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Unrestricted Tribonacci and Tribonacci–Lucas quaternions

Gonca Kızılaslan[1] and Leyla Karabulut[2]

[1] Department of Mathematics, Faculty of Science and Arts,
Kırıkkale University, TR-71450 Kırıkkale, Turkey
e-mail: goncakizilaslan@gmail.com

[2] Department of Mathematics, Graduate School of Natural and Applied Sciences,
Kırıkkale University, TR-71450 Kırıkkale, Turkey
e-mail: leyla.karabulut2006@outlook.com

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Abstract: We define a generalization of Tribonacci and Tribonacci–Lucas quaternions with arbitrary Tribonacci numbers and Tribonacci–Lucas numbers coefficients, respectively. We get generating functions and Binet's formulas for these quaternions. Furthermore, several sum formulas and a matrix representation are obtained.
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1 Introduction

Quaternions have several applications in mathematics, see [13,29,37]. The real quaternion algebra $\mathbf{H}$ has a basis $\{\mathbf{1},\ \mathbf{i},\ \mathbf{j},\ \mathbf{k}\}$ where

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1} \quad \text{and} \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

Thus an element $q \in \mathbf{H}$ can be written as $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. It is a division algebra which is not commutative. The real quaternion $\Re(q) - \Im(q)$, where $\Re(q)$ denotes the real part and

$\Im(q)$ denotes the imaginary part of the quaternion, gives the conjugate of q and we denote this quaternion by q^*. The multiplication of q and q^* gives the norm of q and it is denoted by $N(q)$. Hence for $q \neq 0$, its inverse will be

$$q^{-1} = q^* N(q)^{-1}.$$

Many researchers are interested in quaternions with components chosen in number sequences. One of them is given in [18] and called as Fibonacci quaternions. The n-th Fibonacci quaternion Q_n is defined by

$$Q_n = F_n + F_{n+1}\mathbf{i} + F_{n+2}\mathbf{j} + F_{n+3}\mathbf{k}, \quad n \geq 0,$$

where F_n is the n-th Fibonacci number. The n-th Lucas quaternion $\mathcal{L}_n$ is defined by

$$\mathcal{L}_n = L_n + L_{n+1}\mathbf{i} + L_{n+2}\mathbf{j} + L_{n+3}\mathbf{k}, \quad n \geq 0,$$

where L_n is the n-th Lucas number. After that, these structures attracted a lot of attention and various properties of such sequences were studied by many authors, see [4, 5, 11, 12, 14–17, 19–23, 25, 27, 28, 30–34, 36].

In [6], a generalized Tribonacci sequence $\{V_n\}_{n\geq 0}$ is defined with initial conditions $V_0 = a$, $V_1 = b$, $V_2 = c$ and by the recurrence

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3},\ n \geq 3$$

for a, b, c integers and $r, s, t \in \mathbb{R}$. Then the well-known Tribonacci sequence denoted by $\{T_n\}_n$ is obtained for $(r, s, t) = (1, 1, 1)$ and $(V_0, V_1, V_2) = (1, 1, 2)$, see [9, 10, 24]. On the other hand, the Tribonacci– Lucas sequence $\{K_n\}_n$ is obtained for $(r, s, t) = (1, 1, 1)$ and $(V_0, V_1, V_2) = (3, 1, 3)$, see [38]. For $w = \frac{-1+i\sqrt{3}}{2}$ and

$$\begin{aligned}
\alpha &= \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3} \\
\beta &= \frac{1 + w\sqrt[3]{19 + 3\sqrt{33}} + w^2\sqrt[3]{19 - 3\sqrt{33}}}{3} \\
\gamma &= \frac{1 + w^2\sqrt[3]{19 + 3\sqrt{33}} + w\sqrt[3]{19 - 3\sqrt{33}}}{3},
\end{aligned}$$

the Binet formulæ are

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \tag{1}$$

$$K_n = \alpha^n + \beta^n + \gamma^n, \tag{2}$$

see [38]. Using the sequence $\{V_n\}_{n\geq 0}$, a quaternion sequence of order 3 is defined by,

$$Q_{v,n} = V_n + V_{n+1}\mathbf{i} + V_{n+2}\mathbf{j} + V_{n+3}\mathbf{k},\ n \geq 0,$$

see [6]. In [1], several properties of the classical Tribonacci quaternion sequence $\{TQ_n\}_{n\geq 0}$, which is defined by

$$TQ_n = T_n + T_{n+1}\mathbf{i} + T_{n+2}\mathbf{j} + T_{n+3}\mathbf{k},\ n \geq 0, \tag{3}$$

with Tribonacci number coefficients and the classical Tribonacci–Lucas quaternion sequence $\left\{T\tilde{Q}_n\right\}_{n\geq 0}$, which is defined by

$$T\tilde{Q}_n = K_n + K_{n+1}\mathbf{i} + K_{n+2}\mathbf{j} + K_{n+3}\mathbf{k},\ n \geq 0, \tag{4}$$

with Tribonacci–Lucas number coefficients are presented. The Tribonacci quaternion sequence $\{TQ_n\}_{n\geq 0}$ has the following generating function

$$G(x) = \frac{x + \mathbf{i} + (1 + x + x^2)\mathbf{j} + (2 + 2x + x^2)\mathbf{k}}{1 - x - x^2 - x^3}.$$

For $\overline{\alpha} = 1 + \alpha\mathbf{i} + \alpha^2\mathbf{j} + \alpha^3\mathbf{k}, \overline{\beta} = 1 + \beta\mathbf{i} + \beta^2\mathbf{j} + \beta^3\mathbf{k}$ and $\overline{\gamma} = 1 + \gamma\mathbf{i} + \gamma^2\mathbf{j} + \gamma^3\mathbf{k}$, the Binet formulæ of the sequences $\{TQ_n\}_{n\geq 0}$ and $\{T\tilde{Q}_n\}_{n\geq 0}$ can be obtained by using Equations (1) and (2) as

$$\begin{aligned} TQ_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}\overline{\alpha} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}\overline{\beta} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}\overline{\gamma}, \\ T\tilde{Q}_n &= \alpha^n\overline{\alpha} + \beta^n\overline{\beta} + \gamma^n\overline{\gamma}, \end{aligned}$$

respectively, see [1, 6]. In [26, 35], some generalizations of Tribonacci quaternions are defined and several properties of these quaternions are presented. In [7], another generalization of the Tribonacci and Tribonacci–Lucas quaternion sequences is defined in the sense of polynomials.

One of the common features of all these studies is that the coefficients of the studied quaternion sequences consist of consecutive terms of the selected number sequences. There are also some studies which takes the coefficients of quaternions from the number sequences randomly. In [8], a new class of quaternions whose coefficients are arbitrarily selected from the Fibonacci and Fibonacci-Lucas number sequences is defined. For arbitrary integers $p,\ q$ and r, the n-th unrestricted Fibonacci and Lucas quaternions are defined by,

$$\begin{aligned} \mathcal{F}_n^{(p,r,s)} &= F_n + F_{n+p}\mathbf{i} + F_{n+r}\mathbf{j} + F_{n+s}\mathbf{k}, & (5) \\ \mathcal{L}_n^{(p,r,s)} &= L_n + L_{n+p}\mathbf{i} + L_{n+r}\mathbf{j} + L_{n+s}\mathbf{k}, & (6) \end{aligned}$$

respectively, [8]. By definitions, choosing $p = r = s = -n$ we get the n-th Fibonacci and n-th Lucas numbers. For $p = 1,\ r = s = -n$ in (5), we obtain the n-th Gaussian Fibonacci number. We also get the classical n-th Fibonacci and Lucas quaternions from (5) and (6) as follows

$$\begin{aligned} \mathcal{F}_n^{(1,2,3)} &= Q_n, \\ \mathcal{L}_n^{(1,2,3)} &= \mathcal{L}_n, \end{aligned}$$

see [8]. In [2], the coefficients of quaternions are chosen from Pell and Pell-Lucas numbers and in [3] the coefficients of quaternions are chosen from Fibonacci and Lucas hyper-complex numbers randomly. Inspiring from these studies, we will define a new class of quaternions whose coefficients are arbitrarily chosen from the Tribonacci and Tribonacci–Lucas number sequences in this paper. We also investigate several properties and identities of defined sequences.

2 Unrestricted Tribonacci and Tribonacci–Lucas quaternions

In this section, a generalization of the classical Tribonacci quaternions and Tribonacci–Lucas quaternions are defined and their Binet's formulas and generating functions are obtained. Let p, r and s be arbitrary positive integers throughout the paper.

Definition 1. *The n-th unrestricted Tribonacci quaternion $\mathcal{Q}_n^{(p,r,s)}$ and n-th unrestricted Tribonacci–Lucas quaternion $\tilde{\mathcal{Q}}_n^{(p,r,s)}$ are defined by*

$$\mathcal{Q}_n^{(p,r,s)} = T_n + T_{n+p}\mathbf{i} + T_{n+r}\mathbf{j} + T_{n+s}\mathbf{k},$$
$$\tilde{\mathcal{Q}}_n^{(p,r,s)} = K_n + K_{n+p}\mathbf{i} + K_{n+r}\mathbf{j} + K_{n+s}\mathbf{k}.$$

For $n \geq 0$, we have

$$\mathcal{Q}_{n+3}^{(p,r,s)} = \mathcal{Q}_{n+2}^{(p,r,s)} + \mathcal{Q}_{n+1}^{(p,r,s)} + \mathcal{Q}_n^{(p,r,s)},$$

and

$$\tilde{\mathcal{Q}}_{n+3}^{(p,r,s)} = \tilde{\mathcal{Q}}_{n+2}^{(p,r,s)} + \tilde{\mathcal{Q}}_{n+1}^{(p,r,s)} + \tilde{\mathcal{Q}}_n^{(p,r,s)}.$$

By definitions, we get the well-known sequences by taking special values for p, r and s such as:

$$\begin{aligned} \mathcal{Q}_n^{(-n,-n,-n)} &= T_n, \\ \mathcal{Q}_n^{(1,-n,-n)} &= T_n + iT_{n+1}, \\ \mathcal{Q}_n^{(1,2,3)} &= TQ_n, \\ \tilde{\mathcal{Q}}_n^{(1,2,3)} &= T\tilde{Q}_n, \end{aligned}$$

where TQ_n and $T\tilde{Q}_n$ are defined in (3) and (4). We now present the Binet formulas and generating functions of $\mathcal{Q}_n^{(p,r,s)}$ and $\tilde{\mathcal{Q}}_n^{(p,r,s)}$.

Theorem 2.1. $\mathcal{Q}_n^{(p,r,s)}$ *has the following generating function*

$$\mathcal{G}(x) = \frac{\mathcal{Q}_0^{(p,r,s)} + x(\mathcal{Q}_1^{(p,r,s)} - \mathcal{Q}_0^{(p,r,s)}) + x^2(\mathcal{Q}_2^{(p,r,s)} - \mathcal{Q}_1^{(p,r,s)} - \mathcal{Q}_0^{(p,r,s)})}{1 - x - x^2 - x^3}.$$

Proof. Let the generating function of $\mathcal{Q}_n^{(p,r,s)}$ be

$$\mathcal{G}(x) = \mathcal{Q}_0^{(p,r,s)} + \mathcal{Q}_1^{(p,r,s)}x + \mathcal{Q}_2^{(p,r,s)}x^2 + \cdots + \mathcal{Q}_n^{(p,r,s)}x^n + \cdots.$$

Then the result is obtained by computing $\mathcal{G}(x) - x\mathcal{G}(x) - x^2\mathcal{G}(x) - x^3\mathcal{G}(x)$. $\square$

Let

$$\mathcal{H}(x) = \tilde{\mathcal{Q}}_0^{(p,r,s)} + \tilde{\mathcal{Q}}_1^{(p,r,s)}x + \tilde{\mathcal{Q}}_2^{(p,r,s)}x^2 + \cdots + \tilde{\mathcal{Q}}_n^{(p,r,s)}x^n + \cdots.$$

We can also obtain the generating function for $\tilde{\mathcal{Q}}_n^{(p,r,s)}$ by computing $\mathcal{H}(x) - x\mathcal{H}(x) - x^2\mathcal{H}(x) - x^3\mathcal{H}(x)$ as follows:

$$\mathcal{H}(x) = \frac{\tilde{\mathcal{Q}}_0^{(p,r,s)} + x(\tilde{\mathcal{Q}}_1^{(p,r,s)} - \tilde{\mathcal{Q}}_0^{(p,r,s)}) + x^2(\tilde{\mathcal{Q}}_2^{(p,r,s)} - \tilde{\mathcal{Q}}_1^{(p,r,s)} - \tilde{\mathcal{Q}}_0^{(p,r,s)})}{1 - x - x^2 - x^3}.$$

Theorem 2.2. *For* $\underline{\alpha} = 1+\alpha^p\mathbf{i}+\alpha^r\mathbf{j}+\alpha^s\mathbf{k}$, $\underline{\beta} = 1+\beta^p\mathbf{i}+\beta^r\mathbf{j}+\beta^s\mathbf{k}$ *and* $\underline{\gamma} = 1+\gamma^p\mathbf{i}+\gamma^r\mathbf{j}+\gamma^s\mathbf{k}$, *the Binet formulæ are given by*

$$\begin{aligned}
\mathcal{Q}_n^{(p,r,s)} &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}\underline{\alpha} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}\underline{\beta} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}\underline{\gamma}\\
\tilde{\mathcal{Q}}_n^{(p,r,s)} &= \alpha^n\underline{\alpha} + \beta^n\underline{\beta} + \gamma^n\underline{\gamma}.
\end{aligned}$$

Proof. We use the Equations (1) and (2) and the Definition 1 to obtain the corresponding Binet formula.

$$\begin{aligned}
\mathcal{Q}_n^{(p,r,s)} &= T_n + T_{n+p}\mathbf{i} + T_{n+r}\mathbf{j} + T_{n+s}\mathbf{k}\\
&= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}\\
&\quad + \Big(\frac{\alpha^{n+p+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+p+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+p+1}}{(\gamma-\alpha)(\gamma-\beta)}\Big)\mathbf{i}\\
&\quad + \Big(\frac{\alpha^{n+r+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+r+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+r+1}}{(\gamma-\alpha)(\gamma-\beta)}\Big)\mathbf{j}\\
&\quad + \Big(\frac{\alpha^{n+s+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+s+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+s+1}}{(\gamma-\alpha)(\gamma-\beta)}\Big)\mathbf{k}\\
&= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}(1+\alpha^p\mathbf{i}+\alpha^r\mathbf{j}+\alpha^s\mathbf{k}) + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}(1+\beta^p\mathbf{i}+\beta^r\mathbf{j}+\beta^s\mathbf{k})\\
&\quad + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}(1+\gamma^p\mathbf{i}+\gamma^r\mathbf{j}+\gamma^s\mathbf{k})\\
&= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}\underline{\alpha} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}\underline{\beta} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}\underline{\gamma}.
\end{aligned}$$

The Binet formula for $\tilde{\mathcal{Q}}_n^{(p,r,s)}$ can be obtained similarly. □

3 Some identities of unrestricted Tribonacci and Tribonacci–Lucas quaternions

In this part, we will present some identities of $\mathcal{Q}_n^{(p,r,s)}$ and $\tilde{\mathcal{Q}}_n^{(p,r,s)}$. Since p, r, s are arbitrary positive integers we obtain some identities given in [1] for $p = 1, r = 2, s = 3$. We give the proofs of the first identities given in theorems and the other ones can be obtained similarly.

Theorem 3.1.

$$\begin{aligned}
(\mathcal{Q}_n^{(p,r,s)})^2 &= 2T_n\mathcal{Q}_n^{(p,r,s)} - \mathcal{Q}_n^{(p,r,s)}(\mathcal{Q}_n^{(p,r,s)})^*, \qquad (7)\\
\mathcal{Q}_n^{(p,r,s)} + (\mathcal{Q}_n^{(p,r,s)})^* &= 2T_n,\\
\tilde{\mathcal{Q}}_n^{(p,r,s)} &= \mathcal{Q}_n^{(p,r,s)} + 2\mathcal{Q}_{n-1}^{(p,r,s)} + 3\mathcal{Q}_{n-2}^{(p,r,s)}.
\end{aligned}$$

Proof. Using the definition, we can obtain (7) as follows

$$\begin{aligned}
(\mathcal{Q}_n^{(p,r,s)})^2 &= T_n^2 - T_{n+p}^2 - T_{n+r}^2 - T_{n+s}^2 + 2T_n(T_{n+p}\mathbf{i} + T_{n+r}\mathbf{j} + T_{n+s}\mathbf{k})\\
&= 2T_n(T_n + T_{n+p}\mathbf{i} + T_{n+r}\mathbf{j} + T_{n+s}\mathbf{k}) - (T_n^2 + T_{n+p}^2 + T_{n+r}^2 + T_{n+s}^2)\\
&= 2T_n\mathcal{Q}_n^{(p,r,s)} - \mathcal{Q}_n^{(p,r,s)}(\mathcal{Q}_n^{(p,r,s)})^*
\end{aligned}$$

Hence we get the result. □

Theorem 3.2.
$$
\begin{aligned}
\mathcal{Q}_n^{(p+1,r+1,s+1)} &= \mathcal{Q}_{n+1}^{(p,r,s)} - T_{n-1} - T_{n-2}, \\
\tilde{\mathcal{Q}}_n^{(p+1,r+1,s+1)} &= \tilde{\mathcal{Q}}_{n+1}^{(p,r,s)} - K_{n-1} - K_{n-2}.
\end{aligned} \tag{8}
$$

Proof. We give the proof of (8). By definition,

$$
\begin{aligned}
\mathcal{Q}_n^{(p+1,r+1,s+1)} &= T_n + T_{n+p+1}\mathbf{i} + T_{n+r+1}\mathbf{j} + T_{n+s+1}\mathbf{k} \\
&= (T_{n+1} - T_{n-1} - T_{n-2}) + T_{n+1+p}\mathbf{i} + T_{n+1+r}\mathbf{j} + T_{n+1+s}\mathbf{k} \\
&= T_{n+1} + T_{n+1+p}\mathbf{i} + T_{n+1+r}\mathbf{j} + T_{n+1+s}\mathbf{k} - (T_{n-1} + T_{n-2}) \\
&= \mathcal{Q}_{n+1}^{(p,r,s)} - T_{n-1} - T_{n-2}.
\end{aligned}
$$
□

Theorem 3.3.
$$
\begin{aligned}
\mathcal{Q}_{m+n}^{(p,r,s)} &= \mathcal{Q}_m^{(p,r,s)} K_n - \mathcal{Q}_{m-n}^{(p,r,s)} C_n + \mathcal{Q}_{m-2n}^{(p,r,s)}, \\
\tilde{\mathcal{Q}}_{m+n}^{(p,r,s)} &= \tilde{\mathcal{Q}}_m^{(p,r,s)} K_n - \tilde{\mathcal{Q}}_{m-n}^{(p,r,s)} C_n + \underline{C}_{2n-m}, \\
\mathcal{Q}_{n+2m}^{(p,r,s)} &= K_m \mathcal{Q}_{n+m}^{(p,r,s)} - K_{-m} \mathcal{Q}_n^{(p,r,s)} + \mathcal{Q}_{n-2m}^{(p,r,s)}, \\
\mathcal{Q}_{m+n+1}^{(p,r,s)} &= T_{m-2} \mathcal{Q}_n^{(p,r,s)} + (T_{m-3} + T_{m-2}) \mathcal{Q}_{n+1}^{(p,r,s)} + T_{m-1} \mathcal{Q}_{n+2}^{(p,r,s)}, \quad n \geq 0, m \geq 3.
\end{aligned} \tag{9}
$$

where
$$
C_n = \alpha^n \beta^n + \alpha^n \gamma^n + \beta^n \gamma^n
$$

and
$$
\underline{C}_{2n-m} = C_{2n-m} + \mathbf{i} C_{2n-m-1} + \mathbf{j} C_{2n-m-2} + \mathbf{k} C_{2n-m-3}.
$$

Proof. Since the Tribonacci numbers and Tribonacci–Lucas numbers satisfy the following equalities,

$$
\begin{aligned}
T_{m+n} &= T_m K_n - T_{m-n} C_n + T_{m-2n}, \\
K_{m+n} &= K_m K_n - K_{m-n} C_n + C_{2n-m},
\end{aligned}
$$

see [38], we have

$$
\begin{aligned}
\mathcal{Q}_{m+n}^{(p,r,s)} &= T_{m+n} + T_{m+n+p}\mathbf{i} + T_{m+n+r}\mathbf{j} + T_{m+n+s}\mathbf{k} \\
&= (T_m K_n - T_{m-n} C_n + T_{m-2n}) + (T_{m+p} K_n - T_{m+p-n} C_n + T_{m+p-2n})\mathbf{i} \\
&\quad + (T_{m+r} K_n - T_{m+r-n} C_n + T_{m+r-2n})\mathbf{j} + (T_{m+s} K_n - T_{m+s-n} C_n + T_{m+s-2n})\mathbf{k} \\
&= (T_m + T_{m+p}\mathbf{i} + T_{m+r}\mathbf{j} + T_{m+s}\mathbf{k}) K_n - (T_{m-n} + T_{m-n+p}\mathbf{i} + T_{m-n+r}\mathbf{j} + T_{m-n+s}\mathbf{k}) C_n \\
&\quad + (T_{m-2n} + T_{m-2n+p}\mathbf{i} + T_{m-2n+r}\mathbf{j} + T_{m-2n+s}\mathbf{k}) \\
&= \mathcal{Q}_m^{(p,r,s)} K_n - \mathcal{Q}_{m-n}^{(p,r,s)} C_n + \mathcal{Q}_{m-2n}^{(p,r,s)}.
\end{aligned}
$$
□

The following theorem gives some finite sum identities of defined quaternions.

Theorem 3.4.
$$
\begin{aligned}
\sum_{k=0}^{n} \mathcal{Q}_k^{(p,r,s)} &= \frac{\mathcal{Q}_{n+2}^{(p,r,s)} + \mathcal{Q}_n^{(p,r,s)} + \mathcal{Q}_0^{(p,r,s)} - \mathcal{Q}_2^{(p,r,s)}}{2}, \\
\sum_{k=0}^{n} \mathcal{Q}_{2k}^{(p,r,s)} &= \frac{\mathcal{Q}_{2n+1}^{(p,r,s)} + \mathcal{Q}_{2n}^{(p,r,s)} - \mathcal{Q}_1^{(p,r,s)} + \mathcal{Q}_0^{(p,r,s)}}{2},
\end{aligned} \tag{10}
$$

$$\begin{aligned}
\sum_{k=0}^{n}\mathcal{Q}_{2k+1}^{(p,r,s)} &= \frac{\mathcal{Q}_{2n+2}^{(p,r,s)}+\mathcal{Q}_{2n+1}^{(p,r,s)}+\mathcal{Q}_{1}^{(p,r,s)}-\mathcal{Q}_{2}^{(p,r,s)}}{2},\\
\sum_{k=0}^{n}\mathcal{Q}_{3k}^{(p,r,s)} &= \sum_{k=0}^{3n-1}\mathcal{Q}_{k}^{(p,r,s)}+\mathcal{Q}_{0}^{(p,r,s)}\\
&= \frac{\mathcal{Q}_{3n+2}^{(p,r,s)}-\mathcal{Q}_{3n}^{(p,r,s)}+3\mathcal{Q}_{0}^{(p,r,s)}-\mathcal{Q}_{2}^{(p,r,s)}}{2},\\
\sum_{k=0}^{n}\mathcal{Q}_{4k}^{(p,r,s)} &= \frac{\mathcal{Q}_{4n+2}^{(p,r,s)}+\mathcal{Q}_{4n}^{(p,r,s)}+3\mathcal{Q}_{0}^{(p,r,s)}-\mathcal{Q}_{2}^{(p,r,s)}}{4}.
\end{aligned}$$

Proof. We give the proof of (10) by induction on n. The equality holds for $n=0$, since

$$\mathcal{Q}_{0}^{(p,r,s)}=\frac{\mathcal{Q}_{2}^{(p,r,s)}+\mathcal{Q}_{0}^{(p,r,s)}+\mathcal{Q}_{0}^{(p,r,s)}-\mathcal{Q}_{2}^{(p,r,s)}}{2}.$$

Suppose the equality holds for $n=m$. Let $n=m+1$. Then,

$$\sum_{k=0}^{m+1}\mathcal{Q}_{k}^{(p,r,s)}=\sum_{k=0}^{m}\mathcal{Q}_{k}^{(p,r,s)}+\mathcal{Q}_{m+1}^{(p,r,s)}.$$

From the hypothesis, we can write

$$\begin{aligned}
\sum_{k=0}^{m}\mathcal{Q}_{k}^{(p,r,s)}+\mathcal{Q}_{m+1}^{(p,r,s)} &= \frac{\mathcal{Q}_{m+2}^{(p,r,s)}+\mathcal{Q}_{m}^{(p,r,s)}+\mathcal{Q}_{0}^{(p,r,s)}-\mathcal{Q}_{2}^{(p,r,s)}}{2}+\mathcal{Q}_{m+1}^{(p,r,s)}\\
&= \frac{\mathcal{Q}_{m+2}^{(p,r,s)}+\mathcal{Q}_{m}^{(p,r,s)}+\mathcal{Q}_{0}^{(p,r,s)}-\mathcal{Q}_{2}^{(p,r,s)}+2\mathcal{Q}_{m+1}^{(p,r,s)}}{2}\\
&= \frac{\mathcal{Q}_{m+3}^{(p,r,s)}+\mathcal{Q}_{m+1}^{(p,r,s)}+\mathcal{Q}_{0}^{(p,r,s)}-\mathcal{Q}_{2}^{(p,r,s)}}{2}.
\end{aligned}$$

Hence, the results follows by induction. □

Let

$$\begin{aligned}
R_n &= 3T_{n+1}-T_n \quad (n\geq 0)\\
\tilde{R}_n &= R_n+R_{n+1}\mathbf{i}+R_{n+2}\mathbf{j}+R_{n+3}\mathbf{k}
\end{aligned}$$

and let for $n\geq 2$

$$\begin{aligned}
U_n &= T_{n-1}+T_{n-2}\\
\tilde{U}_n &= U_n+U_{n+1}\mathbf{i}+U_{n+2}\mathbf{j}+U_{n+3}\mathbf{k},
\end{aligned}$$

with initial conditions $U_0=U_1=0$. Using the definitions of these sequences, we define the related quaternion sequences as follows:

$$\begin{aligned}
\mathcal{R}_{n}^{(p,r,s)} &= R_n+R_{n+p}\mathbf{i}+R_{n+r}\mathbf{j}+R_{n+s}\mathbf{k},\\
\mathcal{U}_{n}^{(p,r,s)} &= U_n+U_{n+p}\mathbf{i}+U_{n+r}\mathbf{j}+U_{n+s}\mathbf{k}.
\end{aligned}$$

Theorem 3.5.

$$\begin{aligned}
\sum_{k=0}^{n}\mathcal{U}_k^{(p,r,s)} &= \mathcal{Q}_{n+1}^{(p,r,s)}-\mathcal{Q}_0^{(p,r,s)}-1, \qquad (11)\\
\sum_{k=0}^{n}\mathcal{Q}_k^{(p,r,s)} &= \frac{\mathcal{U}_{n+2}^{(p,r,s)}+\mathcal{U}_{n+1}^{(p,r,s)}+\mathcal{Q}_0^{(p,r,s)}-\mathcal{Q}_2^{(p,r,s)}}{2},\\
\sum_{k=0}^{n}\mathcal{R}_k^{(p,r,s)} &= \mathcal{Q}_{n+3}^{(p,r,s)}+2\mathcal{Q}_{n+1}^{(p,r,s)}-\mathcal{Q}_2^{(p,r,s)}-2\mathcal{Q}_0^{(p,r,s)},\\
\sum_{k=0}^{n}\mathcal{U}_{3k}^{(p,r,s)} &= \mathcal{Q}_{3n}^{(p,r,s)}-\mathcal{Q}_0^{(p,r,s)}+\mathcal{U}_0^{(p,r,s)},\\
\sum_{k=0}^{n}\mathcal{U}_{3k+1}^{(p,r,s)} &= \mathcal{Q}_{3n+1}^{(p,r,s)}-\mathcal{Q}_0^{(p,r,s)}+\mathcal{U}_1^{(p,r,s)}-\mathcal{U}_0^{(p,r,s)}.
\end{aligned}$$

Proof. We give the proof of (11).

$$\begin{aligned}
\sum_{k=0}^{n}\mathcal{U}_k^{(p,r,s)} &= \mathcal{U}_0^{(p,r,s)}+\mathcal{U}_1^{(p,r,s)}+\cdots+\mathcal{U}_n^{(p,r,s)}\\
&= (U_0+U_p\mathbf{i}+U_r\mathbf{j}+U_s\mathbf{k})+(U_1+U_{p+1}\mathbf{i}+U_{r+1}\mathbf{j}+U_{s+1}\mathbf{k})\\
&\quad+\cdots+(U_n+U_{p+n}\mathbf{i}+U_{r+n}\mathbf{j}+U_{s+n}\mathbf{k})\\
&= (U_0+U_1+\cdots+U_n)+(U_p+U_{p+1}+\cdots+U_{p+n})\mathbf{i}\\
&\quad+(U_r+U_{r+1}+\cdots+U_{r+n})\mathbf{j}+(U_s+U_{s+1}+\cdots+U_{s+n})\mathbf{k}
\end{aligned}$$

Since $\sum_{k=0}^{n}U_k=T_{n+1}-1$, we get

$$\begin{aligned}
\sum_{k=0}^{n}\mathcal{U}_k^{(p,r,s)} &= (T_{n+1}-1)+[(T_{n+p+1}-1)-(T_p-1)]\mathbf{i}\\
&\quad+[(T_{n+r+1}-1)-(T_r-1)]\mathbf{j}+[(T_{n+s+1}-1)-(T_s-1)]\mathbf{k}\\
&= (T_{n+1}+T_{n+p+1}\mathbf{i}+T_{n+r+1}\mathbf{j}+T_{n+s+1}\mathbf{k})-(T_p\mathbf{i}+T_r\mathbf{j}+T_s\mathbf{k})-1\\
&= \mathcal{Q}_{n+1}^{(p,r,s)}-\mathcal{Q}_0^{(p,r,s)}-1.
\end{aligned}$$

□

4 A matrix representation

For $\mathcal{Q}_n^{(p,r,s)}$, we present a matrix generator as follows.

Theorem 4.1. *Let*

$$U=\begin{bmatrix}0&1&0\\0&0&1\\1&1&1\end{bmatrix} \text{ and } Q=\begin{bmatrix}\mathcal{Q}_2^{(p,r,s)}-\mathcal{Q}_1^{(p,r,s)}-\mathcal{Q}_0^{(p,r,s)} & \mathcal{Q}_1^{(p,r,s)}-\mathcal{Q}_0^{(p,r,s)} & \mathcal{Q}_1^{(p,r,s)}\\ \mathcal{Q}_0^{(p,r,s)} & \mathcal{Q}_2^{(p,r,s)}-\mathcal{Q}_1^{(p,r,s)} & \mathcal{Q}_1^{(p,r,s)}\\ \mathcal{Q}_2^{(p,r,s)} & \mathcal{Q}_2^{(p,r,s)}+\mathcal{Q}_1^{(p,r,s)} & \mathcal{Q}_3^{(p,r,s)}\end{bmatrix}.$$

Then for $k\geq 2$, we have

$$QU^k=\begin{bmatrix}\mathcal{Q}_{k-1}^{(p,r,s)} & \mathcal{Q}_{k-1}^{(p,r,s)}+\mathcal{Q}_{k-2}^{(p,r,s)} & \mathcal{Q}_k^{(p,r,s)}\\ \mathcal{Q}_k^{(p,r,s)} & \mathcal{Q}_k^{(p,r,s)}+\mathcal{Q}_{k-1}^{(p,r,s)} & \mathcal{Q}_{k+1}^{(p,r,s)}\\ \mathcal{Q}_{k+1}^{(p,r,s)} & \mathcal{Q}_{k+1}^{(p,r,s)}+\mathcal{Q}_k^{(p,r,s)} & \mathcal{Q}_{k+2}^{(p,r,s)}\end{bmatrix}.$$

Proof. For $k=2$,

$$QU^2=\begin{bmatrix}\mathcal{Q}_2^{(p,r,s)}-\mathcal{Q}_1^{(p,r,s)}-\mathcal{Q}_0^{(p,r,s)} & \mathcal{Q}_1^{(p,r,s)}-\mathcal{Q}_0^{(p,r,s)} & \mathcal{Q}_1^{(p,r,s)}\\ \mathcal{Q}_0^{(p,r,s)} & \mathcal{Q}_2^{(p,r,s)}-\mathcal{Q}_1^{(p,r,s)} & \mathcal{Q}_1^{(p,r,s)}\\ \mathcal{Q}_2^{(p,r,s)} & \mathcal{Q}_2^{(p,r,s)}+\mathcal{Q}_1^{(p,r,s)} & \mathcal{Q}_3^{(p,r,s)}\end{bmatrix}\begin{bmatrix}0&0&1\\1&1&1\\1&2&2\end{bmatrix}$$
$$=\begin{bmatrix}\mathcal{Q}_1^{(p,r,s)} & \mathcal{Q}_1^{(p,r,s)}+\mathcal{Q}_0^{(p,r,s)} & \mathcal{Q}_2^{(p,r,s)}\\ \mathcal{Q}_2^{(p,r,s)} & \mathcal{Q}_2^{(p,r,s)}+\mathcal{Q}_1^{(p,r,s)} & \mathcal{Q}_2^{(p,r,s)}+\mathcal{Q}_1^{(p,r,s)}+\mathcal{Q}_0^{(p,r,s)}\\ \mathcal{Q}_2^{(p,r,s)}+\mathcal{Q}_1^{(p,r,s)}+\mathcal{Q}_0^{(p,r,s)} & 2\mathcal{Q}_2^{(p,r,s)}+\mathcal{Q}_1^{(p,r,s)}+\mathcal{Q}_0^{(p,r,s)} & 2\mathcal{Q}_2^{(p,r,s)}+2\mathcal{Q}_1^{(p,r,s)}+\mathcal{Q}_0^{(p,r,s)}\end{bmatrix}$$
$$=\begin{bmatrix}\mathcal{Q}_1^{(p,r,s)} & \mathcal{Q}_1^{(p,r,s)}+\mathcal{Q}_0^{(p,r,s)} & \mathcal{Q}_2^{(p,r,s)}\\ \mathcal{Q}_2^{(p,r,s)} & \mathcal{Q}_2^{(p,r,s)}+\mathcal{Q}_1^{(p,r,s)} & \mathcal{Q}_3^{(p,r,s)}\\ \mathcal{Q}_3^{(p,r,s)} & \mathcal{Q}_3^{(p,r,s)}+\mathcal{Q}_2^{(p,r,s)} & \mathcal{Q}_4^{(p,r,s)}\end{bmatrix},$$

which is true. Suppose

$$QU^k=\begin{bmatrix}\mathcal{Q}_{k-1}^{(p,r,s)} & \mathcal{Q}_{k-1}^{(p,r,s)}+\mathcal{Q}_{k-2}^{(p,r,s)} & \mathcal{Q}_k^{(p,r,s)}\\ \mathcal{Q}_k^{(p,r,s)} & \mathcal{Q}_k^{(p,r,s)}+\mathcal{Q}_{k-1}^{(p,r,s)} & \mathcal{Q}_{k+1}^{(p,r,s)}\\ \mathcal{Q}_{k+1}^{(p,r,s)} & \mathcal{Q}_{k+1}^{(p,r,s)}+\mathcal{Q}_k^{(p,r,s)} & \mathcal{Q}_{k+2}^{(p,r,s)}\end{bmatrix}$$

is true. Then

$$\begin{aligned}QU^{k+1} &= QU^kU\\ &= \begin{bmatrix}\mathcal{Q}_{k-1}^{(p,r,s)} & \mathcal{Q}_{k-1}^{(p,r,s)}+\mathcal{Q}_{k-2}^{(p,r,s)} & \mathcal{Q}_k^{(p,r,s)}\\ \mathcal{Q}_k^{(p,r,s)} & \mathcal{Q}_k^{(p,r,s)}+\mathcal{Q}_{k-1}^{(p,r,s)} & \mathcal{Q}_{k+1}^{(p,r,s)}\\ \mathcal{Q}_{k+1}^{(p,r,s)} & \mathcal{Q}_{k+1}^{(p,r,s)}+\mathcal{Q}_k^{(p,r,s)} & \mathcal{Q}_{k+2}^{(p,r,s)}\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\1&1&1\end{bmatrix}\\ &= \begin{bmatrix}\mathcal{Q}_k^{(p,r,s)} & \mathcal{Q}_k^{(p,r,s)}+\mathcal{Q}_{k-1}^{(p,r,s)} & \mathcal{Q}_{k+1}^{(p,r,s)}\\ \mathcal{Q}_{k+1}^{(p,r,s)} & \mathcal{Q}_{k+1}^{(p,r,s)}+\mathcal{Q}_k^{(p,r,s)} & \mathcal{Q}_{k+2}^{(p,r,s)}\\ \mathcal{Q}_{k+2}^{(p,r,s)} & \mathcal{Q}_{k+2}^{(p,r,s)}+\mathcal{Q}_{k+1}^{(p,r,s)} & \mathcal{Q}_{k+3}^{(p,r,s)}\end{bmatrix}.\end{aligned}$$

Hence, the proof is completed. □

5 Conclusion

Quaternion sequences and their several properties have been studied by many researchers. The coefficients of these sequences are commonly chosen from the number sequences with characteristic polynomial of degree two. One of the well known number sequence with characteristic polynomial of degree three is known as Tribonacci sequence. Recently, quaternions with components chosen from Tribonacci numbers have also attracted attention, [1, 6, 7, 26, 35]. We can see that one of the common property of the studies on Tribonacci quaternions is that the coefficients consist of consecutive terms of the Tribonacci number sequence. We define quaternion sequences whose coefficients are arbitrarily chosen from the Tribonacci and Tribonacci–Lucas number sequences based on the ideas in the articles which takes the similar forms for generalized Fibonacci quaternion sequences, see [2, 3, 8]. We obtain some properties and several identities of these sequences.

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