's identity is given by

$$D_{k,n-1}D_{k,n+1} - D_{k,n}^2 = (-1)^n(2D_{k,1} - e_3(k^3 + 2k)). \tag{1.10}$$

- The d'Ocagne's identity is given by

$$D_{k,r+1}D_{k,n} - D_{k,r}D_{k,n+1} = \frac{(-1)^m(\hat{\beta}\hat{\alpha}\beta_1^{n-m} - \hat{\alpha}\hat{\beta}\beta_2^{n-m})}{\beta_1 - \beta_2}. \tag{1.11}$$

We proceed with some concepts that will be needed later. Consider an isotropic vector $(x, y, z) \in \mathbb{C}^3$, where $\mathbb{C}^3$ is the three dimensional space referred to as a system of orthogonal coordinates. Then the vector (x, y, z) satisfies $x^2 + y^2 + z^2 = 0$. Two numbers η_1 and η_2 can be associated with this vector as

$$x = \eta_1^2 - \eta_2^2, \quad y = i(\eta_1^2 + \eta_2^2), \quad z = -2\eta_1\eta_2.$$

By solving the above equations, we get

$$\eta_1 = \pm\sqrt{\frac{x - iy}{2}} \quad \text{and} \quad \eta_2 = \pm\sqrt{\frac{-x - iy}{2}}.$$

This leads to a definition of a spinor as introduced by Cartan [1]

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}. \tag{1.12}$$

A spinor $\tilde{\eta}$ conjugate to η is defined by (Cartan [1])

$$\tilde{\eta} = iA\overline{\eta}, \tag{1.13}$$

where $\overline{\eta}$ is complex conjugate of η and $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

In 1984, Vivarelli [8] defined a linear and injective correspondence between the quaternions and spinors. Let the sets of quaternions and spinors be denoted as $\mathbb{H}$ and $\mathbb{S}$, respectively. Then the correspondence is defined as below.

Definition 1.1. *Let* $\phi : \mathbb{H} \to \mathbb{S}$ *be any correspondence between a quaternion* $q = a + be_1 + ce_2 + de_3 \in \mathbb{H}$ *and a spinor* $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \in \mathbb{S}$*. It is given by*

$$\phi(a + be_1 + ce_2 + de_3) = \begin{bmatrix} d + ia \\ b + ic \end{bmatrix} \equiv \eta. \tag{1.14}$$

Also, Vivarelli [8] has defined the correspondence between the product of two quaternions and a spinor product matrix given by

$$qp \to -i\hat{Q}P, \tag{1.15}$$

where P is the spinor corresponding to the quaternion q and $\hat{Q}$ is the complex unitary square matrix defined as

$$\begin{bmatrix} d + ia & b - ic \\ b + ic & -d + ia \end{bmatrix}. \tag{1.16}$$

Finally, the mate of a spinor η introduced by Castillo [2] is

$$\check{\eta} = -A\overline{\eta}. \tag{1.17}$$

Erişir and Güngör [3] introduced the Fibonacci spinors using Fibonacci quaternions and studied their algebra. In this paper, we generalize the concept of the Fibonacci spinors by introducing the k-Fibonacci and k-Lucas spinors. Starting with the Binet formulas we present their basic properties, such as Cassini's identity, Catalan's identity, d'Ocagne's identity, Vajda's identity, and Honsberger's identity. In addition, we discuss their generating functions. Finally, we obtain sum formulae and relations between k-Fibonacci and k-Lucas spinors.

2 *k*-Fibonacci spinors

In this section, we define the k-Fibonacci spinor sequence $\{FS_{k,n}\}_{n\geq 0}$ and its conjugates. Moreover, we obtain its Binet type formula, generating function and some interesting identities.

Consider the correspondence between the set of k-Fibonacci quaternions denoted as $\mathbb{F}$ and the set of spinors $\mathbb{S}$. Using Definition 1.1, the correspondence $\phi : \mathbb{F} \to \mathbb{S}$ is defined as

$$\phi(F_{k,n} + F_{k,n+1}e_1 + F_{k,n+2}e_2 + F_{k,n+3}e_3) = \begin{bmatrix} F_{k,n+3} + iF_{k,n} \\ F_{k,n+1} + iF_{k,n+2} \end{bmatrix} = FS_{k,n}. \tag{2.1}$$

Note that this transformation is linear and injective but not surjective and hence not bijective.
If $\overline{D}_{k,n} = F_{k,n} - F_{k,n+1}e_1 - F_{k,n+2}e_2 - F_{k,n+3}e_3$ is the conjugate of the quaternion $D_{k,n}$, then the k-Fibonacci spinor $FS^*_{k,n}$ corresponding to $\overline{D}_{k,n}$ is

$$FS^*_{k,n} = \begin{bmatrix} -F_{k,n+3} + iF_{k,n} \\ -F_{k,n+1} - iF_{k,n+2} \end{bmatrix}.$$

Now, by the above defined transformation we introduce a sequence of k-Fibonacci spinors recursively given in the following definition.

Definition 2.1. *For $n \geq 0$, the k-Fibonacci spinor sequence $\{FS_{k,n}\}$ is defined recursively by*

$$FS_{k,n+2} = kFS_{k,n+1} + FS_{k,n}, \tag{2.2}$$

with $FS_{k,0} = \begin{bmatrix} k^2+1 \\ 1+ik \end{bmatrix}$ *and* $FS_{k,1} = \begin{bmatrix} k^3+2k+i \\ k+i(k^2+1) \end{bmatrix}$.

From (2.2) we note that the characteristic equation for k-Fibonacci spinors is same as of k-Fibonacci sequence (1.3) and hence the roots are β_1 and β_2 satisfying the relations in (1.4). Complex conjugate of $FS_{k,n}$ can be written as

$$\overline{FS}_{k,n} = \begin{bmatrix} F_{k,n+3} - iF_{k,n} \\ F_{k,n+1} - iF_{k,n+2} \end{bmatrix}.$$

The spinor conjugate to the k-Fibonacci spinor $FS_{k,n}$ equals

$$\tilde{S}_{k,n} = iA\overline{S}_{k,n} = \begin{bmatrix} F_{k,n+2} + iF_{k,n+1} \\ -F_{k,n} - iF_{k,n+3} \end{bmatrix},$$

where we have used (1.13). Also, from (1.17) the mate of $FS_{k,n}$ is given by

$$\check{S}_{k,n} = -A\overline{S}_{k,n} = \begin{bmatrix} -F_{k,n+1} + iF_{k,n+2} \\ F_{k,n+3} - iF_{k,n} \end{bmatrix}.$$

Theorem 2.1 (Binet formula)**.** *For $n \geq 0$, we have*

$$FS_{k,n} = \frac{1}{\sqrt{k^2+4}} \begin{bmatrix} \beta_1^3 + i \\ \beta_1 + i\beta_1^2 \end{bmatrix} \beta_1^n - \frac{1}{\sqrt{k^2+4}} \begin{bmatrix} \beta_2^3 + i \\ \beta_2 + i\beta_2^2 \end{bmatrix} \beta_2^n. \tag{2.3}$$

Proof. We know that the characteristic equation for the k-Fibonacci spinor sequence $\{FS_{k,n}\}$ is the same as the characteristic equation for the k-Fibonacci sequence. Hence, we can write

$$FS_{k,n} = A\beta_1^n + B\beta_2^n, \tag{2.4}$$

with A and B to be determined. We have

$$FS_{k,0} = A + B = \begin{bmatrix} k^2+1 \\ 1+ik \end{bmatrix} \quad \text{and} \quad FS_{k,1} = A\beta_1 + B\beta_2 = \begin{bmatrix} k^3+2k+i \\ k+i(k^2+1) \end{bmatrix}.$$

After some necessary calculations, we get

$$A = \frac{1}{\sqrt{k^2+4}} \begin{bmatrix} \beta_1^3 + i \\ \beta_1 + i\beta_1^2 \end{bmatrix} \quad \text{and} \quad B = -\frac{1}{\sqrt{k^2+4}} \begin{bmatrix} \beta_2^3 + i \\ \beta_2 + i\beta_2^2 \end{bmatrix}.$$

This completes the proof. □

With the help of the Binet formula, we can also extend the k-Fibonacci spinor sequence $\{FS_{k,n}\}$ in the negative direction. This is shown in the next theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$FS_{k,-n} = (-1)^n \begin{bmatrix} F_{k,n-3} - iF_{k,n} \\ F_{k,n-1} - iF_{k,n-2} \end{bmatrix}. \tag{2.5}$$

Proof. Using the fact that $\beta_1\beta_2 = -1$, we have $\beta_1^{-n} = (-1)^n\beta_2^n$ and $\beta_2^{-n} = (-1)^n\beta_1^n$. Upon replacing n by $-n$ in (2.4), we obtain

$$\begin{aligned} FS_{k,-n} &= A\beta_1^{-n} + B\beta_2^{-n} \\ &= A(-1)^n\beta_2^n + B(-1)^n\beta_1^n \\ &= (-1)^n(A\beta_2^n + B\beta_1^n). \end{aligned}$$

Since $A = \frac{1}{\sqrt{k^2+4}} \begin{bmatrix} \beta_1^3 + i \\ \beta_1 + i\beta_1^2 \end{bmatrix}$ and $B = -\frac{1}{\sqrt{k^2+4}} \begin{bmatrix} \beta_2^3 + i \\ \beta_2 + i\beta_2^2 \end{bmatrix}$, we finally get

$$\begin{aligned} FS_{k,-n} &= (-1)^n \begin{bmatrix} [-(k^3+2k)F_{k,n} + (k^2+1)F_{k,n+1}] - iF_{k,n} \\ (-kF_{k,n} + F_{k,n+1}) + i[-(k^2+1)F_{k,n} + kF_{k,n+1}] \end{bmatrix} \\ &= (-1)^n \begin{bmatrix} F_{k,n-3} - iF_{k,n} \\ F_{k,n-1} - iF_{k,n-2} \end{bmatrix}. \end{aligned}$$ □

Theorem 2.3 (Cassini's identity)**.** *For $n \in \mathbb{N}$, we have*

$$\hat{S}_{k,n-1}FS_{k,n+1} - \hat{S}_{k,n}FS_{k,n} = (-1)^n \begin{bmatrix} -2F_{k,1} + iF_{k,4} \\ -2F_{k,3} + i2F_{k,2} \end{bmatrix}. \tag{2.6}$$

Proof. For $n, m \in \mathbb{N}$ let $D_{k,n} = F_{k,n} + F_{k,n+1}e_1 + F_{k,n+2}e_2 + F_{k,n+3}e_3$ and $D_{k,m} = F_{k,m} + F_{k,m+1}e_1 + F_{k,m+2}e_2 + F_{k,m+3}e_3$ be two k-Fibonacci quaternions, respectively. Then by (1.15), we can write

$$D_{k,n}D_{k,m} = -i\hat{S}_{k,n}FS_{k,m},$$

where $\hat{S}_{k,n} = \begin{bmatrix} F_{k,n+3} + iF_{k,n} & F_{k,n+1} - iF_{k,n+2} \\ F_{k,n+1} + iF_{k,n+2} & -F_{k,n+3} + iF_{k,n} \end{bmatrix}$ and $FS_{k,m} = \begin{bmatrix} F_{k,m+3} + iF_{k,m} \\ F_{k,m+1} + iF_{k,m+2} \end{bmatrix}$. Thus, we have

$$D_{k,n-1}D_{k,n+1} - D_{k,n}^2 = -i\hat{S}_{k,n-1}FS_{k,n+1} + i\hat{S}_{k,n}FS_{k,n} = -i(\hat{S}_{k,n-1}FS_{k,n+1} - \hat{S}_{k,n}FS_{k,n}).$$

Now using Cassini's identity for the k-Fibonacci quaternions (1.10), we have

$$-i(\hat{S}_{k,n-1}FS_{k,n+1} - \hat{S}_{k,n}FS_{k,n}) = (-1)^n \begin{bmatrix} F_{k,4} + i2F_{k,1} \\ 2F_{k,2} + i2F_{k,3} \end{bmatrix}$$

or equivalently

$$\hat{S}_{k,n-1}FS_{k,n+1} - \hat{S}_{k,n}FS_{k,n} = (-1)^n \begin{bmatrix} -2F_{k,1} + iF_{k,4} \\ -2F_{k,3} + i2F_{k,2} \end{bmatrix}.$$ □

Note that if we substitute $k = 1$ in (2.6), we get Cassini's identity for the ordinary Fibonacci spinors as derived in [3].

Theorem 2.4 (Catalan's identity)**.** *For* $n \in \mathbb{N}$*, we have*

$$\hat{S}_{k,n-r}FS_{k,n+r} - \hat{S}_{k,n}FS_{k,n} = (-1)^{n-r+1}\begin{bmatrix} -2F_{k,r}^2 + i\big(2F_{k,r}F_{k,r+3} - (k^2+2)F_{k,2r}\big) \\ 2F_{k,r}F_{k,r+2} + i2F_{k,r}F_{k,r+1} \end{bmatrix}. \quad (2.7)$$

Proof. We already know that

$$D_{k,n-r}D_{k,n+r} - D_{k,n}^2 = -i\hat{S}_{k,n-r}FS_{k,n+r} + i\hat{S}_{k,n}FS_{k,n} = -i(\hat{S}_{k,n-r}FS_{k,n+r} - \hat{S}_{k,n}FS_{k,n}).$$

Now using Catalan's identity for the k-Fibonacci quaternions (1.9), we get

$$-i(\hat{S}_{k,n-r}FS_{k,n+r} - \hat{S}_{k,n}FS_{k,n}) = (-1)^{n-r+1}\begin{bmatrix} \big(2F_{k,r}F_{k,r+3} - (k^2+2)F_{k,2r}\big) + i2F_{k,r}^2 \\ 2F_{k,r}F_{k,r+1} + i2F_{k,r}F_{k,r+2} \end{bmatrix}$$

or equivalently

$$\hat{S}_{k,n-r}FS_{k,n+r} - \hat{S}_{k,n}FS_{k,n} = (-1)^{n-r+1}\begin{bmatrix} -2F_{k,r}^2 + i\big(2F_{k,r}F_{k,r+3} - (k^2+2)F_{k,2r}\big) \\ -2F_{k,r}F_{k,r+2} + i2F_{k,r}F_{k,r+1} \end{bmatrix}. \qquad \square$$

Theorem 2.5 (d'Ocagne's identity)**.** *For* $n, r \in \mathbb{N}$*, we have*

$$\hat{S}_{k,r+1}FS_{k,n} - \hat{S}_{k,r}FS_{k,n+1} = (-1)^r\begin{bmatrix} -2F_{k,n-r} + i\big(F_{k,n-r+3} + F_{k,n-r-1}\big) \\ -2F_{k,n-r+2} + i2F_{k,n-r+1} \end{bmatrix}. \quad (2.8)$$

Proof. By the expression (1.11), we write

$$D_{k,r+1}D_{k,n} - D_{k,r}D_{k,n+1} = \frac{(-1)^r(\hat{\beta}\hat{\alpha}\beta_1^{n-m} - \hat{\alpha}\hat{\beta}\beta_2^{n-m})}{\beta_1 - \beta_2}.$$

Using the values of $\hat{\alpha}\hat{\beta}$ and $\hat{\beta}\hat{\alpha}$ and Definition 1.5, we rewrite the above expression as

$$D_{k,r+1}D_{k,n} - D_{k,r}D_{k,n+1} = (-1)^r\big(2D_{k,n-r} + (F_{k,n-r-1} - F_{k,n-r+3})e_3\big).$$

Now, we have

$$D_{k,r+1}D_{k,n} - D_{k,r}D_{k,n+1} = -i\hat{S}_{k,r+1}FS_{k,n} + i\hat{S}_{k,r}FS_{k,n+1} = -i(\hat{S}_{k,r+1}FS_{k,n} - \hat{S}_{k,r}FS_{k,n+1}).$$

And thus, we get

$$-i(\hat{S}_{k,r+1}FS_{k,n} - \hat{S}_{k,r}FS_{k,n+1}) = (-1)^r\begin{bmatrix} \big(F_{k,n-r+3} + F_{k,n-r-1}\big) + i2F_{k,n-r} \\ 2F_{k,n-r+1} + i2F_{k,n-r+2} \end{bmatrix}$$

or equivalently

$$\hat{S}_{k,r+1}FS_{k,n} - \hat{S}_{k,r}FS_{k,n+1} = (-1)^r\begin{bmatrix} -2F_{k,n-r} + i\big(F_{k,n-r+3} + F_{k,n-r-1}\big) \\ -2F_{k,n-r+2} + i2F_{k,n-r+1} \end{bmatrix}. \qquad \square$$

Theorem 2.6 (Honsberger's identity)**.** *For* $n, r \in \mathbb{N}$*, we have*

$$\hat{S}_{k,n+1}FS_{k,r} + \hat{S}_{k,n}FS_{k,r-1} = \begin{bmatrix} -\big(F_{k,n+r} + F_{k,n+r+2} + F_{k,n+r+4} + F_{k,n+r+6}\big) + i2F_{k,n+r+3} \\ -2F_{k,n+r+2} + i2F_{k,n+r+1} \end{bmatrix}. \quad (2.9)$$

Proof. Here, we make use of the identity

$$D_{k,n+1}D_{k,r} + D_{k,n}D_{k,r-1} = 2D_{k,n+r} + \big(F_{k,n+r} + F_{k,n+r+2} + F_{k,n+r+4} + F_{k,n+r+6}\big)$$

in conjunction with (1.15), to get

$$D_{k,n+1}D_{k,r} + D_{k,n}D_{k,r-1} = -i\hat{S}_{k,n+1}FS_{k,r} + i\hat{S}_{k,n}FS_{k,r-1} = -i(\hat{S}_{k,n+1}FS_{k,r} + \hat{S}_{k,n}FS_{k,r-1}).$$

This produces

$$-i(\hat{S}_{k,n+1}FS_{k,r} + \hat{S}_{k,n}FS_{k,r-1}) = \begin{bmatrix} 2F_{k,n+r+3} + i\big(F_{k,n+r} + F_{k,n+r+2} + F_{k,n+r+4} + F_{k,n+r+6}\big) \\ 2F_{k,n+r+1} + i2F_{k,n+r+2} \end{bmatrix}$$

and the proof is completed. □

Theorem 2.7 (Vajda's identity)**.** *For $n, m, r \in \mathbb{N}$, we have*

$$\hat{S}_{k,n+m}FS_{k,n+r} - \hat{S}_{k,n}FS_{k,n+m+r} = (-1)^n F_{k,m} \begin{bmatrix} -2F_{k,r} + i\big(F_{k,r+3} - k^2 L_{k,r}\big) \\ -2F_{k,r+2} + i2F_{k,r+1} \end{bmatrix}. \tag{2.10}$$

Proof. Using (1.15), we may write

$$D_{k,n+m}D_{k,n+r} - D_{k,n}D_{k,n+m+r} = -i(\hat{S}_{k,n+m}FS_{k,n+r} - \hat{S}_{k,n}FS_{k,n+m+r})$$

as well as

$$D_{k,n+m}D_{k,n+r} - D_{k,n}D_{k,n+m+r} = (-1)^n F_{k,m}\big(2D_{k,r} - k^2 L_{k,r} e_3\big).$$

Thus,

$$-i(\hat{S}_{k,n+m}FS_{k,n+r} - \hat{S}_{k,n}FS_{k,n+m+r}) = (-1)^n F_{k,m} \begin{bmatrix} \big(F_{k,r+3} - k^2 L_{k,r}\big) + i2F_{k,r} \\ 2F_{k,r+1} + i2F_{k,r+2} \end{bmatrix}$$

and the proof is completed. □

Theorem 2.8 (Generating function)**.** *The generating function for the k-Fibonacci spinor sequence $\{FS_{k,n}\}$ is*

$$G_k(x) = \frac{1}{1 - kx - x^2} \begin{bmatrix} k^2 + xk + 1 + ix \\ 1 + i(k + x) \end{bmatrix}. \tag{2.11}$$

Proof. By definition, the generating function for the k-Fibonacci spinor sequence $\{FS_{k,n}\}$ is

$$G_k(x) = \sum_{n=0}^{\infty} FS_{k,n} x^n.$$

If we multiply the recurrence relation (2.2) by x^{n+2} and sum from zero to infinity, we get

$$\sum_{n=0}^{\infty} FS_{k,n+2}x^{n+2} - k\sum_{n=0}^{\infty} FS_{k,n+1}x^{n+2} - \sum_{n=0}^{\infty} FS_{k,n}x^{n+2} = 0.$$

This is simply

$$(G_k(x) - FS_{k,0} - FS_{k,1}x) - xk(G_k(x) - FS_{k,0}) - x^2 G_k(x) = 0$$

or

$$G_k(x) - xkG_k(x) - x^2 G_k(x) = FS_{k,0} + (FS_{k,1} - kFS_{k,0})x.$$

The proof is completed upon inserting the initial values. □

Theorem 2.9 (Finite sum formulae)**.** *The sums of the first n terms of the k-Fibonacci spinor sequence $\{FS_{k,n}\}$ equal*

1. $\sum_{j=1}^{n} FS_{k,j} = \frac{1}{k}[FS_{k,n+1} + FS_{k,n} - (FS_{k,0} + FS_{k,1})]$,
2. $\sum_{j=1}^{n} FS_{k,2j} = \frac{1}{k}[FS_{k,2n+1} - FS_{k,1}]$,
3. $\sum_{j=1}^{n} FS_{k,2j-1} = \frac{1}{k}[FS_{k,2n} - FS_{k,0}]$.

Proof. These finite sum identities can be proved by using the corresponding sum identities for k-Fibonacci sequences (see [5, 6]). For instance, the first identity follows from $\sum_{j=1}^{n} F_{k,j} = \frac{1}{k}[F_{k,n+1} + F_{k,n} - 1]$, while the second identity uses $\sum_{j=1}^{n} F_{k,2j} = \frac{1}{k}[F_{k,2n+1} - 1]$, and the third identity follows from $\sum_{j=1}^{n} F_{k,2j-1} = \frac{1}{k}[F_{k,2n}]$. □

3 k-Lucas spinors

In this section, we introduce the k-Lucas spinor sequence $\{LS_{k,n}\}$ analogous to the k-Fibonacci spinor sequence $\{FS_{k,n}\}$ and state the basic properties of this sequence like Binet formula, Cassini's identity, Catalan's identity and so on.

Let $P_{k,n} = L_{k,n} + L_{k,n+1}e_1 + L_{k,n+2}e_2 + L_{k,n+3}e_3$ be the k-Lucas quaternion. Then the corresponding k-Lucas spinor is given by

$$\phi(L_{k,n} + L_{k,n+1}e_1 + L_{k,n+2}e_2 + L_{k,n+3}e_3) = \begin{bmatrix} L_{k,n+3} + iL_{k,n} \\ L_{k,n+1} + iL_{k,n+2} \end{bmatrix} = LS_{k,n}. \tag{3.1}$$

The k-Lucas spinor $LS^*_{k,n}$ corresponding to the conjugate $\overline{P}_{k,n} = L_{k,n} - L_{k,n+1}e_1 - L_{k,n+2}e_2 - L_{k,n+3}e_3$ of $P_{k,n}$ is

$$LS^*_{k,n} = \begin{bmatrix} -L_{k,n+3} + iL_{k,n} \\ -L_{k,n+1} - iL_{k,n+2} \end{bmatrix}.$$

Definition 3.1. *For $n \geq 0$, k-Lucas spinor sequence $\{LS_{k,n}\}_{n\geq 0}$ is defined recursively by*

$$LS_{k,n+2} = kLS_{k,n+1} + LS_{k,n}, \tag{3.2}$$

with the initial values

$$LS_{k,0} = \begin{bmatrix} (k^3 + 3k) + i2 \\ k + i(k^2 + 2) \end{bmatrix} \quad \textit{and} \quad LS_{k,1} = \begin{bmatrix} (k^4 + 4k^2 + 2) + ik \\ (k^2 + 2) + i(k^3 + 3k) \end{bmatrix}.$$

The complex conjugate of $LS_{k,n}$ is $\overline{LS}_{k,n} = \begin{bmatrix} L_{k,n+3} - iL_{k,n} \\ L_{k,n+1} - iL_{k,n+2} \end{bmatrix}$.

The spinor conjugate to the k-Lucas spinor $LS_{k,n}$ is $\tilde{LS}_{k,n} = iA\overline{LS}_{k,n} = \begin{bmatrix} L_{k,n+2} + iL_{k,n+1} \\ -L_{k,n} - iL_{k,n+3} \end{bmatrix}$.

Finally, the mate of $LS_{k,n}$ can be written as $\check{LS}_{k,n} = -A\overline{LS}_{k,n} = \begin{bmatrix} -L_{k,n+1} + iL_{k,n+2} \\ L_{k,n+3} - iL_{k,n} \end{bmatrix}$.

Theorem 3.1 (Binet formula)**.** *For $n \geq 0$, we have*

$$LS_{k,n} = \begin{bmatrix} \beta_1^3 + i \\ \beta_1 + i\beta_1^2 \end{bmatrix} \beta_1^n + \begin{bmatrix} \beta_2^3 + i \\ \beta_2 + i\beta_2^2 \end{bmatrix} \beta_2^n. \tag{3.3}$$

Proof. Consider the expression $LS_{k,n} = A\beta_1^n + B\beta_2^n$. Then using the initial values in (3.2), we obtain

$$A + B = \begin{bmatrix} (k^3 + 3k) + i2 \\ k + i(k^2 + 2) \end{bmatrix} \quad \text{and} \quad A\beta_1 + B\beta_2 = \begin{bmatrix} (k^4 + 4k^2 + 2) + ik \\ (k^2 + 2) + i(k^3 + 3k) \end{bmatrix}.$$

Solving for A and B, respectively, gives the Binet form as required. □

The extension of the k-Lucas spinor sequence $LS_{k,n}$ to negative subscripts is accomplished in the same way as it was done for the k-Fibonacci spinor sequence.

Theorem 3.2. *For $n \geq 0$, we have*

$$LS_{k,-n} = (-1)^n \begin{bmatrix} -L_{k,n-3} + iL_{k,n} \\ L_{k,n-1} + iL_{k,n-2} \end{bmatrix}. \tag{3.4}$$

Proof. Use the fact $\beta_1\beta_2 = -1$ in Theorem 3.1. □

Theorem 3.3 (Cassini's identity)**.** *For $n \in \mathbb{N}$, we have*

$$\hat{S}'_{k,n-1} LS_{k,n+1} - \hat{S}'_{k,n} LS_{k,n} = (-1)^{n+1}(k^2 + 4) \begin{bmatrix} -2 + i(k^3 + 2k) \\ -2(k^2 + 1) + i2k \end{bmatrix}. \tag{3.5}$$

Proof. From (1.15), we have

$$P_{k,n-1}P_{k,n+1} - P_{k,n}^2 = -i\hat{S}'_{k,n-1} LS_{k,n+1} + i\hat{S}'_{k,n} LS_{k,n} = -i(\hat{S}'_{k,n-1} LS_{k,n+1} - \hat{S}'_{k,n} LS_{k,n}).$$

Using the Cassini's identity $P_{k,n-1}P_{k,n+1} - P_{k,n}^2 = (-1)^{n+1}(k^2 + 4)(2D_{k,1} - (k^3 + 2k)e_3)$, we may write

$$\hat{S}'_{k,n-1} LS_{k,n+1} - \hat{S}'_{k,n} LS_{k,n} = (-1)^{n+1}(k^2 + 4) \begin{bmatrix} -2F_{k,1}^2 + i\big(2F_{k,1}F_{k,4} - (k^2 + 2)F_{k,2}\big) \\ -2F_{k,1}F_{k,3} + i2F_{k,1}F_{k,2} \end{bmatrix}$$

or

$$\hat{S}'_{k,n-1} LS_{k,n+1} - \hat{S}'_{k,n} LS_{k,n} = (-1)^{n+1}(k^2 + 4) \begin{bmatrix} -2 + i(k^3 + 2k) \\ -2(k^2 + 1) + i2k \end{bmatrix}.$$ □

Theorem 3.4 (Catalan's identity)**.** *For $n \in \mathbb{N}$, we have*

$$\hat{S}'_{k,n-r} LS_{k,n+r} - \hat{S}'_{k,n} LS_{k,n} = (-1)^{n-r+2}(k^2 + 4) \begin{bmatrix} -2F_{k,r}^2 + i\big(2F_{k,r}F_{k,r+3} - (k^2 + 2)F_{k,2r}\big) \\ -2F_{k,r}F_{k,r+2} + i2F_{k,r}F_{k,r+1} \end{bmatrix}. \tag{3.6}$$

Proof. The Catalan identity for the k-Lucas quaternions is given by

$$P_{k,n-r}P_{k,n+r} - P_{k,n}^2 = (-1)^{n-r+2}(k^2+4)(2F_{k,r}D_{k,r} - L_{k,2}F_{k,2r}e_3).$$

Thus,

$$\begin{aligned}\hat{S}'_{k,n-r}LS_{k,n+r} - \hat{S}'_{k,n}LS_{k,n} &= i(-1)^{n-r+2}(k^2+4)\begin{bmatrix}\big(2F_{k,r}F_{k,r+3} - (k^2+2)F_{k,2r}\big) + i2F_{k,r}^2 \\ 2F_{k,r}F_{k,r+1} + i2F_{k,r}F_{k,r+2}\end{bmatrix} \\ &= (-1)^{n-r+2}(k^2+4)\begin{bmatrix}-2F_{k,r}^2 + i\big(2F_{k,r}F_{k,r+3} - (k^2+2)F_{k,2r}\big) \\ -2F_{k,r}F_{k,r+2} + i2F_{k,r}F_{k,r+1}\end{bmatrix},\end{aligned}$$

as required. □

Theorem 3.5 (d'Ocagne's identity)**.** *For $n, r \in \mathbb{N}$, we have*

$$\hat{S}'_{k,r+1}LS_{k,n} - \hat{S}'_{k,r}LS_{k,n+1} = (-1)^{r+1}(k^2+4)\begin{bmatrix}-2F_{k,n-r} + i\big(F_{k,n-r+3} + F_{k,n-r-1}\big) \\ -2F_{k,n-r+2} + i2F_{k,n-r+1}\end{bmatrix}. \quad (3.7)$$

Proof. From (1.15), we have

$$P_{k,r+1}P_{k,n} - P_{k,r}P_{k,n+1} = -i\hat{S}'_{k,r+1}LS_{k,n} + i\hat{S}'_{k,r}LS_{k,n+1} = -i(\hat{S}'_{k,r+1}LS_{k,n} - \hat{S}'_{k,r}LS_{k,n+1}).$$

Next, the d'Ocagne's identity for the k-Lucas quaternions is

$$P_{k,r+1}P_{k,n} - P_{k,r}P_{k,n+1} = (-1)^{r+1}(k^2+4)\big(2D_{k,n-r} + (F_{k,n-m-1} - F_{k,n-m+3})e_3\big).$$

Combining these two identities yields

$$-i(\hat{S}'_{k,r+1}LS_{k,n} - \hat{S}'_{k,r}LS_{k,n+1}) = (-1)^{r+1}(k^2+4)\begin{bmatrix}\big(F_{k,n-r+3} + F_{k,n-r-1}\big) + i2F_{k,n-r} \\ 2F_{k,n-r+1} + i2F_{k,n-r+2}\end{bmatrix}$$

which is the stated result. □

Theorem 3.6 (Honsberger's identity)**.** *For $n, r \in \mathbb{N}$, we have*

$$\hat{S}'_{k,n+1}LS_{k,r} + \hat{S}'_{k,n}LS_{k,r-1} = (k^2+4)\begin{bmatrix}-\big(F_{k,n+r} + F_{k,n+r+2} + F_{k,n+r+4} + F_{k,n+r+6}\big) + i2F_{k,n+r+3} \\ -2F_{k,n+r+2} + i2F_{k,n+r+1}\end{bmatrix}. \quad (3.8)$$

Proof. Using the identity

$$P_{k,n+1}P_{k,r} + P_{k,n}P_{k,r-1} = (k^2+4)\big[2D_{k,n+r} + \big(F_{k,n+r} + F_{k,n+r+2} + F_{k,n+r+4} + F_{k,n+r+6}\big)\big]$$

and (1.15), we have

$$\begin{aligned}P_{k,n+1}P_{k,r} + P_{k,n}P_{k,r-1} &= -i\hat{S}'_{k,n+1}LS_{k,r} + i\hat{S}'_{k,n}LS_{k,r-1} \\ &= -i(\hat{S}'_{k,n+1}LS_{k,r} + \hat{S}'_{k,n}LS_{k,r-1}).\end{aligned}$$

So,
$$
\begin{aligned}
&-i(\hat{S}'_{k,n+1}LS_{k,r}+\hat{S}'_{k,n}LS_{k,r-1})\\
&=(k^2+4)\begin{bmatrix}2F_{k,n+r+3}+i\big(F_{k,n+r}+F_{k,n+r+2}+F_{k,n+r+4}+F_{k,n+r+6}\big)\\ 2F_{k,n+r+1}+i2F_{k,n+r+2}\end{bmatrix}\\
&=(k^2+4)\begin{bmatrix}-\big(F_{k,n+r}+F_{k,n+r+2}+F_{k,n+r+4}+F_{k,n+r+6}\big)+i2F_{k,n+r+3}\\ -2F_{k,n+r+2}+i2F_{k,n+r+1}\end{bmatrix}. \qquad\square
\end{aligned}
$$

Theorem 3.7 (Vajda's identity)**.** *For $n, m, r \in \mathbb{N}$, we have*
$$
\hat{S}'_{k,n+m}LS_{k,n+r}-\hat{S}'_{k,n}LS_{k,n+m+r}=(-1)^n(k^2+4)F_{k,m}\begin{bmatrix}-2F_{k,r}+i\big(F_{k,r+3}-k^2L_{k,r}\big)\\ -2F_{k,r+2}+i2F_{k,r+1}\end{bmatrix}. \tag{3.9}
$$

Proof. From (1.15) it is easy to see that
$$
P_{k,n+m}P_{k,n+r}-P_{k,n}P_{k,n+m+r}=-i(\hat{S}'_{k,n+m}LS_{k,n+r}-\hat{S}'_{k,n}LS_{k,n+m+r})
$$
and
$$
P_{k,n+m}P_{k,n+r}-P_{k,n}P_{k,n+m+r}=(-1)^n(k^2+4)F_{k,m}\big(2D_{k,r}-k^2L_{k,r}e_3\big).
$$
Hence, we can conclude that
$$
-i(\hat{S}'_{k,n+m}LS_{k,n+r}-\hat{S}'_{k,n}LS_{k,n+m+r})=(-1)^n(k^2+4)F_{k,m}\begin{bmatrix}\big(F_{k,r+3}-k^2L_{k,r}\big)+i2F_{k,r}\\ 2F_{k,r+1}+i2F_{k,r+2}\end{bmatrix}. \quad\square
$$

Theorem 3.8 (Generating function)**.** *The generating function for the k-Lucas spinors $\{LS_{k,n}\}$ is*
$$
H_k(x)=\frac{1}{1-kx-x^2}\begin{bmatrix}[(k^3+3k)+x(k^2+2)]+i(2-kx)\\ (k+2x)+i(k^2+2+kx)\end{bmatrix}. \tag{3.10}
$$

Proof. The proof is essentially a copy of the proof of Theorem 2.8 (with different initial values) and omitted. □

Theorem 3.9 (Finite sum formulae)**.** *The sums of the first n terms of the k-Lucas spinor sequence $\{LS_{k,n}\}$ are given by*

1. $\sum_{j=1}^{n} LS_{k,j}=\frac{1}{k}\big[LS_{k,n+1}+LS_{k,n}-2(FS_{k,0}+FS_{k,1})\big]$,
2. $\sum_{j=1}^{n} LS_{k,2j}=\frac{1}{k}[LS_{k,2n+1}-2FS_{k,0}]$,
3. $\sum_{j=1}^{n} LS_{k,2j-1}=\frac{1}{k}[LS_{k,2n}-2FS_{k,1}]$.

Proof. Using the relation $LS_{k,n}=FS_{k,n-1}+FS_{k,n+1}$ and the sum 1. in Theorem 2.9, we may write
$$
\begin{aligned}
\sum_{j=1}^{n} LS_{k,j}&=\sum_{j=1}^{n} FS_{k,j-1}+\sum_{j=1}^{n} FS_{k,j+1}\\
&=\frac{1}{k}\big[FS_{k,n}+FS_{k,n-1}-(FS_{k,0}+FS_{k,1})\big]+\frac{1}{k}\big[FS_{k,n+2}+FS_{k,n+1}-(FS_{k,0}+FS_{k,1})\big]\\
&=\frac{1}{k}\big[FS_{k,n}+FS_{k,n+2}+FS_{k,n-1}+FS_{k,n+1}-2(FS_{k,0}+FS_{k,1})\big]\\
&=\frac{1}{k}\big[LS_{k,n+1}+LS_{k,n}-2(FS_{k,0}+FS_{k,1})\big].
\end{aligned}
$$
The remaining two identities can be easily derived using $LS_{k,n}=FS_{k,n-1}+FS_{k,n+1}$ and the sum identities 2. and 3. in Theorem 2.9. □

4 Combinatorial properties

We conclude with some identities relating k-Fibonacci and k-Lucas spinors. These identities can be proved by using the relations of k-Fibonacci and k-Lucas numbers. So, we only state these identities and omit the proofs.

Theorem 4.1. *The following identities hold true:*

1. $\hat{S}_{k,n}FS_{k,n} = \begin{bmatrix} -(2F_{k,n}^2 - (k^2+2)F_{k,2n+3}) + i\big(2F_{k,n}F_{k,n+3}\big) \\ -(2F_{k,n}F_{k,n+2}) + i(2F_{k,n}F_{k,n+1}) \end{bmatrix}$,
2. $\hat{S}'_{k,n}LS_{k,n} = \begin{bmatrix} -(2L_{k,n}^2 - (k^2+2)(k^2+4)F_{k,2n+3}) + i\big(2L_{k,n}L_{k,n+3}\big) \\ -(2L_{k,n}L_{k,n+2}) + i(2L_{k,n}L_{k,n+1}) \end{bmatrix}$,
3. $\hat{S}_{k,n}FS^*_{k,n} = i\begin{bmatrix} (k^2+2)F_{k,2n+3} \\ 0 \end{bmatrix}$,
4. $\hat{S}'_{k,n}LS^*_{k,n} = i\begin{bmatrix} (k^2+2)(k^2+4)F_{k,2n+3} \\ 0 \end{bmatrix}$,
5. $FS_{k,n} + FS^*_{k,n} = i\begin{bmatrix} 2F_{k,n} \\ 0 \end{bmatrix}$,
6. $LS_{k,n} + LS^*_{k,n} = i\begin{bmatrix} 2L_{k,n} \\ 0 \end{bmatrix}$.

Theorem 4.2. *The following identities hold true:*

1. $FS_{k,n-1} + FS_{k,n+1} = LS_{k,n}$,
2. $LS_{k,n-1} + LS_{k,n+1} = 5FS_{k,n}$,
3. $FS_{k,n} + LS_{k,n} = 2FS_{k,n+1}$,
4. $FS_{k,n+m} + (-1)^m FS_{k,n-m} = L_{k,m}FS_{k,n}$,
5. $FS_{k,n+m} - (-1)^m FS_{k,n-m} = F_{k,m}LS_{k,n}$,
6. $LS_{k,n+m} + (-1)^m LS_{k,n-m} = L_{k,m}LS_{k,n}$,
7. $LS_{k,n+m} - (-1)^m LS_{k,n-m} = 5F_{k,m}FS_{k,n}$,
8. $kFS_{k,n+m} = F_{k,m+1}FS_{k,n+1} - F_{k,m-1}FS_{k,n-1}$,
9. $FS_{k,m+n} = F_{k,m}FS_{k,n+1} + F_{k,m-1}FS_{k,n}$.

Acknowledgement

The authors thank the two anonymous referees for a careful reading of the paper and for comments that improved the quality of our manuscript. The first & second authors acknowledge the University Grant Commission (UGC), India, for providing a fellowship for this research work.

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