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A class of solutions of the equation $d\left(n^2\right) = d\left(\varphi\left(n\right)\right)$

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Abstract: For any positive integer n let $d\left(n\right)$ and $\varphi\left(n\right)$ be the number of divisors of n and the Euler's phi function of n, respectively. In this paper we present some notes on the equation $d\left(n^2\right) = d\left(\varphi\left(n\right)\right)$. In fact, we characterize a class of solutions that have at most three distinct prime factors. Moreover, we show that Dickson's conjecture implies that $d\left(n^2\right) = d\left(\varphi\left(n\right)\right)$ infinitely often.

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1 Introduction

Let $d(n)$ be the divisor function, which counts the number of positive divisors of n, i.e., if n has the prime factorization $n = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k}$ with distinct primes $q_1, q_2, \ldots, q_k$ and positive integers $a_1, a_2, \ldots, a_k$, then

$$d(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1).$$

Let $\varphi(n)$ be the Euler function, which counts the number of positive integers $m \leq n$ with $(m, n) = 1$. It is well-known that

$$\varphi(n) = q_1^{a_1-1}(q_1 - 1) q_2^{a_2-1}(q_2 - 1) \cdots q_k^{a_k-1}(q_k - 1).$$

Various diophantine equations involving the divisor function and Euler's phi function were investigated by many authors (see [1, 2, 6–8, 10, 11]). In [4, Problem 705, page 78], it is shown that $\varphi(d(n)) = d(\varphi(n))$ has infinitely many solutions; while in [12, pages 110–111], it is shown that $d(n) = \varphi(n)$ has the only solutions $1, 3, 8, 10, 24$ and 30, where $d(n) < \varphi(n)$ for $n \geq 31$. Using these multiplicative functions, we are interested here in problems involving the number of positive divisors of $\varphi(n)$. In fact, in the present work, we compare the value of the divisor function to its value at Euler's functions. More precisely, we aim to prove that the diophantine equation

$$d\left(n^2\right) = d(\varphi(n)) \tag{1}$$

has infinitely many integer solutions as well as we identify large families of solutions. The first few terms are:

$$1, 5, 57, 74, 202, 292, 394, 514, 652, 1354, 2114, 2125, \ldots.$$

For this purpose, define

$$\mathbb{S} := \left\{n \in \mathbb{N} : d\left(n^2\right) = d(\varphi(n))\right\}.$$

In this paper, we characterize the elements of $\mathbb{S}$ that have at most three distinct prime factors. The problem is interesting because it can force us to solve some diophantine equations involving prime numbers. Note also that the proofs are all on the elementary side and depend on long case by case analysis type arguments.

Recall that the Fermat numbers are the sequence (F_n) of positive integers defined by

$$F_n = 2^{2^n} + 1, n = 0, 1, \ldots.$$

If a particular F_m is prime it is called a Fermat prime. The only known Fermat primes are F_0, F_1, F_2, F_3 and F_4 and it has been conjectured that there are only finitely many. On the other hand, if $p = 2^k + 1$ is a prime then $k = 2^n$ for some n and p is a Fermat prime.

It is well-known that $d(n) = 2$ if and only if n is prime and that $d(n)$ is prime if and only if $n = p^{q-1}$, where p and q are both prime. Note also that if n is a prime power, namely $n = p^a$ with $p \geq 2$ and $a \geq 1$, then $n \in \mathbb{S}$ implies $(2a + 1) = d(p - 1)\, a$. But the last equation is only true for $a = 1$ and $p = 5$. Hence, $n = 5$. Observe first of all that there is a connection between Fermat primes and the solutions of the equation (1), where F_1 is the unique prime solution. Moreover, if $n = q_1^{a_1} q_2^{a_2} \cdots q_k^{a_k} \in \mathbb{S}$, where $q_1 < q_2 < \cdots < q_k$ are primes and $a_1, a_2, \ldots, a_k$ are positive integers, then a_k must be odd. In fact, since $(q_1 \cdots q_{k-1}(q_1 - 1) \cdots (q_k - 1), q_k) = 1$ we conclude that

$$d(\varphi(n)) = d\left(q_1^{a_1-1} q_2^{a_2-1} \cdots q_k^{a_k-1}(q_1 - 1)(q_2 - 1) \cdots (q_k - 1)\right) a_k,$$

and so a_k must be odd since $d(n^2) = \prod_{i=1}^{k}(2a_i + 1)$.

Now, let n as above and put

$$\prod_{i=1}^{k}(q_i - 1) = 2^{x_1+x_2+\cdots+x_k} \cdot q_1^{\alpha_1^{(2)}+\alpha_1^{(3)}+\cdots+\alpha_1^{(k)}} \cdots q_{k-2}^{\alpha_{k-2}^{(k-1)}+\alpha_{k-2}^{(k)}} \cdot q_{k-1}^{\alpha_{k-1}^{(k)}} \cdot m,$$

where $x_1, x_2, \ldots, x_k, m \geq 1$ and $\alpha_1^{(2)}, \ldots, \alpha_1^{(k)}, \alpha_2^{(3)}, \ldots, \alpha_2^{(k)}, \ldots, \alpha_{k-2}^{(k-1)}, \alpha_{k-2}^{(k)}, \alpha_{k-1}^{(k)}$ are non-negative integers with $(2q_1q_2 \cdots q_{k-1}, m) = 1$. Thus in order to prove that n satisfies (1), it suffices to confirm that the exponents of the prime factors of n and the above variables satisfy the following diophantine equation:

$$\prod_{i=1}^{k}(2a_i + 1) = \left(1 + \sum_{i=1}^{k} x_i\right)\left(a_1 + \sum_{i=2}^{k} \alpha_1^{(i)}\right)\left(a_2 + \sum_{i=3}^{k} \alpha_2^{(i)}\right) \cdots \left(a_{k-1} + \alpha_{k-1}^{(k)}\right) a_k d(m). \tag{2}$$

In particular, if $k = 3$ and n is odd then we need to solve the diophantine equation

$$(2a + 1)(2b + 1)(2c + 1) = (x_1 + x_2 + x_3 + 1)(a_1 + \alpha_1 + \alpha_2)(b + \alpha_3)\, c \cdot d(m),$$

where $m, x_1, x_2, x_3 \geq 1$ and $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with $(2q_1q_2, m) = 1$.

Now we are in a position to state the main results of the paper.

2 Solutions having two distinct prime factors

Assume that $n = q_1^a q_2^b \in \mathbb{S}$, where q_1, q_2 are distinct primes with $2 \leq q_1 < q_2$ and $a, b \geq 1$. Since $((q_1 - 1)(q_2 - 1), q_2) = 1$, we obtain

$$(2a + 1)(2b + 1) = d\left((q_1 - 1)(q_2 - 1)\, q_1^{a-1}\right) b. \tag{3}$$

2.1 n is square-free

We have the following result:

Proposition 2.1. *The only square-free solutions of the form q_1q_2 are:*

i) $n = 3 \cdot 19$.

ii) $n = 2F_3$*, where* $F_3 = 257$.

iii) $n = 2(4p^2 + 1)$*, where p and $4p^2 + 1$ are simultaneously primes.*

We need the following lemma.

Lemma 2.1. *Let p be a prime number with $p \geq 5$. Then the number $2p^{2a} + 1$ is composite for every $a \geq 1$. In particular, if $p \equiv 1 (\mathrm{mod}\ 3)$, then the number $2p^a + 1$ is composite for every $a \geq 2$.*

Proof. First, it is clear that if $p = 3k + 1$ with $k \geq 2$, then $2p^{2a} + 1 = 2(3k+1)^{2a} \equiv 0(\bmod 3)$. That is, $2p^{2a} + 1$ is a multiple of 3. But, if $p = 3k + 2$ for some $k \geq 1$, then $p = 3k' - 1$ with $k' = k + 1$ and so

$$2p^{2a} + 1 = 2(3k' - 1)^{2a} + 1 = 2\left[\sum_{i=1}^{2n} (-1)^{2n-i} \binom{i}{2n} (3k')^i\right] + 1 \equiv 0(\bmod 3),$$

which is also a multiple of 3. By the same way, if $a \geq 2$ and $p \equiv 1(\bmod 3)$, then $2p^a + 1$ is a multiple of 3. This completes the proof. $\square$

Remark 2.2. *Let p be a prime number with $p \geq 5$ and let $a \geq 1$. Similar to what we have shown in Lemma 2.1, if r is odd then the number $2^r p^{2a} + 1$ is composite. However, if r is odd and $p \equiv 1(\bmod 3)$, then the number $2^r p^a + 1$ is also composite.*

Proof of Proposition 2.1. Suppose that $n = pq$, where p and q are odd primes with $p < q$. By (3), we have

$$9 = d\left((p-1)(q-1)\right). \tag{4}$$

We put $p - 1 = 2^s m_1$ and $q - 1 = 2^{s'} m_2$, where m_1, m_2 are odd and $s, s' \geq 1$. From (4), we obtain $9 = (s + s' + 1)\, d(m_1 m_2)$. We distinguish the following cases:

Case 1. $s + s' = 2$ and $d(m_1 m_2) = 3$. That is, $s = s' = 1$ and $m_1 m_2 = r^2$ for some prime $r \geq 3$. On the other hand, since $p < q$ we conclude that $m_1 = 1$ and $m_2 = r^2$. Hence, $p = 3$ and $q = 2r^2 + 1$. But, by Lemma 2.1, the number $2r^2 + 1$ is a multiple of 3 for $r \geq 5$, in which case $n = 3 \cdot 19$ is the only solution of this form.

Case 2. $s + s' = 8$ and $d(m_1 m_2) = 1$. That is, $m_1 = m_2 = 1$. Therefore, p and q are Fermat primes and hence s, s' are powers of 2. This case is not valid since $p < q$.

Now, assume that $n = 2q$ with $q \geq 3$ is prime. By (3), $9 = d(q-1)$ from which it follows that $q - 1$ is either 2^8 or $2^2 p^2$ for some prime $p \geq 3$. Hence, $n = 2 \cdot 257 = 2F_3$ or $n = 2(4p^2 + 1)$ with p and $4p^2 + 1$ are simultaneously primes.

This completes the proof. $\square$

2.2 n is not square-free with n odd

Assume that n is **odd**. We have the following results:

Proposition 2.3. *Let $n = q_1^a q_2$, where $3 \leq q_1 < q_2$ and $a \geq 2$. If $n \in \mathbb{S}$, then n is one of the numbers:*

- $n = F_1^3 F_2$.
- $n = 3^{5t-3}(2^3 \cdot 3^t + 1)$, *where $t \geq 2$ and $2^3 \cdot 3^t + 1$ is prime.*
- $n = 5^{5t-3}(2^2 \cdot 5^t + 1)$, *where $t \geq 2$ and $2^2 \cdot 5^t + 1$ is prime.*
- $n = 3^{t-1}(2 \cdot 3^t + 1)$, *where $t \geq 4$ and $2 \cdot 3^t + 1$ is prime.*

Proof. By (3), we have

$$3(2a+1) = d\left((q_1 - 1)(q_2 - 1)\, q_1^{a-1}\right). \tag{5}$$

There are two cases:

Case 1. $(q_2-1,q_1)=1$. We put $q_1-1=2^s m_1$ and $q_2-1=2^{s'}m_2$, where $(2,m_1m_2)=1$ and $s,s'\geq 1$. From (5), we have $3=a\left(d\left(m_1m_2\right)\left(s+s'+1\right)-6\right)$. Since $a\geq 2$, it follows that $a=3$ and $s+s'=6$. Hence, $m_1=m_2=1$ and so we must have $s=2$ and $s'=4$. That is, $q_1=5$ and $q_2=17$, in which case $n=5^3\cdot 17$.

Case 2. $(q_2-1,q_1)=q_1$. As above, we put $q_1-1=2^s m_1$ and $q_2-1=2^{s'}q_1^t m_2$, where $(2,m_1m_2)=1$ and $s,s',t\geq 1$. By (5) we have

$$3\left(2a+1\right)=d\left(m_1m_2\right)\left(s+s'+1\right)\left(a+t\right). \tag{6}$$

It is clear that $d\left(m_1m_2\right)$ cannot be ≥ 3, otherwise

$$3\left(2a+1\right)\geq 3\left(a+t\right)\left(s+s'+1\right)\geq 9\left(a+1\right)>6a+3,$$

a contradiction. Moreover, if $d\left(m_1m_2\right)=2$, then the right-hand side of (6) is even, while its left-hand side is odd, also a contradiction. Therefore, $d\left(m_1m_2\right)=1$ and so by (6) once again, $3\left(2a+1\right)=\left(2^i+s'+1\right)\left(a+t\right)$ for some $i\geq 0$. Note also that $2^i+s'+1$ cannot be ≥ 6, otherwise $3\left(2a+1\right)\geq 6\left(a+t\right)>6a+3$, a contradiction. Consequently, we have either $2^i+s'=4$ or $2^i+s'=2$.

(i) $2^i+s'=4$. There are two possibilities:
- $i=0$ and $s'=3$. It follows that $a=5t-3$, $q_1=3$ and $q_2=2^3\cdot 3^t+1$, thus $n=3^{5t-3}\left(2^3\cdot 3^t+1\right)$, where $2^3\cdot 3^t+1$ is prime. For example, for $t=2$, we get $n=3^7\cdot 73$.
- $i=1$ and $s'=2$. That is, $a=5t-3$, $q_1=5$ and $q_2=2^2\cdot 5^t+1$, thus $n=5^{5t-3}\left(2^2\cdot 5^t+1\right)$, where $2^2\cdot 5^t+1$ is prime. For example, for $t=2$, we have $n=5^7\cdot 101$.

(ii) $2^i+s'=2$. That is, $i=0$, $s'=1$ and so $a=t-1$. Hence, $q_1=3$ and $q_2=2\cdot 3^t+1$. Consequently, $n=3^{t-1}\left(2\cdot 3^t+1\right)$, where $\left(2\cdot 3^t+1\right)$ is prime. For example, for $t=4$, we have $n=3^3\cdot 163$.

This completes the proof. □

Proposition 2.4. *The number $n=F_1F_2^3$ is the only solution of the form $q_1q_2^b$, where $3\leq q_1<q_2$ and $b\geq 2$.*

Proof. Assume that $n=q_1q_2^b\in\mathbb{S}$, where $3\leq q_1<q_2$ and $b\geq 2$. Applying (3), we obtain $3\left(2b+1\right)=d\left(\left(q_1-1\right)\left(q_2-1\right)\right)b$, from which it follows that $b\left(d\left(\left(q_1-1\right)\left(q_2-1\right)\right)-6\right)=3$, and so $d\left(\left(q_1-1\right)\left(q_2-1\right)\right)=7$ and $b=3$. Or, equivalently, $\left(q_1-1\right)\left(q_2-1\right)=2^6$. Thus, $q_1=5$ and $q_2=17$, in which case $n=5\cdot 17^3$. □

Theorem 2.5. *Let $n=q_1^aq_2^b$, where $3\leq q_1<q_2$ and $a,b\geq 2$. If $n\in\mathbb{S}$, then $n=3^a\left(2\cdot 3^t+1\right)^b$, where $2\cdot 3^t+1$ is prime and $ab+2a+2b+1=3bt$.*

Proof. By (3), we have

$$\left(2a+1\right)\left(2b+1\right)=d\left(\left(q_1-1\right)\left(q_2-1\right)q_1^{a-1}\right)b. \tag{7}$$

There are two cases:

Case 1. $(q_2-1,q_1)=1$. We put $q_1-1=2^x m_1$ and $q_2-1=2^y m_2$, where $x,y\geq 1$ and $(2,m_1m_2)=1$. It then follows from (7) that

$$(2a+1)(2b+1)=(x+y+1)\,d(m_1m_2)\,ab. \tag{8}$$

It is clear that $d(m_1m_2)$ cannot be ≥ 3. Otherwise, $(x+y+1)\,a\geq 3a>2a+1$ and $d(m_1m_2)\,b\geq 3b>2b+1$, a contradiction. Moreover, if $d(m_1m_2)=2$, then the right-hand side of (8) is even, while its left-hand side is odd, also a contradiction. Therefore, $m_1=m_2=1$ and so by (8) once again, $(2a+1)(2b+1)=(2^j+2^i+1)\,ab$ for some $j>i\geq 0$. Note also that 2^j+2^i+1 cannot be ≥ 6, and hence $i=0$ and $j=1$. That is, $(2a+1)(2b+1)=4ab$, which is impossible.

Case 2. $(q_2-1,q_1)=q_1$. We put $q_1-1=2^x m_1$ and $q_2-1=2^y q_1^t m_2$, where $x,y,t\geq 1$ and $(2,m_1m_2)=1$. By (7), $(2a+1)(2b+1)=(x+y+1)\,d(m_1m_2)\,(a+t)\,b$ from which it is follows that $m_1=m_2=1$ and $b\geq 3$. Moreover, we see that $x+y$ is even and $x+y\leq 4$. Therefore, if $x+y=4$, then

$$a(b-2)=b(2-5t)+1. \tag{9}$$

This is impossible since the left-hand side of (9) is positive, while its right-hand side is negative. If $x+y=2$, then $x=y=1$. It follows that $n=3^a\left(2\cdot 3^t+1\right)^b$, where $2\cdot 3^t+1$ is prime and $ab+2a+2b+1=3bt$. For example, for $a=10$, $b=7$ and $t=5$, we obtain $n=3^{10}\left(2\cdot 3^5+1\right)^7=3^{10}\cdot 487^7$. □

2.3 n is not square-free with n even

Now, assume that n is **even**. We have the following notes:

Lemma 2.2. *2^x-3 is divisible by 5 if and only if $x\equiv 3(\mathrm{mod}\ 4)$.*

Proof. Clearly, $2^{4k}\equiv 1(\mathrm{mod}\ 5)$ for every $k\geq 0$. Hence, $2^x\equiv 3(\mathrm{mod}\ 5)$ if and only if $x\equiv 3(\mathrm{mod}\ 4)$. □

Proposition 2.6. *Let $n=2^a q_2$, where $q_2\geq 3$ and $a\geq 2$. If $n\in\mathbb{S}$, then n is one of the numbers:*

- *$n=2^{5t-3}\left(2^t\cdot p^4+1\right)$, where p is an odd prime with $2^t\cdot p^4+1$ is also prime.*
- *$n=2^{t-1}\left(2^t\cdot p^2+1\right)$, where p is an odd prime with $2^t\cdot p^2+1$ is also prime.*
- *$n=2^{\left(2^i-3\right)/5}F_i$, where $i\equiv 3(\mathrm{mod}\ 4)$ and F_i is a Fermat prime.*

Proof. By (3), we have $3(2a+1)=d\left((q-1)\,2^{a-1}\right)$. Put $q-1=2^t m$, where $(2,m)=1$ and $t\geq 1$. It follows that $3(2a+1)=d(m)(a+t)$, and hence

$$(6-d(m))\,a=d(m)\,t-3. \tag{10}$$

It is clear from (10) that $d(m)$ is odd and cannot be ≥ 6. Now, we consider separately the following possibilities:

- $d(m)=5$. It follows that $m=p^4$, where $p\geq 3$ is prime and $a=5t+3$. Thus, $n=2^{5t-3}\left(2^t\cdot p^4+1\right)$, where $2^t\cdot p^4+1$ is also prime. For example, for $t=1$ and $p=3$, we get $n=2^2\left(2\cdot 3^4+1\right)=2^2\cdot 163$.

- $d(m) = 3$. It follows that $m = p^2$, where $p \geq 3$ is prime and $a = t - 1$. Thus, $n = 2^{t-1}(2^t \cdot p^2 + 1)$, where $2^t \cdot p^2 + 1$ is also prime. For example, for $t = p = 3$, we have $n = 2^2(2^3 \cdot 3^2 + 1) = 2^2 \cdot 73$.
- $d(m) = 1$. Then $t = 2^i$ for some $i \geq 0$, and by (10) we have $a = (2^i - 3)/5$. By Lemma 2.2, $i \equiv 3 (\mathrm{mod}\ 4)$. Hence, $n = 2^{(2^i-3)/5} F_i$, where F_i is a Fermat prime.

This completes the proof. $\square$

Proposition 2.7. *Let $n = 2q_2^b$, where q_2 is odd prime and $b \geq 2$. Then $n \notin \mathbb{S}$.*

Proof. By (3), we have $3(2b+1) = d(q_2 - 1)b$. We put $q_2 - 1 = 2^s m$, where $(2, m) = 1$ and $s \geq 1$. It follows that $3(2b+1) = d(m)(s+1)b$, and so

$$3 = (d(m)(s+1) - 6)b. \tag{11}$$

By (11), we must have $b = 3$, $d(m) = 7$ and $s = 0$ or $d(m) = 1$ and so $s = 2^i = 6$ for some i. Thus there is no solution in both cases. $\square$

Proposition 2.8. *Let $n = 2^a q_2^b$, where $q_2 \geq 3$ and $a, b \geq 2$. If $n \in \mathbb{S}$, then n is one of the numbers:*

- *$n = 2^a(2^s \cdot p^2 + 1)^b$, where p and $2^s \cdot p^2 + 1$ are simultaneously primes with $ab + 2a + 2b + 1 = 3bs$,*
- *$n = 2^a\left(2^{(3ab+2a+2b+1)/b} + 1\right)^b$, where b divides $2a+1$ and $2^{(3ab+2a+2b+1)/b} + 1$ is prime.*

Proof. By (3), we have $(2a+1)(2b+1) = d(2^{a-1}(q_2 - 1))b$. If we put $q_2 - 1 = 2^s m$, where $(2, m) = 1$ and $s \geq 1$, it follows that

$$(2b+1)(2b+1) = d(m)(s+a)b. \tag{12}$$

By (12), $d(m)$ cannot be ≥ 6; otherwise, $d(m)(s+a)b \geq (3b)(2(a+s)) > (2a+1)(2b+1)$, a contradiction. Moreover, $d(m)$ cannot be even. So the rest cases are:

- $d(m) = 5$. By (12), we have $b(2 - 5s) = a(b-2) - 1$, which has no sense.
- $d(m) = 3$. Then $m = p^2$ for some prime $p \geq 3$, and by (12) we have $ab + 2a + 2b + 1 = 3bs$. Thus, $n = 2^a(2^s \cdot p^2 + 1)^b$ where p and $2^s \cdot p^2 + 1$ are simultaneously primes with $ab + 2a + 2b + 1 = 3bs$. For example, for $a = 13$, $b = 9$ and $s = 6$, we get $n = 2^{13} \cdot 577^9$.
- $d(m) = 1$. It follows that $q = 2^{2^i} + 1$ for some $i \geq 0$, and so $n = 2^a F_i^b$, where $3ab + 2a + 2b + 1 = 2^i b$ by (12). Or equivalently, $n = 2^a\left(2^{(3ab+2a+2b+1)/b} + 1\right)^b$, where b divides $2a+1$ and $2^{(3ab+2a+2b+1)/b} + 1$ is prime.

The proof is finished. $\square$

3 Solutions having three distinct prime factors

Let $n = q_1^a q_2^b q_3^c \in \mathbb{S}$, where q_1, q_2, q_3 are distinct primes with $2 \leq q_1 < q_2 < q_3$ and $a, b, c \geq 1$. Be definition, we see that

$$(2a+1)(2b+1)(2c+1) = d\left((q_1 - 1)(q_2 - 1)(q_3 - 1)\, q_1^{a-1} q_2^{b-1}\right) c. \tag{13}$$

We consider separately the cases n is square-free and n is not square-free (odd and even).

3.1 n is square-free

First, assume that n is **square-free odd**.

Proposition 3.1. *The only possible solutions of the form $q_1q_2q_3$ are:*

- $F_1F_3F_4$, $F_0F_2 \cdot 73$, $11 \cdot F_2 \cdot 41$.
- $F_1F_2\left(4p^2+1\right)$, *where p is an odd prime with $4p^2+1$ is prime.*
- $F_1\left(2p+1\right)\left(2p^5+1\right)$, *where p is an odd prime with $2p+1$ and $2p^5+1$ are primes.*
- $F_1\left(2^2p+1\right)\left(2^4p+1\right)$, *where p is an odd prime with 2^2p+1 and 2^4p+1 are primes.*
- $F_2\left(2p+1\right)\left(2^3p+1\right)$, *where p is an odd prime with $2p+1$ and 2^3p+1 are primes.*

For the proof we need the following lemma:

Lemma 3.1. *If p is prime greater than 3, then the numbers $2^{2\alpha+1}p+1$ and $2^{2\beta}p+1$ with $\alpha \geq 0$ and $\beta \geq 1$ cannot be simultaneously primes.*

Proof. First we note that $2^{2\alpha+1} \equiv 2(\text{mod } 3)$ and $2^{2\beta} \equiv 1(\text{mod } 3)$. Thus if p is of the form $3k+1$, then $2^{2\alpha+1}p+1 = 2^{2\alpha+1}(3k+1)+1 \equiv 0(\text{mod } 3)$ and if p is of the form $3k+2$, then $2^{2\beta}p+1 = 2^{2\beta}(3k+2)+1 \equiv 0(\text{mod } 3)$. □

Proof of Proposition 3.1. We put $q_1 - 1 = 2^x m_1$, $q_2 - 1 = 2^y m_2$, and $q_3 - 1 = 2^z m_3$, where m_i is odd and $x, y, z \geq 1$. Let $m = m_1m_2m_3$. Applying (13), we have $27 = d\left(m\right)\left(x+y+z+1\right)$. We distinguish two cases:

Case 1. $d\left(m\right) = 1$. Then and $2^i + 2^j + 2^k = 26$ for some $0 \leq i < j < k$, which is only true for $i = 1$, $j = 3$ and $k = 4$. Hence, $n = F_1F_3F_4$.

Case 2. $d\left(m\right) = 3$ and $x+y+z = 8$. It follows that $m = p^2$, where $p \geq 3$ is prime. Also we consider the following subcases:

Case 2.1. $m_1 = m_2 = 1$ and $m_3 = p^2$. Since $x < y$, by Lemma 2.1, there are only two possibilities:

- $n = 3 \cdot 17 \cdot 73$.
- $n = 5 \cdot 17 \cdot (4p^2+1)$, where p and $4p^2+1$ are both primes. For example, $p = 3$.

Case 2.2. $m_1 = 1$ and $m_2 = m_3 = p$. It follows that $x = 2^i$, where $i \geq 0$ and $y < z$. By Lemma 3.1, we have three possibilities:

- $n = 5(2p+1)(2^5p+1)$, where p, $2p+1$ and 2^5p+1 are simultaneously primes. For example, for $p = 11$, we obtain $n = 5 \cdot 23 \cdot 353$.
- $n = 5(4p+1)(16p+1)$, where p, $4p+1$ and $16p+1$ are simultaneously primes. For example, for $p = 7$, we have $n = 5 \cdot 29 \cdot 113$.
- $n = 17(2p+1)(2^3p+1)$, where p, $2p+1$ and 2^3p+1 are simultaneously primes. For example, for $p = 11$, we have $n = 17 \cdot 23 \cdot 89$.

Case 2.3. $m_1 = m_2 = p$ and $m_3 = 1$. This case is not valid. We have the same for $m_1 = m_3 = 1$ and $m_2 = p^2$ or $m_2 = m_3 = 1$ and $m_1 = p^2$.

Case 2.4. $m_1 = m_3 = p$ and $m_2 = 1$. It follows that $y = 2^i$, where $i \geq 0$ and $x < z$. Thus we must have $p = 5$. Hence, $n = 11 \cdot 17 \cdot 41$. □

Second, assume that n is **square-free even**.

Proposition 3.2. *The only possible solutions of the form* $2q_2q_3$ *are:*

- $2 \cdot 3 \cdot 1153$, $2 \cdot 19 \cdot 1459$.
- $2 \cdot 5\,(2^6p^2+1)$, *where* p *is an odd prime with* 2^6p^2+1 *is prime.*
- $2 \cdot 17\,(2^4p^2+1)$, *where* p *is an odd prime with* 2^4p^2+1 *is prime.*
- $2\,(2p+1)\,(2^7p+1)$, *where* p *is an odd prime with* $2p+1$ *and* 2^7p+1 *are primes.*
- $2\,(2^2p+1)\,(2^6p+1)$, *where* p *is an odd prime with* 2^2p+1 *and* 2^6p+1 *are primes.*
- $2\,(2^3p+1)\,(2^5p+1)$, *where* p *is an odd prime with* 2^3p+1 *and* 2^5p+1 *are primes.*
- $2\,(2p+1)\,(2p^7+1)$, *where* p *is an odd prime with* $2p+1$ *and* $2p^7+1$ *are primes.*
- $2\,(2p^3+1)\,(2p^5+1)$, *where* p *is an odd prime with* $2p^3+1$ *and* $2p^5+1$ *are primes.*
- $2\,(2p_1+1)\,(2p_1p_2^2+1)$, *where* p_1, p_2, $2p_1+1$ *and* $2p_1p_2^2+1$ *are simultaneously primes.*

For the proof, we need the following lemma:

Lemma 3.2. *If* p_1 *and* p_2 *are primes greater than 3, then* $2p_1^2p_2^2+1$ *is composite.*

Proof. This follows immediately from the proof of Lemma 2.1. □

Proof of Proposition 3.2. We put $q_1 - 1 = 2^x m_1$ and $q_2 - 1 = 2^y m_2$, where $(2, m_1m_2) = 1$ and $x, y \geq 1$. By (13), $27 = d\,(m_1m_2)\,(x+y+1)$. There are three cases to consider.

Case 1. $d\,(m_1m_2) = 1$. That is, $2^i + 2^j = 26$ for some $i, j \geq 0$. This is impossible.

Case 2. $d\,(m_1m_2) = 3$ and $x+y = 8$. It follows that $m_1m_2 = p^2$, where $p \geq 3$ is prime. We have three subcases:

Case 2.1. $m_1 = 1$ and $m_2 = p^2$. Then the solutions are given by:

- $n = 2 \cdot 3(2^7p^2+1)$ with p and 2^7p^2+1 are both prime, and by Lemma 2.1, $p = 3$ is the only prime with this property, in which case $n = 2 \cdot 3 \cdot 1153$.
- $n = 2 \cdot (5(2^6p^2+1)$ with p and 2^6p^2+1 are both prime. For example, for $p = 3$, we have $n = 2 \cdot 5 \cdot 577$.
- $n = 2 \cdot 17(16p^2+1)$ with p and $16p^2+1$ are both prime. For example, for $p = 5$, we have $n = 2 \cdot 17 \cdot 401$.

Case 2.2. $m_1 = m_2 = p$. Then $x < y$, and the solutions are:

- $2 \cdot (2p+1)(2^7p+1)$, where p, $2p+1$ and 2^7p+1 are primes. For example, for $p = 5$, we get $n = 2 \cdot 11 \cdot 641$.
- $n = 2(4p+1)(2^6p+1)$, where p, $4p+1$ and 2^6p+1 are primes. For example, for $p = 7$, we get $n = 2 \cdot 29 \cdot 449$.
- $n = 2(8p+1)(2^5p+1)$, where p, $8p+1$ and 2^5p+1 are primes. For example, for $p = 11$, we have $n = 2 \cdot 89 \cdot 353$.

Case 2.3. $m_1 = p^2$ and $m_2 = 1$. This case is not valid.

Case 3. $d(m_1 m_2) = 9$ and $x + y = 2$. We have two subcases:

Case 3.1. $m_1 m_2 = p^8$, where $p \geq 3$ is prime. Then the solutions are:

- $n = 2(2p+1)(2p^7+1)$, where p, $2p+1$ and $2p^7+1$ are primes.
- $n = 2(2p^2+1)(2p^6+1)$, where p, $2p^2+1$ and $2p^6+1$ are primes, which is only true for $p = 3$, and so $n = 2 \cdot 19 \cdot 1459$.
- $n = 2(2p^3+1)(2p^5+1)$, where p, $2p^3+1$ and $2p^5+1$ are primes. For example, for $p = 29$, we have $n = 2 \cdot 48779 \cdot 41022299$.

Case 3.2. $m_1 m_2 = p_1^2 p_2^2$, where p_1, p_2 are odd primes with $p_1 < p_2$. By Lemma 3.2, the number $2 \cdot 3 \cdot (2p_1^2 p_2^2 + 1)$ is composite and by Lemma 2.1, we obtain

- $n = 2(2p_1+1)(2p_1p_2^2+1)$ with p_1, p_2, $2p_1+1$ and $2p_1p_2^2+1$ are primes. For example, for $p_1 = 3$ and $p_2 = 5$, $n = 2 \cdot 7 \cdot 151$.
- $n = 2(2p_2+1)(2p_1^2p_2+1)$ with p_1, p_2, $2p_2+1$ and $2p_1^2p_2+1$ are primes. For example, for $p_1 = 3$ and $p_2 = 11$ we obtain $n = 2 \cdot 23 \cdot 199$.

This completes the proof. □

3.2 n is not square-free with n odd

Assume that n is **not square-free** with n is **odd**. Here, we characterize all odd solutions having only one power prime.

Proposition 3.3. *The only possible solutions of the form $q_1^a q_2 q_3$, where $a \geq 2$ and $3 \leq q_1 < q_2 < q_3$ are:*

- $5^3(2p+1)(8p+1)$, *where p, $2p+1$and $8p+1$ are prime numbers.*
- $q_1^{17(t+t')-9}(2^y q_1^t+1)(2^z q_1^{t'}+1)$, *where q_1 is a Fermat prime and y, z are positive integers with $(q_1, y+z) \in \{(3,15),(5,14),(17,12),(257,8)\}$ and $2^y q_1^t+1$, $2^z q_1^{t'}+1$ are primes.*
- $q_1^{5(t+t')-3}(2^y q_1^t+1)(2^z q_1^{t'}+1)$, *where q_1 is a Fermat prime and y, z, t, t' are positive integers with $(q_1, y+z) \in \{(3,13),(5,12),(17,10),(257,6)\}$ and $2^y q_1^t+1$, $2^z q_1^{t'}+1$ are primes.*
- $q_1^{\frac{13(t+t')-9}{5}}(2^y q_1^t+1)(2^z q_1^{t'}+1)$, *where q_1 is a Fermat prime and y, z, t, t' are positive integers, with $(q_1, y+z) \in \{(3,11),(5,10),(17,8),(257,4)\}$ and $2^y q_1^t+1$, $2^z q_1^{t'}+1$ are primes.*
- $q_1^{\frac{11(t+t')-9}{7}}(2^y q_1^t+1)(2^z q_1^{t'}+1)$, *where q_1 is a Fermat prime and y, z, t, t' are positive integers with $(q_1, y+z) \in \{(3,9),(5,8),(17,6)\}$ and $2^y q_1^t+1$, $2^z q_1^{t'}+1$ are primes.*
- $q_1^{t+t'-1}(2^y q_1^t+1)(2^z q_1^{t'}+1)$, *where q_1 is a Fermat prime and y, z, t, t' are positive integers with $(q_1, y+z) \in \{(3,7),(5,6),(17,4)\}$ and $2^y q_1^t+1$, $2^z q_1^{t'}+1$ are primes.*
- $q_1^{\frac{7(t+t')-9}{11}}(2^y q_1^t+1)(2^z q_1^{t'}+1)$, *where q_1 is a Fermat prime and y, z, t, t' are positive integers with $(q_1, y+z) \in \{(3,5),(5,4)\}$ and $2^y q_1^t+1$, $2^z q_1^{t'}+1$ are primes.*
- $q_1^{\frac{5(t+t')-9}{13}}(2^y q_1^t+1)(2^z q_1^{t'}+1)$, *where q_1 is a Fermat prime and y, z, t, t' are positive integers with $(q_1, y+z) \in \{(3,3),(5,2)\}$ and $2^y q_1^t+1$, $2^z q_1^{t'}+1$ are primes.*

- $n = (2^x q+1)^{5(t+t')-3}\left(2^y(2^x q+1)^t q+1\right)\left(2^z(2^x q+1)^{t'}+1\right)$, *where* q, $2^x q+1$, $2^y(2^x q+1)^t q+1$, $2^z(2^x q+1)^{t'}+1$ *are primes and* x, y, z, t, t' *are positive integers with* $x+y+z=4$.
- $n = (2^x q+1)^{5(t+t')-3}\left(2^y(2^x q+1)^t+1\right)\left(2^z(2^x q+1)^{t'} q+1\right)$, *where* q, $2^x q+1$, $2^y(2^x q+1)^t+1$, $2^z(2^x q+1)^{t'} q+1$ *are primes and* x, y, z, t, t' *are positive integers with* $x+y+z=4$.
- $n = (2^x+1)^{5(t+t')-3}\left(2^y(2^x+1)^t q+1\right)\left(2^z(2^x+1)^{t'} q+1\right)$, *where* q, 2^x+1, $2^y(2^x q+1)^t q+1$, $2^z(2^x q+1)^{t'} q+1$ *are primes and* x, y, z, t, t' *are positive integers with* $x+y+z=4$.
- $n = (2^x+1)^{5(t+t')-3}\left(2^y(2^x+1)^t q^2+1\right)\left(2^z(2^x+1)^{t'}+1\right)$, *where* q, 2^x+1, $2^y(2^x q^2+1)^t q+1$, $2^z(2^x q+1)^{t'}+1$ *are primes and* x, y, z, t, t' *are positive integers with* $x+y+z=4$.
- $n = (2^x+1)^{5(t+t')-3}\left(2^y(2^x+1)^t+1\right)\left(2^z(2^x+1)^{t'} q^2+1\right)$, *where* q, 2^x+1, $2^y(2^x q+1)^t+1, 2^z(2^x q+1)^{t'} q^2+1$ *are primes and* x, y, z, t, t' *are positive integers with* $x+y+z=4$.

Proof. Let $n = q_1^a q_2 q_3 \in \mathbb{S}$. Therefore, by (13) we get

$$9\,(2a+1) = d\left(\left(q_1-1\right)\left(q_2-1\right)\left(q_3-1\right) q_1^{a-1}\right). \tag{14}$$

There are two cases:
Case 1. Suppose that $\left(\left(q_2-1\right)\left(q_3-1\right), q_1\right) = 1$. It follows from (14) that

$$9 = \left(d\left(\left(q_1-1\right)\left(q_2-1\right)\left(q_3-1\right)\right) - 18\right) a. \tag{15}$$

We distinguish the following subcases:

Case 1.1. $a = 9$. From (15), $d\left(\left(q_1-1\right)\left(q_2-1\right)\left(q_3-1\right)\right) = 19$. We must have

$$\left(q_1-1\right)\left(q_2-1\right)\left(q_3-1\right) = 2^{18},$$

and hence q_1, q_2 and q_3 are Fermat primes. This is impossible.

Case 1.2. $a = 3$. From (15), $d\left(\left(q_1-1\right)\left(q_2-1\right)\left(q_3-1\right)\right) = 21$. If

$$\left(q_1-1\right)\left(q_2-1\right)\left(q_3-1\right) = 2^{20},$$

then q_1, q_2 and q_3 are Fermat primes, which is impossible. Thus, we must have

$$\left(q_1-1\right)\left(q_2-1\right)\left(q_3-1\right) = 2^6 q^2,$$

where q is odd prime, with $(q_1, q) = 1$. By Lemma 2.1, $8q^2+1$ and $2q^2+1$ are composite, and by Lemma 3.1, $2q+1$ and $2^4 q+1$ are not simultaneously primes and the same for $2^2 q+1$ and $2^3 q+1$, so we conclude that n is of the form: $n = 5^3(2q+1)(8q+1)$, where $q, 2q+1$ and $8q+1$ are prime numbers with $q > 5$. For example, if $q = 11$ then $n = 5^3 \cdot 23 \cdot 89$.

Case 2. Suppose that $((q_2-1)(q_3-1), q_1) = q_1$. We put $q_1 - 1 = 2^x m_1$, $q_2 - 1 = 2^y q_1^t m_2$ and $q_3 - 1 = 2^z q_1^{t'} m_3$, where $x, y, z \geq 1$, $\max(t, t') \geq 1$ and $(2p_1, m_1 m_2 m_3) = 1$. Let us take $m = m_1 m_2 m_3$. It follows from (14) that

$$(18 - d(m)(x+y+z+1))\, a = d(m)(x+y+z+1)(t+t') - 9. \tag{16}$$

As above, $d(m)(x+y+z+1)$ cannot be ≥ 18. Thus, we distinguish the following subcases:

Case 2.1. $d(m)(x+y+z+1) = 17$. Thus, $d(m) = 1$ and so $x+y+z = 16$. Or equivalently, $m_1 = m_2 = m_3 = 1$, q_1 is a Fermat prime and by (16), $a = 17(t+t') - 9$. Therefore, we get

$$n = q_1^{17(t+t')-9}\left(2^y \cdot q_1^t + 1\right)\left(2^z \cdot q_1^{t'} + 1\right),$$

where $2^y \cdot q_1^t + 1$ and $2^z \cdot q_1^{t'} + 1$ are primes with $2^y \cdot q_1^t < 2^z \cdot q_1^{t'}$, and we have

$$(q_1, y+z) \in \{(3,15), (5,14), (17,12), (257,8)\}.$$

For example, for $q_1 = 3$, $y = 2$, $z = 13$, $t = 2$ and $t' = 5$ we obtain $n = 3^{110} \cdot 37 \cdot 1990657$.

Case 2.2. $d(m)(x+y+z+1) = 15$. We will consider separately the two cases $d(m) = 1$ and $d(m) = 3$.

- $d(m) = 1$. Then $x+y+z = 14$, q_1 is a Fermat prime and by (16), $a = 5(t+t') - 3$. Thus, we have

  $$n = q_1^{5(t+t')-3}\left(2^y \cdot q_1^t + 1\right)\left(2^z \cdot q_1^{t'} + 1\right),$$

  where $2^y \cdot q_1^t + 1$ and $2^z \cdot q_1^{t'} + 1$ are primes with $2^y \cdot q_1^t < 2^z \cdot q_1^{t'}$, and we have $(q_1, y+z) \in \{(3,13), (5,12), (17,10), (257,6)\}$. For example, for $x = 1$, $y = 2$, $z = 11$, $t = 1$ and $t' = 2$ we obtain $n = 3^{12} \cdot 13 \cdot 18433$.

- $d(m) = 3$. Then $x+y+z = 4$ and $m = q^2$, where q is an odd prime with $(q_1, q) = 1$. Thus, by (16), $a = 5(t+t') - 3$. In this case, n is one of the numbers:
  - $n = (2^x q + 1)^{5(t+t')-3}\left(2^y q_1^t q + 1\right)\left(2^z q_1^{t'} + 1\right)$, where $2^x q + 1$, $2^y q_1^t q + 1$ and $2^z q_1^{t'} + 1$ are primes. For example, for $x = y = 1$, $z = 2$, $q = 3$, $t = 1$ and $t' = 2$, we get $n = 7^{12} \cdot 43 \cdot 197$.
  - $n = (2^x q + 1)^{5(t+t')-3}\left(2^y q_1^t + 1\right)\left(2^z q_1^{t'} q + 1\right)$, where $2^x q + 1$, $2^y q_1^t + 1$ and $2^z q_1^{t'} q + 1$ are primes. For example, for $x = z = 1$, $y = 2$, $q = 3$, $t = 1$ and $t' = 1$, we get $n = 7^7 \cdot 29 \cdot 43$.
  - $n = (2^x + 1)^{5(t+t')-3}\left(2^y q_1^t q + 1\right)\left(2^z q_1^{t'} q + 1\right)$, where $2^x + 1$, $2^y q_1^t q + 1$ and $2^z q_1^{t'} q + 1$ are primes. For example, for $x = z = 1$, $y = 2$, $q = 13$, $t = 0$ and $t' = 1$, we get $n = 3^2 \cdot 53 \cdot 79$.
  - $n = (2^x + 1)^{5(t+t')-3}\left(2^y q_1^t q^2 + 1\right)\left(2^z q_1^{t'} + 1\right)$, where $2^x + 1$, $2^y q_1^t q^2 + 1$ and $2^z q_1^{t'} + 1$ are primes. For example, for $x = y = 1$, $z = 2$, $q = 5$, $t = 1$ and $t' = 6$, we get $n = 3^{32} \cdot 151 \cdot 2917$.
  - $n = (2^x + 1)^{5(t+t')-3}\left(2^y q_1^t + 1\right)\left(2^z q_1^{t'} q^2 + 1\right)$, where $2^x + 1$, $2^y q_1^t + 1$ and $2^z q_1^{t'} q^2 + 1$ are primes. For example, for $y = z = t = 1$, $x = 2$, $q = 3$ and $t' = 0$, we get $n = 5^2 \cdot 11 \cdot 19$.

Case 2.3. $d(m)(x+y+z+1) = 13$. Then $d(m) = 1$ and $x+y+z = 12$. Thus, $m_1 = m_2 = m_3 = 1$ and q_1 is a Fermat prime. By (16), $5a = 13(t+t') - 9$. That is,

$$n = (q_1)^{\frac{13(t+t')-9}{5}} \left(2^y q_1^t + 1\right) \left(2^z q_1^{t'} + 1\right),$$

where 2^x+1, $2^y q_1^t + 1$ and $2^z q_1^{t'} + 1$ are primes with $2^y q_1^t < 2^z q_1^{t'}$ and $5 \mid 13(t+t') - 9$, and we have $(q_1, y+z) \in \{(3,11),(5,10),(17,8),(257,4)\}$. For example, for $x = t = 1$, $y = 5$, $z = 6$ and $t' = 2$ we get $n = 3^6 \cdot 97 \cdot 577$.

Case 2.4. $d(m)(x+y+z+1) = 11$. We also have $m = 1$ and $x+y+z = 10$. So, q_1 is a Fermat prime and by (16), $7a = 11(t+t') - 9$. Thus,

$$n = q_1^{\frac{11(t+t')-9}{7}} \left(2^y q_1^t + 1\right) \left(2^z q_1^{t'} + 1\right),$$

where $2^y q_1^t + 1$ and $2^z q_1^{t'} + 1$ are primes with $2^y q_1^t < 2^z q_1^{t'}$ and $7 \mid 11(t+t') - 9$, and we have $(q_1, x+y) \in \{(3,9),(5,8),(17,6)\}$. For example, for $x = t = 1$, $y = 2$, $z = 7$ and $t' = 3$ we have $n = 3^5 \cdot 13 \cdot 3457$.

Case 2.5. $d(m)(x+y+z+1) = 9$. Here, we must have $d(m) = 1$ and $x+y+z = 8$, from which it follows that $m = 1$ and q_1 is a Fermat prime. Thus, by (16), $a = t+t'-1$. Hence,

$$n = q_1^{t+t'-1} \left(2^y q_1^t + 1\right) \left(2^z q_1^{t'} + 1\right),$$

where $2^y q_1^t + 1$ and $2^z q_1^{t'} + 1$ are primes with $2^y q_1^t < 2^z q_1^{t'}$, and we have

$$(q_1, x+y) \in \{(3,7),(5,6),(17,4)\}.$$

For example, for $x = t = 1$, $y = 2$, $z = 5$ and $t' = 4$ we have $n = 3^4 \cdot 13 \cdot 2593$.

Case 2.6. $d(m)(x+y+z+1) = 7$. As above $m = 1$ and $x+y+z = 6$. By (16), $11a = 7(t+t') - 9$. That is,

$$n = q_1^{(7(t+t')-9)/11} \left(2^y q_1^t + 1\right) \left(2^z q_1^{t'} + 1\right),$$

where $2^y q_1^t + 1$ and $2^z q_1^{t'} + 1$ are primes with $2^y q_1^t < 2^z q_1^{t'}$ and $11 \mid 7(t+t') - 9$, and we have $(q_1, x+y) \in \{(3,5),(5,4)\}$. For example, for $x = y = 1$, $z = t' = 4$ and $t = 2$ we get $n = 3^3 \cdot 19 \cdot 1297$.

Case 2.7. $d(m)(x+y+z+1) = 5$. We must have $m = 1$ and $x+y+z = 4$. By (16), we deduce that $13a = 5(t+t') - 9$, and so

$$n = q_1^{\frac{5(t+t')-9}{13}} \left(2^y q_1^t + 1\right) \left(2^z q_1^{t'} + 1\right),$$

where $2^y q_1^t + 1$ and $2^z q_1^{t'} + 1$ are primes with $2^y q_1^t < 2^z q_1^{t'}$ and $13 \mid 5(t+t') - 9$, and we have $(q_1, x+y) \in \{(3,3),(5,2)\}$. For example, for $x = t = 1$, $y = 1$, $z = 2$ and $t' = 6$ we get $n = 3^2 \cdot 7 \cdot 2917$. $\square$

Proposition 3.4. *The only possible solutions of the form $q_1q_2^bq_3$, where $b \geq 2$ and $3 \leq q_1 < q_2 < q_3$ are:*

- $5(2q+1)^3(8q+1)$, *where $q, 2q+1, 8q+1$ are primes with $(q, 2q+1) = 1$.*
- $q_1q_2^{17t-9}(2^zq_2^t+1)$, *where q_1, q_2 are Fermat primes and z, t are positive integers with $(q_1, q_2, z) \in \{(3,5,13),(3,17,11),(3,257,7),(5,17,10),(5,257,6),(17,257,4)\}$ and $2^zq_2^t+1$ is prime.*
- $n = q_1q_2^{5t-3}(2^z \cdot q_2^t+1)$, *where q_1, q_2 are Fermat primes and z, t are positive integers with $(q_1, q_2, z) \in \{(3,5,11),(3,17,9),(3,257,5),(5,17,8),(5,257,4),(17,257,2)\}$ and $2^zq_2^t+1$ is prime.*
- $n = q_1q_2^{\frac{13t-9}{5}}(2^z \cdot q_2^t+1)$, *where q_1, q_2 are Fermat primes and z, t are positive integers with $(q_1, q_2, z) \in \{(3,5,9),(3,17,7),(3,257,3),(5,17,6),(5,257,2)\}$ and $2^zq_2^t+1$ is prime.*
- $n = q_1q_2^{\frac{11t-9}{7}}(2^z \cdot q_2^t+1)$, *where q_1, q_2 are Fermat primes and z, t are positive integers with $(q_1, q_2, z) \in \{(3,5,7),(3,17,5),(3,257,1),(5,17,4)\}$ and $2^zq_2^t+1$ is prime.*
- $n = q_1q_2^{t-1}(2^z \cdot q_2^t+1)$, *where q_1, q_2 are Fermat primes and z, t are positive integers with $(q_1, q_2, z) \in \{(3,5,5),(3,17,3),(5,17,2)\}$ and $2^zq_2^t+1$ is prime.*
- $n = q_1q_2^{\frac{7t-9}{11}}(2^z \cdot q_2^t+1)$, *where q_1, q_2 are Fermat primes and z, t are positive integers with $(q_1, q_2, z) \in \{(3,5,3),(3,17,1)\}$ and $2^zq_2^t+1$ is prime.*
- $n = (2^x+1)(2^yq+1)^{5t-3}(2^z(2^yq+1)^tq+1)$, *where $q, 2^x+1, 2^yq+1, 2^z(2^yq+1)^tq+1$ are primes.*
- $n = (2^x+1)(2^yq^2+1)^{5t-3}(2^z(2^yq^2+1)^t+1)$, *where q, 2^x+1, 2^yq^2+1 and $2^z(2^yq^2+1)^t+1$ are primes.*
- $n = 3 \cdot 5^{5t-3}(2 \cdot 5^tq^2+1)$, *where q and $2 \cdot 5^tq^2$ are primes.*
- $n = 3 \cdot 5^{\frac{5t-9}{13}}(2 \cdot 5^t+1)$, *where $2 \cdot 5^t+1$ is prime.*

Proof. Let $n = q_1q_2^bq_3$, where $b \geq 2$ and $3 \leq q_1 < q_2 < q_3$. Assume further that $n \in \mathbb{S}$. It follows that

$$9(2b+1) = d\left((q_1-1)(q_2-1)(q_3-1)q_2^{b-1}\right). \qquad (17)$$

There are two cases:

Case 1. Assume that $((q_3-1), q_2) = 1$. By (17), we have

$$9 = (d((q_1-1)(q_2-1)(q_3-1)) - 18)b. \qquad (18)$$

We distinguish the following subcases:

Case 1.1. $b=9$. From (18), $d\left(\left(q_1-1\right)\left(q_2-1\right)\left(q_3-1\right)\right)=19$. We must have

$$\left(q_1-1\right)\left(q_2-1\right)\left(q_3-1\right)=2^{18},$$

and hence q_1, q_2 and q_3 are Fermat numbers. This is impossible.

Case 1.2. $b=3$. By (18), $d\left((q_1-1)(q_2-1)(q_3-1)\right)=21$. If $(q_1-1)(q_2-1)(q_3-1)=2^{20}$, then q_1, q_2 and q_3 are Fermat numbers, which is impossible. Thus, we must have

$$\left(q_1-1\right)\left(q_2-1\right)\left(q_3-1\right)=2^6 q^2,$$

where q is odd prime with $(q_2, q)=1$. By Lemma 2.1, $8q^2+1$ and $2q^2+1$ are composite and by Lemma 3.1, $2q+1$ and 2^4q+1 are not simultaneously primes. A similar argument holds for 2^2q+1 and 2^3q+1 , so we conclude that n is of the form:

$$n=5(2q+1)^3(8q+1),$$

where $2q+1$ and $8q+1$ are simultaneously primes. For example, if $q=5$ then $n=5\cdot 11^3\cdot 41$.

Case 2. Assume that $\left(\left(q_3-1\right), q_2\right)=q_2$. We put $q_1-1=2^x m_1$, $q_2-1=2^y m_2$ and $q_3-1=2^z q_2^t m_3$, where $x, y, z\geq 1$, $t\geq 1$ and $(2q_2, m_1m_2m_3)=1$. Put $m=m_1m_2m_3$. It follows from (17) that

$$\left(18-d\left(m\right)\left(x+y+z+1\right)\right)b=\left(d\left(m\right)\left(x+y+z+1\right)t-9\right). \tag{19}$$

Note that $d\left(m\right)\left(x+y+z+1\right)$ cannot ≥ 18. Thus, we distinguish the following subcases:

Case 2.1. $d\left(m\right)\left(x+y+z+1\right)=17$. Then $d\left(m\right)=1$, and so $x+y+z=16$. Or equivalently, $m_1=m_2=m_3=1$, q_1 and q_2 are Fermat numbers and by (19), $b=17t-9$. Therefore, $n=q_1q_2^{17t-9}\left(2^z q_2^t+1\right)$, where $2^z q_2^t+1$ is prime with

$$(q_1, q_2, z)\in\{(3,5,13),(3,17,11),(3,257,7),(5,17,10),(5,257,6),(17,257,4)\}.$$

For example, for $x=1$, $y=2$, $z=13$, $t=1$ we have $n=3\cdot 5^8\cdot 40961$.

Case 2.2. $d\left(m\right)\left(x+y+z+1\right)=15$. We will consider separately the two cases $d\left(m\right)=1$ and $d\left(m\right)=3$.

- When $d\left(m\right)=1$. Then $x+y+z=14$, q_1 and q_2 are Fermat numbers and by (19), $b=5t-3$. Thus, $n=q_1q_2^{5t-3}\left(2^z\cdot q_2^t+1\right)$, where $2^z\cdot q_2^t+1$ is prime with

$$(q_1, q_2, z)\in\{(3,5,11),(3,17,9),(3,257,5),(5,17,8),(5,257,4),(17,257,2)\}.$$

  For example, for $x=1$, $y=2$, $z=11$, we have $t=15$ which is the first value with this property. That is, $n=3\cdot 5^{72}\cdot 62500000000001$.

- When $d\left(m\right)=3$. Then $x+y+z=4$ and $m=q^2$, where q is an odd prime with $(q_2, q)=1$. Thus, by (19), $b=5t-3$. Hence,
  - $n=\left(2^x+1\right)\left(2^y q+1\right)^{5t-3}\left(2^z q_2^t q+1\right)$. For example, for $x=z=1$, $y=2$ and $t=1$ we have $n=3\cdot 13^2\cdot 79$.
  - $n=\left(2^x+1\right)\left(2^y q^2+1\right)^{5t-3}\left(2^z q_2^t+1\right)$. For example, for $x=y=1$, $z=2$ and $t=3$ we have $n=3\cdot 19^{12}\cdot 27437$.

- $n = (2^x + 1)(2^y + 1)^{5t-3}(2^z q_2^t q^2 + 1)$, we must have $x = z = 1$, $y = 2$, hence $n = (3)(5)^{5t-3}(2 \cdot 5^t q^2 + 1)$. For example for $q = 7$ and $t = 1$ we get $n = 3 \cdot 5^2 \cdot 491$.

Case 2.3. $d(m)(x + y + z + 1) = 13$. Then $d(m) = 1$ and $x + y + z = 12$. Thus, $m_1 = m_2 = m_3 = 1$ and q_1, q_2 are Fermat primes. By (19), $5b = 13t - 9$. That is,

$$n = q_1 q_2^{(13t-9)/5}\left(2^z \cdot q_2^t + 1\right),$$

where $5 \mid 13t - 9$ and $2^z \cdot q_2^t + 1$ is prime and we have

$$(q_1, q_2, z) \in \{(3, 5, 9), (3, 17, 7), (3, 257, 5), (5, 17, 6), (5, 257, 2)\}.$$

Case 2.4. $d(m)(x + y + z + 1) = 11$. We also have $m = 1$ and $x+y+z = 10$. So, q_1 and q_2 are Fermat numbers and by (19), $7b = 11t - 9$. Thus, $n = q_1 q_2^{(11t-9)/7}(2^z \cdot q_2^t + 1)$, where $7 \mid 11t - 9$ and $2^z \cdot q_2^t + 1$ is prime and

$$(q_1, q_2, z) \in \{(3, 5, 7), (3, 17, 5), (3, 257, 1), (5, 17, 4)\}.$$

For example, if $x = 1$, $y = 2$, $z = 7$ and $t = 333$ then $n = 3 \cdot 5^{522} \cdot (2^7 \cdot 5^{333} + 1)$.

Case 2.5. $d(m)(x + y + z + 1) = 9$. Here, we must have $m = 1$ and $x + y + z = 8$, from which it follows that q_1 and q_2 are Fermat primes. Thus, by (19), $b = t - 1$ and so

$$n = q_1 q_2^{t-1}\left(2^z \cdot q_2^t + 1\right),$$

where $2^z \cdot q_2^t + 1$ is prime and $(q_1, q_2, z) \in \{(3, 5, 5), (3, 17, 3), (5, 17, 2)\}$. For example, for $x = 1$, $y = 2$, $z = 5$. For $t = 3$, we get $n = 3 \cdot 5^2 \cdot 4001$.

Case 2.6. $d(m)(x + y + z + 1) = 7$. Obviously $m = 1$ and $x + y + z = 6$. By (16), $11b = 7t - 9$. That is, $n = q_1 q_2^{(7t-9)/11}(2^z \cdot q_2^t + 1)$, where $11 \mid 7t - 9$ and $2^z \cdot q_2^t + 1$ is prime and we have

$$(q_1, q_2, z) \in \{(3, 5, 3), (3, 17, 1)\}.$$

Case 2.7. $d(m)(x + y + z + 1) = 5$. We must have $m = 1$ and $x + y + z = 4$. By (19), we deduce that $13b = 5t - 9$, and so $n = 3 \cdot 5^{(5t-9)/13}(2 \cdot 5^t + 1)$, where $13 \mid 5t - 9$ and $2 \cdot 5^t + 1$ is prime. After computation, $t = 3699$ is the first value with this property. That is, $n = 3 \cdot 5^{1422} \cdot (2 \cdot 5^{3699} + 1)$. □

Proposition 3.5. *The only solutions of the form $q_1 q_2 q_3^c$, where $c \geq 2$ and $3 \leq q_1 < q_2 < q_3$ are: $3 \cdot 5 \cdot 73^3$, $3 \cdot 17 \cdot 19^3$ and $5(2q + 1)(8q + 1)^3$, where q, $2q + 1$ and $8q + 1$ are primes.*

Proof. Let $n = q_1 q_2 q_3^c$, where $c \geq 2$ and $3 \leq q_1 < q_2 < q_3$. Since $n \in \mathbb{S}$, then

$$9 = (d((q_1 - 1)(q_2 - 1)(q_3 - 1)) - 18)\, c. \tag{20}$$

We distinguish the following two cases:

Case 1. $c = 9$ and $d\left(\left(q_1 - 1\right)\left(q_2 - 1\right)\left(q_3 - 1\right)\right) = 19$. It follows that

$$\left(q_1 - 1\right)\left(q_2 - 1\right)\left(q_3 - 1\right) = 2^{18}$$

and so q_1, q_2, q_3 are Fermat numbers, which is impossible.

Case 2. $c = 3$ and $d\left(\left(q_1 - 1\right)\left(q_2 - 1\right)(q_3 - 1)\right) = 21$As above, if $(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^{20}$, then q_1, q_2, q_3 are also Fermat number, which is impossible. But, if $(q_1 - 1)(q_2 - 1)(q_3 - 1) = 2^6 q^2$, with q is odd prime, then by applying Lemma 3.1, $2q+1$ and $16q+1$ are not simultaneously primes (the same for $4q + 1$ and $8q + 1$). Then n is one of the numbers:

- $n = 3 \cdot 5 \cdot \left(8q^2 + 1\right)^3$, where $8q^2 + 1$ is prime. By Lemma 2.1, $q = 3$ is the only prime with this property, hence $n = 3 \cdot 5 \cdot 73^3$.
- $n = 3 \cdot 17 \cdot \left(2q^2 + 1\right)^3$, where $2q^2 + 1$ is prime. From Lemma 2.1, $q = 3$ is the only prime with this property, hence $n = 3 \cdot 17 \cdot 19^3$.
- $n = 5 \cdot (2q + 1)\left(8q + 1\right)^3$, where $2q + 1$ and $8q + 1$ are prime numbers. The first primes with these properties are $q = 5, 11, 29, 131, 179, 239, 431, 491, \ldots$. □

3.3 n is not square-free with n even

Now, assume that n is **not square-free** with n **is even**. Here, we characterize all even solutions having only one power prime.

Proposition 3.6. *The only possible solutions of the form $2^a pq$, where $a \geq 2$ and $3 \leq p < q$ are:*

- $2^{17(x+y)-9}(2^x m_1 + 1)(2^y m_2 + 1)$, *where* $2^x m_1 + 1$, $2^y m_2 + 1$ *are primes with* $2^x m_1 < 2^y m_2$ *and* $m_1 m_2 = r^{16}$ *such that* r *is an odd prime.*
- $2^{5(x+y)-3}(2^x m_1 + 1)(2^y m_2 + 1)$, *where* $2^x m_1 + 1$, $2^y m_2 + 1$ *are primes with* $2^x m_1 < 2^y m_2$ *and* $m_1 m_2 = r^{14}$ *or* $m_1 m_2 = r_1^4 r_2^2$ *such that* r, r_1, r_2 *are odd primes.*
- $2^{\frac{13(x+y)-9}{5}}(2^x m_1 + 1)(2^y m_2 + 1)$, *where* $2^x m_1 + 1$, $2^y m_2 + 1$ *are primes with* $2^x m_1 < 2^y m_2$ *and* $m_1 m_2 = r^{12}$ *such that* r *is an odd prime.*
- $2^{\frac{11(x+y)-9}{7}}(2^x m_1 + 1)(2^y m_2 + 1)$, *where* $2^x m_1 + 1$, $2^y m_2 + 1$ *are primes with* $2^x m_1 < 2^y m_2$ *and* $m_1 m_2 = r^{10}$ *such that* r *is an odd prime.*
- $2^{(x+y)-1}(2^x m_1 + 1)(2^y m_2 + 1)$, *where* $2^x m_1 + 1$, $2^y m_2 + 1$ *are primes with* $2^x m_1 < 2^y m_2$ *and* $m_1 m_2 = r^8$ *or* $m_1 m_2 = r_1^2 r_2^2$ *such that* r, r_1, r_2 *are odd primes.*
- $2^{\frac{7(x+y)-9}{11}}(2^x m_1 + 1)(2^y m_2 + 1)$, *where* $2^x m_1 + 1$, $2^y m_2 + 1$ *are primes with* $2^x m_1 < 2^y m_2$ *and where* $m_1 m_2 = r^6$ *such that* r *is an odd prime.*
- $2^{\frac{5(x+y)-9}{13}}(2^x m_1 + 1)(2^y m_2 + 1)$, *where* $2^x m_1 + 1$, $2^y m_2 + 1$ *are primes with* $2^x m_1 < 2^y m_2$ *and* $m_1 m_2 = r^4$, *such that* r *is an odd prime.*
- $2^{\frac{(x+y)-3}{5}}(2^x m_1 + 1)(2^y m_2 + 1)$, *where* $2^x m_1 + 1$, $2^y m_2 + 1$ *are primes with* $2^x m_1 < 2^y m_2$ *and* $m_1 m_2 = r^2$ *such that* r *is an odd prime.*
- $2^{\frac{(x+y)-1}{9}}(2^x + 1)(2^y + 1)$, *where* $2^x + 1$ *and* $2^y + 1$ *are primes with* $x < y$.

Proof. Let $n = 2^a pq$, where $a \geq 2$ and $3 \leq p < q$. Since $n \in \mathbb{S}$, then

$$9\,(2a+1) = d\left((p-1)\,(q-1)\,2^{a-1}\right). \tag{21}$$

We put $p-1 = 2^x m_1, q-1 = 2^y m_2$, where $x, y \geq 1$ and $(2, m_1 m_2) = 1$, it follow from (21) that

$$(18 - d(m_1 m_2))a = d(m_1 m_2)(x+y) - 9. \tag{22}$$

We observe that $d(m_1 m_2)$ is odd and cannot be ≥ 18, so we distinguish the following cases:

Case 1. $d(m_1 m_2) = 17$. It follows that $m_1 m_2 = r^{16}$, where $r \geq 3$ is prime and so, by (22), $a = 17(x+y) - 9$. Then $n = 2^{17(x+y)-9}(2^x m_1 + 1)(2^y m_2 + 1)$, where $2^x m_1 + 1$ and $2^y m_2 + 1$ are primes with $2^x m_1 < 2^y m_2$. For example, for $m_1 = r$, $m_2 = r^{15}$, $y = 2$, $x = 2$ and $r = 3$, we get $n = 2^{59} \cdot 13 \cdot 57395629$.

Case 2. $d(m_1 m_2) = 15$. It follows that $m_1 m_2 = r^{14}$ or $m_1 m_2 = r_1^4 r_2^2$, where r_1 and r_2 are distinct odd primes and by (22), $a = 5(x+y) - 3$. Therefore,

$$n = 2^{5(x+y)-3}(2^x m_1 + 1)(2^y m_2 + 1),$$

where $2^x m_1 + 1$ and $2^y m_2 + 1$ are primes with $2^x m_1 < 2^y m_2$. For example, for $m_1 = 5^2$, $m_2 = 3^4$, $x = 2$, $y = 1$, $r_1 = 5$ and $r_2 = 3$ we have $= 2^{12} \cdot 101 \cdot 163$.

Case 3. $d(m_1 m_2) = 13$. It follows that $m_1 m_2 = r^{12}$, where $r \geq 3$ is prime and by (22), $5a = 13(x+y) - 9$. Thus we obtain $n = 2^{(13(x+y)-9)/5}(2^x m_1 + 1)(2^y m_2 + 1)$, where $5|13(x+y) - 9$, $2^x m_1 + 1$ and $2^y m_2 + 1$ are primes with $2^x m_1 < 2^y m_2$. For example, for $x = 1$, $y = 2$ and $m_1 = m_2 = 3^6$ we obtain $n = 2^6 \cdot 1459 \cdot 2917$.

Case 4. $d(m_1 m_2) = 11$. Therefore, $m_1 m_2 = r^{10}$, where $r \geq 3$ is prime. From (22), $7a = 11(x+y) - 9$. Hence, $n = 2^{(11(x+y)-9)/7}(2^x m_1 + 1)(2^y m_2 + 1)$, where $7|11(x+y) - 9$, $2^x m_1 + 1$ and $2^y m_2 + 1$ are primes with $2^x m_1 < 2^y m_2$. For example, for $x = 2$, $y = 16$, $m_1 = 3^6$ and $m_2 = 3^4$ we obtain $n = 2^{27} \cdot 2917 \cdot 5308\,417$.

Case 5. $d(m_1 m_2) = 9$. It follows that $m_1 m_2 = r^8$ or $m_1 m_2 = r_1^2 r_2^2$, where r_1 and r_2 are distinct odd primes. By (22), $a = (x+y) - 1$. Hence, $n = 2^{(x+y)-1}(2^x m_1 + 1)(2^y m_2 + 1)$, where $2^x m_1 + 1$ and $2^y m_2 + 1$ are primes with $2^x m_1 < 2^y m_2$. For example, for $x = 1$, $y = 4$ and $m_1 = m_2 = 3^4$ we have $n = 2^4 \cdot 163 \cdot 1297$. Also, for $x = 1$, $y = 2$, $m_1 = 3^2$ and $m_2 = 5^2$ we get $n = 2^2 \cdot 19 \cdot 101$.

Case 6. $d(m_1 m_2) = 7$. That is, $m_1 m_2 = r^6$, where $r \geq 3$ is prime, and by (22), $11a = 7(x+y) - 9$. Hence, $n = 2^{(7(x+y)-9)/11}(2^x m_1 + 1)(2^y m_2 + 1)$, where $11|7(x+y) - 9$, $(2^x m_1 + 1)$ and $(2^y m_2 + 1)$ are primes with $2^x m_1 < 2^y m_2$. For example, for $x = 2$, $y = 4$, $m_1 = 3^2$ and $m_2 = 3^4$ we get $n = 2^3 \cdot 37 \cdot 1297$.

Case 7. $d(m_1 m_2) = 5$. Then $m_1 m_2 = r^4$, where $r \geq 3$ is prime and by (22), $13a = 5(x+y) - 9$. Hence, $n = 2^{(5(x+y)-9)/13}(2^x m_1 + 1)(2^y m_2 + 1)$, where $13|5(x+y) - 9$, $(2^x m_1 + 1)$ and $(2^y m_2 + 1)$ are primes with $2^x m_1 < 2^y m_2$. For example, for $x = 1$, $y = 6$ and $m_1 = m_2 = 3^2$ we have $n = 2^2 \cdot 19 \cdot 577$.

Case 8. $d(m_1 m_2) = 3$. Then $m_1 m_2 = r^2$, where $r \geq 3$ is prime. By (22), $5a = (x+y) - 3$. Hence, $n = 2^{((x+y)-3)/5}(2^x m_1 + 1)(2^y m_2 + 1)$, where $5|(x+y) - 3$, $2^x m_1 + 1$ and $2^y m_2 + 1$ are primes with $2^x m_1 < 2^y m_2$. For example, for $x = 1$, $y = 12$ and $m_1 = 1$, $m_2 = 11^2$ we have $n = 2^2 \cdot 3 \cdot 495617$.

Case 9. $d(m_1 m_2) = 1$. That $m_1 = m_2 = 1$ and by (22), $9a = (x+y) - 1$. Hence,

$$n = 2^{(x+y-1)/9}(2^x + 1)(2^y + 1),$$

where $9|(x+y) - 1$, $(2^x + 1)$ and $(2^y + 1)$ are Fermat primes with $2^x + 1 < 2^y + 1$. $\square$

Proposition 3.7. *The only possible solutions of the form $2p^b q$, where $b \geq 2$ and $3 \leq p < q$ are:*

- $2F_1^9 F_4$, $2F_2^3 F_4$, $2 \cdot 19^3 \cdot 163$.
- $2 \cdot 5^3(2^4 r^2 + 1)$, *where r and $2^4 r^2 + 1$ are primes.*
- $2 \cdot 17^3(2^2 r^2 + 1)$, *where r and $2^2 r^2 + 1$ are primes.*
- $2(2r+1)^3(2^5 r + 1)$, *where r, $2r+1$ and $2^5 r + 1$ are primes.*
- $2(2r+1)^3(2r^5 + 1)$, *where r, $2r+1$ and $2r^5 + 1$ are primes.*
- $2(2^x+1)^{17t-9}(2^y(2^x+1)^t + 1)$, *where $2^x + 1$, $2^y(2^x+1)^t + 1$ are primes and x, y are positive integers with $x + y = 16$.*
- $2(2^x+1)^{5t-3}(2^y(2^x+1)^t + 1)$, *where $2^x + 1$, $2^y(2^x+1)^t + 1$ are primes and x, y are positive integers with $x + y = 14$.*
- $2 \cdot 3^{5t-3}(2^3 \cdot 3^t \cdot r^2 + 1)$, *where r and $2^3 \cdot 3^t \cdot r^2 + 1$ are primes with $(3, r) = 1$.*
- $2 \cdot 5^{5t-3}(2^2 \cdot 5^t \cdot r^2 + 1)$, *where r and $2^3 \cdot 5^t \cdot r^2 + 1$ are primes with $(5, r) = 1$.*
- $2(2r+1)^{5t-3}(2^3 r(2r+1)^t + 1)$, *where r, $2r+1$ and $2^3 r(2r+1)^t + 1$ are primes.*
- $2(4r+1)^{5t-3}(2^2 r(4r+1)^t + 1)$, *where r, $4r+1$ and $2^2 r(4r+1)^t + 1$ are primes.*
- $2(8r+1)^{5t-3}(2r(8r+1)^t + 1)$, *where r, $8r+1$ and $2r(8r+1)^t + 1$ are primes.*
- $2(4r^2+1)^{5t-3}(4(4r^2+1)^t + 1)$, *where r, $4r^2+1$ and $4(4r^2+1)^t + 1$ are primes.*
- $2 \cdot 3^{5t-3}(2 \cdot 3^t \cdot r^4 + 1)$, *where r and $2 \cdot 3^t \cdot r^4 + 1$ are primes with $(3, r) = 1$.*
- $2(2r+1)^{5t-3}(2r^3(2r+1)^t + 1)$, *where r, $2r+1$ and $2r^3(2r+1)^t + 1$ are primes.*
- $2 \cdot 19^{5t-3}(2 \cdot 19^t r^2 + 1)$, *where r and $2 \cdot 19^t r^2 + 1$ are primes with $(19, r) = 1$.*
- $2(2r^3+1)^{5t-3}(2r(2r^3+1)^t + 1)$, *where r, $2r^3+1$ and $2r(2r^3+1)^t + 1$ are primes.*
- $2(2r^4+1)^{5t-3}(2(2r^4+1)^t + 1)$, *where r, $2r^4+1$ and $2(2r^4+1)^t + 1$ are primes.*
- $2(2^x+1)^{\frac{13t-9}{5}}(2^y(2^x+1)^t + 1)$, *where $2^x + 1, 2^y(2^x+1)^t + 1$ are primes and x, y, t are positive integers with $x + y = 12$.*
- $2(2^x+1)^{\frac{11t-9}{7}}(2^y(2^x+1)^t + 1)$, *where $2^x + 1, 2^y(2^x+1)^t + 1$ are primes and x, y, t are positive integers with $x + y = 10$.*
- $2(2^x+1)^{t-1}(2^y(2^x+1)^t + 1)$, *where $2^x + 1, 2^y(2^x+1)^t + 1$ are primes and x, y, t are positive integers with $x + y = 8$.*
- $2 \cdot 3^{t-1}(2 \cdot 3^t r^2 + 1)$ *where r and $2 \cdot 3^t r^2 + 1$ are primes with $(3, r) = 1$.*
- $2(2r+1)^{t-1}(2r(2r+1)^t + 1)$, *where r, $2r+1$ and $2r(2r+1)^t + 1$ are primes.*
- $2(2^x+1)^{\frac{7t-9}{11}}(2^y(2^x+1)^t + 1)$ *where $2^x + 1, 2^y(2^x+1)^t + 1$ are primes and x, y, t are positive integers with $x + y = 6$.*
- $2(2^x+1)^{\frac{5t-9}{13}}(2^y(2^x+1)^t + 1)$, *where $2^x + 1$, $2^y(2^x+1)^t + 1$ are primes and x, y, t are positive integers with $x + y = 4$.*

Proof. Let $n = 2p^b q$, where $b \geq 2$ and $3 \leq p < q$. Since $n \in \mathbb{S}$, we have

$$9\,(2b+1) = d\left((p-1)\,(q-1)\,p^{b-1}\right) \tag{23}$$

Case 1. Assume that $(q-1, p) = 1$. It follows from (23) that $9 = (d\,((p-1)\,(q-1)) - 18)\,b$. We distinguish the following subcases:

Case 1.1. $b = 9$ and $d\,((p-1)\,(q-1)) = 19$. Thus we must have $(p-1)\,(q-1) = 2^{18}$ and p, q are Fermat primes. Hence, $n = 2F_1^9 F_4$.

Case 1.2. $b = 3$ and $d\,((p-1)\,(q-1)) = 21$. Here we have the following possibilities:

- $(p-1)\,(q-1) = 2^{20}$ and p, q are Fermat number. As above, $n = 2F_2^3 F_4$.
- $(p-1)\,(q-1) = 2^6 r^2$, where $r \geq 3$ is prime. By Lemma 2.1, $2^2 \cdot r + 1$ and $2^4 \cdot r^2 + 1$ cannot be simultaneously primes. Thus, n is one of the numbers:
  - $n = 2 \cdot 5^3 (2^4 r^2 + 1)$, where $2^4 \cdot r^2 + 1$ is prime. For example, for $r = 29$ we have $n = 2 \cdot 5^3 \cdot 13457$.
  - $n = 2 \cdot 17^3 (2^2 r^2 + 1)$, where $2^2 \cdot r^2 + 1$ is prime. For example, for $r = 3$ we have $n = 2 \cdot 17^3 \cdot 37$.
  - $n = 2(2r+1)^3 (2^5 r + 1)$, where $2 \cdot r + 1$ and $2^5 \cdot r + 1$ are both prime. For example, for $r = 3$ we have $n = 2 \cdot 7^3 \cdot 97$.
- $(p-1)\,(q-1) = 2^2 r^6$, where $r \geq 3$ is prime with $(r, q) = 1$. Thus, n is one of the numbers:
  - $n = 2(2 \cdot r + 1)^3 (2 \cdot r^5 + 1)$, where $2 \cdot r + 1$ and $2r^5 + 1$ are both prime. For example, for $r = 3$ we get $n = 2 \cdot 7^3 \cdot 487$.
  - $n = 2(2 \cdot r^2 + 1)^3 (2 \cdot r^4 + 1)$, where $2 \cdot r^2 + 1$ and $2 \cdot r^4 + 1$ are both prime. By Lemma 2.1, $r = 3$ is the only solution for this case, hence we get $n = 2 \cdot 19^3 \cdot 163$.

Case 2. Assume that $(q-1, p) = p$. We put $p - 1 = 2^x m_1$ and $q - 1 = 2^y p^t m_2$, where $x, y \geq 1$, $t \geq 1$ and $(2p, m_1 m_2) = 1$. Let $m = m_1 m_2$, it follows from (23) that

$$(18 - d\,(m)\,(x+y+1))\,b = (d\,(m)\,(x+y+1)\,t - 9)\,. \tag{24}$$

We observe that $d\,(m)\,(x+y+1)$ is odd and cannot be ≥ 18, so we have the following possibilities:

Case 2.1. $d\,(m)\,(x+y+1) = 17$. That is, $m = 1$ and $x + y = 16$. So, p is a Fermat prime. By (24), $b = 17t - 9$ and therefore $n = 2(2^x+1)^{17t-9}(2^y(2^x+1)^t + 1)$, where $(2^x + 1)$ and $(2^y(2^x+1)^t + 1)$ are primes. For example, for $x = 1$, $t = 4$ and $y = 15$ we have $n = 2 \cdot 3^{59} \cdot 2654209$.

Case 2.2. $d\,(m)\,(x+y+1) = 15$. There are three possibilities:

- $d\,(m) = 1$ and $x + y = 14$. So, $m_1 = m_2 = 1$ and p is a Fermat prime. By (24), $b = 5t - 3$, in which case $n = 2(2^x+1)^{5t-3}(2^y(2^x+1)^t + 1)$, where $2^x + 1$ and $2^y(2^x+1)^t + 1$ are primes. For example, for $x = 1$, $y = 13$, $t = 5$ we have $n = 2 \cdot 3^{22} \cdot 1990657$.
- $d\,(m) = 3$ and $x + y = 4$. Therefore, $m = r^2$, where $r \geq 3$ is prime. From (24), $b = 5t - 3$. By Lemma 2.1, $2 \cdot 73^t + 1$ is composite. Also, by Remark 2.2, the number $2^3 \cdot 19^t + 1$ is composite. Thus, n is one of the numbers:

- $\circ$ $n = 2 \cdot 3^{5t-3}(2^3 \cdot 3^t \cdot r^2 + 1)$, where $2^3 \cdot 3^t \cdot r^2 + 1$ is prime. For example, for $r = 7$ and $t = 2$ we obtain $n = 2 \cdot 3^7 \cdot 3529$.
- $\circ$ $n = 2 \cdot 5^{5t-3}(2^2 \cdot 5^t \cdot r^2 + 1)$, where $2^2 \cdot 5^t \cdot r^2 + 1$ is prime. For example, for $r = 7$ and $t = 4$ we obtain $n = 2 \cdot 5^{17} \cdot 122501$.
- $\circ$ $n = 2(2r+1)^{5t-3}(2^3 \cdot r \cdot (2r+1)^t + 1)$, where $2r + 1$ and $2^3 \cdot r \cdot (2r+1)^t + 1$ are primes. For example, for $r = t = 3$ we have $n = 2 \cdot 7^{12} \cdot 8233$.
- $\circ$ $n = 2(4r+1)^{5t-3}(2^2 \cdot r \cdot (4r+1)^t + 1)$, where $4r + 1$ and $2^2 \cdot r \cdot (4r+1)^t + 1$ are primes. For example, for $r = 3$ and $t = 1$ we have $n = 2 \cdot 13^2 \cdot 157$.
- $\circ$ $n = 2(8r+1)^{5t-3}(2 \cdot r \cdot (8r+1)^t + 1)$, where $8r + 1$ and $2 \cdot r \cdot (8r+1)^t + 1$ are primes. For example, for $r = 5$ and $t = 2$ we get $n = 2 \cdot 41^7 \cdot 16811$.
- $\circ$ $n = 2(4r^2+1)^{5t-3}(4 \cdot (4r^2+1)^t + 1)$, where $4r^2 + 1$ and $4 \cdot r \cdot (4r^2+1)^t + 1$ are primes. For example, for $r = 7$ and $t = 6$, we get $n = 2 \cdot 197^{27} \cdot 233806913236517$.

- $d(m) = 5$ and $x = y = 1$. Thus $m = r^4$, where $r \geq 3$ is prime. It follows from (24) that $b = 5t - 3$. Therefore, by Lemma 2.1, n is one of the numbers:
  - $\circ$ $n = 2 \cdot 3^{5t-3}(2 \cdot 3^t \cdot r^4 + 1)$, where $2 \cdot 3^t \cdot r^4 + 1$ is prime. For example, if $r = 5$ and $t = 2$ then $n = 2 \cdot 3^7 \cdot 11251$.
  - $\circ$ $n = 2(2r+1)^{5t-3}(2 \cdot r^3 \cdot (2r+1)^t + 1)$, where $2r + 1$ and $2 \cdot r^3 \cdot (2r+1)^t + 1$ are primes. For example, for $r = 3$ and $t = 1$ we have $n = 2 \cdot 7^2 \cdot 379$.
  - $\circ$ $n = 2 \cdot 19^{5t-3}(2 \cdot 19^t \cdot r^2 + 1)$, where $2 \cdot 19^t \cdot r^2 + 1$ is prime. For example, if $r = 3$ and $t = 29$, then $n = 2 \cdot 19^{142} \cdot 218336795902605993201009018384568383223$.
  - $\circ$ $n = 2(2r^3+1)^{5t-3}(2 \cdot r \cdot (2r^3+1)^t + 1)$, where $2r^3 + 1$ and $2 \cdot r \cdot (2r^3+1)^t + 1$ are primes. For example, if $r = 5$ and $t = 12$, then
  $$n = 2 \cdot 251^{57} \cdot 62529457064557415999535378011.$$
  - $\circ$ $n = 2(2r^4+1)^{5t-3}(2 \cdot (2r^4+1)^t + 1)$, where $2r^4 + 1$ and $2 \cdot (2r^4+1)^t + 1$ are primes.

Case 2.3. $d(m)(x + y + 1) = 13$. It follows that $m = 1$ and $x + y = 12$, which gives that p is a Fermat prime. By (24), $5b = 13t - 9$ and so $n = 2(2^x+1)^{(13t-9)/5}(2^y(2^x+1)^t + 1)$, where $2^x + 1$ and $2^y(2^x+1)^t + 1$ are primes, with $5|13t - 9$.

Case 2.4. $d(m)(x + y + 1) = 11$. Obviously, $m = 1$ and $x + y = 12$. So, p is a Fermat prime. By (24), $7b = 11t - 9$ and therefore $n = 2(2^x+1)^{(11t-9)/7}(2^y(2^x+1)^t + 1)$, where $2^x + 1$ and $2^y(2^x+1)^t + 1$ are both prime, with $7|11t - 9$.

Case 2.5. $d(m)(x + y + 1) = 9$. There are two possibilities to consider:

- $d(m) = 1$ and so $x + y = 8$. Thus, $m_1 = m_2 = 1$ and q_2 is a Fermat prime By (24), $b = t - 1$, from which it follows that $n = 2(2^x+1)^{t-1}(2^y(2^x+1)^t + 1)$, where $2^x + 1$ and $2^y(2^x+1)^t + 1$ are prime. For example, for $x = 2$, $y = 6$ and $t = 14$ we have $n = 2 \cdot 5^{13} \cdot 390625000001$.
- $d(m) = 3$ and $x + y = 2$. So, $m_1 m_2 = r^2$, where r is odd prime and $(p, r) = 1$, $x = y = 1$ and $b = t - 1$. Therefore, by Lemma 2.1, it follows that n is one of the numbers:

- $n = 2 \cdot 3^{t-1}(2 \cdot 3^t \cdot r^2 + 1)$, where $2 \cdot 3^t \cdot r^2 + 1$ is prime. For example, for $r = 5$ and $t = 4$ we get $n = 2 \cdot 3^3 \cdot 4051$.
- $n = 2(2r+1)^{t-1}(2 \cdot r \cdot (2r+1)^t + 1)$, where $2r+1$ and $2 \cdot r \cdot (2r+1)^t + 1$ are primes. For example for $r = 3$ and $t = 4$, we have $n = 2 \cdot 7^3 \cdot 14407$.
- $n = 2 \cdot 19^{t-1}(2 \cdot 19^t + 1)$. By Lemma 2.1, the number $2 \cdot 19^t + 1$ is divisible by 3.

Case 2.6. $d(m)(x+y+1) = 7$, it follows that $d(m) = 1$, and $(x+y+1) = 7$. So, $m_1 = m_2 = 1$, p is Fermat numbers, $x + y = 6$. From (24), $11b = 7t - 9$ and hence

$$n = 2(2^x + 1)^{(7t-9)/11}(2^y(2^x+1)^t + 1),$$

where $2^x + 1$ and $2^y(2^x+1)^t + 1$ are primes, with $11|7t - 9$. For example, for $x = 4$, $y = 2$ and $t = 6$ we have $n = 2 \cdot 17^3 \cdot 96550277$.

Case 2.7. $d(m)(x+y+1) = 5$. It follows that $m_1 = m_2 = 1$ and $x + y = 4$. So, p is a Fermat prime and by (24), $13b = 5t - 9$. Therefore, $n = 2(2^x+1)^{(5t-9)/13}(2^y(2^x+1)^t + 1)$, where $(2^x + 1)$ and $(2^y(2^x+1)^t + 1$ are both prime with $13|5t - 9$. For example, for $x = 1$, $y = 3$ and $t = 7$ we have $n = 2 \cdot 3^2 \cdot 17497$. $\square$

Proposition 3.8. *The only possible solutions of the form $2pq^c$, with $c \geq 2$ and $3 \leq p < q$ are:*

- $2F_1F_4^9$, $2F_2F_4^3$, $2 \cdot 3 \cdot 1459^3$, $2 \cdot 19 \cdot 163^3$.
- $2 \cdot 5 \cdot (2^4r^2 + 1)^3$, *where r and $2^4r^2 + 1$ are odd primes.*
- $2 \cdot 17 \cdot (2^2r^2 + 1)^3$, *where r and $2^2r^2 + 1$ are odd primes.*
- $2(4r+1)(2^4r + 1)^3$, *where r, $4r + 1$ and $2^4r + 1$ are odd primes.*
- $2(2r+1)(2r^5 + 1)^3$, *where r, $2r + 1$ and $2r^5 + 1$ are odd primes.*

Proof. Let $n = 2pq^c$ where $c \geq 2$ and $3 \leq p < q$. Since $n \in \mathbb{S}$, we have

$$9 = (d((p-1)(q-1)) - 18)\,c.$$

We distinguish the following cases:

Case 1. $d((p-1)(q-1)) = 19$ and $c = 9$. It follows that $(p-1)(q-1) = 2^{18}$ and so p, q are Fermat primes. Then $n = 2F_1F_4^9$ is the only solution.

Case 2. $d((p-1)(q-1)) = 21$ and $c = 3$. Here $(p-1)(q-1)$ is either 2^{20}, $2^6 \cdot r^2$ or $2^2 \cdot r^6$ where $r \geq 3$ is prime. We study these subcases separately.

Case 2.1. $(p-1)(q-1) = 2^{20}$. Then p, q are Fermat primes, in which case $n = 2F_2F_4^3$.

Case 2.2. $(p-1)(q-1) = 2^6 \cdot r^2$. Then n is one of the numbers:

- $n = 2 \cdot 5(2^4 \cdot r^2 + 1)^3$, where $2^4 \cdot r^2 + 1$ is also prime. For example, for $r = 5$ we have $n = 2 \cdot 5 \cdot 401^3$.
- $n = 2 \cdot 17(2^2 \cdot r^2 + 1)^3$, where $2^2 \cdot r^2 + 1$ is prime. For example, for $r = 3$ we get $n = 2 \cdot 17 \cdot 37^3$.
- $n = 2(2^2 \cdot r + 1)(2^4 \cdot r + 1)^3$, where $2^2 \cdot r + 1$ and $2^4 \cdot r + 1$ are prime. For example, for $r = 7$ we get $n = 2 \cdot 29 \cdot 113^3$.

Case 2.3. $(p-1)(q-1) = 2^2 \cdot r^6$. Then n is one of the numbers:

- $n = 2 \cdot 3(2 \cdot r^6 + 1)^3$, where $2 \cdot r^6 + 1$ is prime. By Lemma 2.1, $r = 3$ is the only possible value, i.e., $n = 2 \cdot 3 \cdot 1459^3$.
- $n = 2(2 \cdot r^2 + 1)(2 \cdot r^4 + 1)^3$, where $2 \cdot r^2 + 1$ and $2 \cdot r^4 + 1$ are primes. By Lemma 2.1, we get $r = 3$ and so $n = 2 \cdot 19 \cdot 163^3$.
- $n = 2(2 \cdot r + 1)(2 \cdot r^5 + 1)^3$, where r, $2 \cdot r + 1$ and $2 \cdot r^5 + 1$ are primes. The first primes r with these properties are $r = 3, 23, 29, 53, 251, 443, 953, \ldots$. □

4 Are there infinitely many n such that $d\left(n^2\right) = d\left(\varphi\left(n\right)\right)$?

The crucial question that remains: Is the set $\mathbb{S}$ infinite? The answer to this question seems difficult because we have, in the previous section, a system of polynomials in which, for a given prime p, each polynomial must takes in p a value which is also a prime number.

Recall that Dickson's Conjecture was formulated by Leonard Dickson in [5]: *Let $s \geq 1$ and let $f_i(x) = a_i \cdot x + b_i$ with a_i, b_i integers, $b_i \geq 1$ for $i = 1, 2, \ldots, s$. If there does not exist any integer $n > 1$ dividing all the products $f_1(k) f_2(k) \cdots f_s(k)$, for every integer k, then there exist infinitely many natural numbers m such that all numbers $f_1(m), f_2(m), \ldots, f_s(m)$ are prime.*

As in [3], let us take the system of integer valued polynomials whose leading coefficients are positive:

$$\begin{cases} f_1(x) = x, \\ f_2(x) = 4x + 1, \\ f_3(x) = 16x + 1. \end{cases}$$

Assume further that there exists an integer $n > 1$ which is a common divisor for the integers

$$f_1(k) f_2(k) f_3(k), \text{ for } k \in \mathbb{Z}.$$

That is, n is a common divisor for the integers $k(4k+1)(16k+1)$, $k \in \mathbb{Z}$. Then n is a common divisor for the integers $(n+1)(4n+5)(16n+17)$. Since $n \nmid (n+1)$, n divide $4n+5$ or n divide $16n+17$. This means that $n = 5$ or $n = 17$. But either $n = 5$ or $n = 17$ does not divide $f_1(2) f_2(2) f_3(2) = 2 \cdot 3^3 \cdot 11$. So there is no integer $n > 1$ which is a common divisor for the integers $f_1(k) f_2(k) f_3(k)$, $k \in \mathbb{Z}$. Consequently, we have the following result:

Theorem 4.1. *Assuming Dickson's conjecture, there exist infinitely many primes p such that $4p+1$ and $16p+1$ are primes.*

Corollary 4.1. *There exist infinitely many positive integers n such that $n \in \mathbb{S}$.*

Proof. Recall that the integer $n = p(4p+1)(16p+1)$ with p, $4p+1$ and $16p+1$ primes are in $\mathbb{S}$. Since, by the above theorem there exist infinitely many primes p such that $4p+1$ and $16p+1$ are primes. Thus, we have infinitely many integers $n = p(4p+1)(16p+1) \in \mathbb{S}$. □

We also use Dickson's conjecture to create families of prime numbers

$$\begin{cases} f_1(x) &= a_1x+1 \\ f_2(x) &= a_2x+1 \\ &\vdots \\ f_s(x) &= a_sx+1 \end{cases}$$

where $a_1, \ldots, a_s$ are positive integers. We can easily check that the above polynomials verify Dickson's hypothesis. Indeed, suppose that there exists an integer $n > 1$ such that n is a common divisor of all the integers $f_1(k) f_2(k) \cdots f_s(k)$, $k \in \mathbb{Z}$. Then $n \mid f_1(0) f_2(0) \cdots f_s(0)$, i.e., $n \mid 1$ which implies that $n = 1$. Then there exist infinitely many positive integers n such that $f_1(n), f_2(n), \ldots, f_s(n)$ are simultaneously primes.

5 Miscellaneous examples

In the following, we present some examples of solutions that cannot be deduced from the previous theorems and propositions.

Example 5.1. *The set* $\mathbb{S}$ *contains the following numbers:*

1. $n = F_1^a \cdot F_2^b \cdot F_3$, $n = F_1^a \cdot F_2 \cdot F_3^b$ *and* $n = F_1 \cdot F_2^a \cdot F_3^b$, *where* F_n *is the* n*-th Fermat prime and* $(a, b) = (3, 7)$ *or* $(7, 3)$.
2. $n = F_1^a \cdot F_2^b \cdot F_3 \cdot F_4$, $n = F_1^a \cdot F_2 \cdot F_3^b \cdot F_4$, $F_1^a \cdot F_2 \cdot F_3 \cdot F_4^b$, $F_1 \cdot F_2^a \cdot F_3^b \cdot F_4$, $F_1 \cdot F_2^a \cdot F_3 \cdot F_4^b$ *and* $F_1 \cdot F_2 \cdot F_3^a \cdot F_4^b$, *where* $(a, b) = (3, 7)$.

Example 5.2. *We have:*

1. *Let* p, q, r *be distinct primes such that* $2p+1$, $4q+1$ *and* $2pqr^2+1$ *are prime. Then* $n = 2 \cdot 17 \cdot (2p+1)(4q+1)(2pqr^2+1) \in \mathbb{S}$.
2. *Let* $q_1, q_1, \ldots, q_k$ *be distinct primes such that* $4q_1+1, \ldots, 4q_k+1$ *and* $4q_1 \cdots q_k + 1$ *are prime for some* $k \geq 1$. *If* $n = 2(4q_1+1) \cdots (4q_k+1)(4q_1 \cdots q_k + 1) \in \mathbb{S}$, *then* $k = 3$. *For example, for* $(q_1, q_2, q_3) = (7, 13, 57)$ *we get*

$$n = 2(4q_1+1)(4q_2+1)(4q_3+1)(4q_1q_2q_3+1) = 2 \cdot 29 \cdot 53 \cdot 229 \cdot 20749 \in \mathbb{S}.$$

3. *If* $n \geq 7$ *is the product of safe primes**, *then* $n \notin \mathbb{S}$.

Let $q_1, q_1, \ldots, q_k$ be Sophie Germain primes for some $k \geq 1$ such that $2q_1 \cdots q_k + 1$ is also prime.

*A prime p is said to be a *Sophie Germain prime* if $2p+1$ is also a prime, in which case, this last prime is called a *safe prime*. It has been conjectured that there are infinitely many Sophie Germain primes, but this remains unproved, see [9].

Example 5.3. *We have:*

- *If* $n = 2 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \in \mathbb{S}$, *then* $k = 7$. *For example, if*
$$(q_1, q_2, q_3, q_4, q_5, q_6, q_7) = (3, 5, 11, 23, 29, 41, 131),$$
then $n = 2 \cdot (2q_1 + 1) \cdots (2q_7 + 1) (2q_1 \cdots q_7 + 1) = 253470367109666245154 \in \mathbb{S}$.
- *If* $n = 2 \cdot F_0 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \in \mathbb{S}$, *then* $k = 24$. *For example, let* q_i *(*$1 \le i \le 24$*) be the following Sophie Germain primes:* 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131, 173, 179, 191, 233, 239, 251, 281, 293, 359, 719, 1439, 1481, 3413. *After computation, the number:*
$$\begin{aligned} n &= 2 \cdot F_0 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \\ &= 2 \cdot 3 \cdot 7 \cdot 11 \cdot 23 \cdot 47 \cdot 59 \cdot 83 \cdot 107 \cdot 167 \cdot 179 \cdot 227 \cdot 263 \\ &\quad \cdot 347 \cdot 359 \cdot 383 \cdot 467 \cdot 479 \cdot 503 \cdot 563 \cdot 587 \cdot 719 \cdot 1439 \cdot 2879 \cdot 2963 \\ &\quad \cdot 6827 \cdot 668385166547574839150402388419262454473804930401971. \end{aligned}$$
is an element of $\mathbb{S}$.
- *If* $n = 2 \cdot F_0 \cdot F_2 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \in \mathbb{S}$, *then* $k = 74$. *Let us take* $q_1, q_2, \ldots, q_{73}$ *be the first odd Sophie Germain primes. That is,* $(q_1, q_2, \ldots, q_{73}) = (3, 5, \ldots, 2945)$. *Then the result holds for* $q_{74} = 3863$.
- *If* $n = 2 \cdot F_0 \cdot F_1 \cdot F_2 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \in \mathbb{S}$, *then* $k = 234$.
- *If* $n = 2 \cdot F_0 \cdot F_1 \cdot F_2 \cdot F_3 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \in \mathbb{S}$, *then* $k = 712$.
- *If* $n = 2 \cdot F_0 \cdot F_1 \cdot F_2 \cdot F_3 \cdot F_4 \cdot (2q_1 + 1) \cdots (2q_k + 1) (2q_1 \cdots q_k + 1) \in \mathbb{S}$, *then* $k = 2153$.

6 Conclusion

As a conclusion, the different results that we have proved give rise to diophantine equations that deserve to be studied. Here, we give some examples.

1. In Proposition 2.1, we need to find primes p such that $4p^2 + 1$ is also prime.
2. In Theorem 2.5, we need to solve the system:
$$\begin{cases} 2 \cdot 3^t + 1 & \text{is prime} \\ ab + 2a + 2b + 1 = 3bt & \end{cases},$$
where a, b, t are non-negative integers.
3. In Proposition 2.6, we need to solve the system:
$$\begin{cases} p & \text{is prime} \\ 2^t p^4 + 1 & \text{is prime} \\ t & \text{positive integer} \end{cases}.$$
4. In Proposition 2.8, we need to solve the system:
$$\begin{cases} p & \text{is prime} \\ 2^s p^2 + 1 & \text{is prime} \\ ab + 2a + 2b + 1 = 3bs & \end{cases},$$
where a, b, s are positive integers.

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