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New properties of divisors of natural number

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Abstract: The divisors of a natural number are very important for several areas of mathematics, representing a promising field in number theory. This work sought to analyze new relations involving the divisors of natural numbers, extending them to prime numbers. These are relations that may have an interesting application for counting the number of divisors of any natural number and understanding the behavior of prime numbers. They are not a primality test, but they can be a possible tool for this and could also be useful for understanding the Riemann's zeta function that is strongly linked to the distribution of prime numbers.
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1 Introduction

The divisors establish, together with the divisor function, a link with the Riemann's zeta function, Eisenstein's series, Lambert's series, Euler's totient function among others [6]. Ordinary integers were studied by [2] who used integer divisors and their representation in prime factors in the definition and [5] studied the averages between the divisors of whole numbers, bringing some very interesting results.

One of the most intriguing problems in number theory is finding the divisors of an integer or its number as this will have implications for cryptography and other areas of knowledge. The divisors of an integer are important because they also relate to the partition of an integer [1].

Given the importance of understanding the behavior of the divisors of any natural number, we establish as the objective of this work the analysis of some relations involving these divisors, extending the analyzes to the prime numbers.

2 Results

Theorem 1. *Let $N \in \mathbb{N}$ with D divisors $d_1, d_2, \ldots, d_D$ being $d_1 = 1$ and $d_D = N$ and $d_1 < d_2 < \cdots < d_D$. If $H = d_1^1 + d_2^2 + \cdots + d_D^D$, we can say that*

$$\lfloor \log_N H \rfloor = D. \tag{1}$$

Proof. Let us remember two properties of floor function:

$$\lfloor k + x \rfloor = k + \lfloor x \rfloor, \forall k \in \mathbb{Z},$$
$$\lfloor \log_p (x \cdot y) \rfloor = \lfloor \log_p x + \log_p y \rfloor, \text{ if } x, y > 0.$$

If N is a prime number (two divisors) we have:

$$\begin{aligned}\lfloor \log_N (1 + N^2) \rfloor &= \left\lfloor \log_N N^2 \cdot \left(\frac{1}{N^2} + 1\right) \right\rfloor \\ &= \left\lfloor \log_N N^2 + \log_N \left(\frac{1}{N^2} + 1\right) \right\rfloor \\ &= 2 + \left\lfloor \log_N \left(\frac{1}{N^2} + 1\right) \right\rfloor.\end{aligned}$$

If $z = \frac{1}{N^2} + 1$, we have $1 < z < N$ and we conclude that $\lfloor \log_N z \rfloor = 0$ because, by [7]

$$\lfloor \log_b x \rfloor = \lfloor \log_b \lfloor x \rfloor \rfloor \Longrightarrow \lfloor \log_N z \rfloor = \lfloor \log_N \lfloor z \rfloor \rfloor = \lfloor \log_N 1 \rfloor = 0.$$

Therefore we have:

$$\lfloor \log_N (1 + N^2) \rfloor = 2 + 0 = 2 = D.$$

If we do this for any natural number, we have:

$$H = d_1^1 + d_2^2 + \cdots + d_D^D = d_D^D \cdot \left(\frac{d_1^1}{d_D^D} + \frac{d_2^2}{d_D^D} + \cdots + 1\right).$$

Using the previous reasoning, we can do:

$$\begin{aligned}\lfloor \log_N H \rfloor &= \left\lfloor \log_N d_D^D \cdot \left(\frac{d_1^1}{d_D^D} + \frac{d_2^2}{d_D^D} + \cdots + 1\right) \right\rfloor \\ &= \left\lfloor \log_N d_D^D + \log_N \left(\frac{d_1^1}{d_D^D} + \frac{d_2^2}{d_D^D} + \cdots + 1\right) \right\rfloor \\ &= D + \left\lfloor \log_N \left(\frac{d_1^1}{d_D^D} + \frac{d_2^2}{d_D^D} + \cdots + 1\right) \right\rfloor.\end{aligned}$$

However, if $z = \frac{d_1^1}{d_D^D} + \frac{d_2^2}{d_D^D} + \cdots + 1$, we have $1 < z < N$ and we conclude that:

$$\lfloor \log_N z \rfloor = 0 \Longrightarrow \lfloor \log_N H \rfloor = D + 0 = D.$$

This completes the proof. □

Graphically we confirm this hypothesis (see Figure 1).

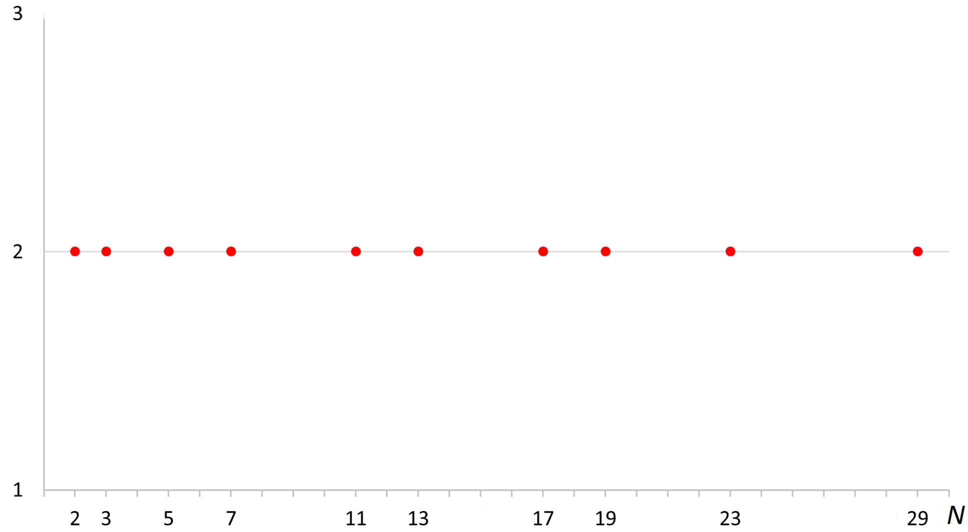

Figure 1. Visualization of $f(N) = \lfloor \log_N (1 + N^2) \rfloor$ when N is a prime number

Therefore, it is seen that the expression does not depend on the value of N (and its divisors). The result of the expression will always be equal to the number of divisors of N.

Example 1. If we use $N = 12$, whose divisors are $1, 2, 3, 4, 6, 12$,

$$H = 1^1 + 2^2 + 3^3 + 4^4 + 6^5 + 12^6 = 2994048,$$

$$\lfloor \log_{12} 2994048 \rfloor = \lfloor 6.001085 \rfloor = 6 = D.$$

Theorem 2. *Let $N \in \mathbb{N}$ with D divisors $d_1, d_2, \ldots, d_D$ being $d_1 = 1$ and $d_D = N$ and let $d_1 < d_2 < \cdots < d_D$. If $H = d_1^1 + d_2^2 + \cdots + d_D^D$ and $P = d_1 \cdot d_2 \cdot d_3 \cdots d_D$, then we have:*

$$\lfloor \log_P H \rfloor = 2. \tag{2}$$

Proof. We know that $P = N^{\frac{D}{2}}$ and $\lfloor \log_N H \rfloor = \lfloor D + \log_N z \rfloor$, with $1 < z < N \Longrightarrow \log_N H = D + \log_N z$. Therefore:

$$\begin{aligned}
\lfloor \log_P H \rfloor &= \left\lfloor \log_{N^{\frac{D}{2}}} H \right\rfloor \\
&= \left\lfloor \frac{2}{D} \cdot \log_N H \right\rfloor \\
&= \left\lfloor \frac{2}{D} \cdot (D + \log_N z) \right\rfloor \\
&= \left\lfloor 2 + \frac{2}{D} \cdot \log_N z \right\rfloor \\
&= 2 + \left\lfloor \frac{2}{D} \cdot \log_N z \right\rfloor \\
&= 2 + 0 \\
&= 2.
\end{aligned}$$

This completes the proof. $\square$

Graphically from $N \geq 2$, we have:

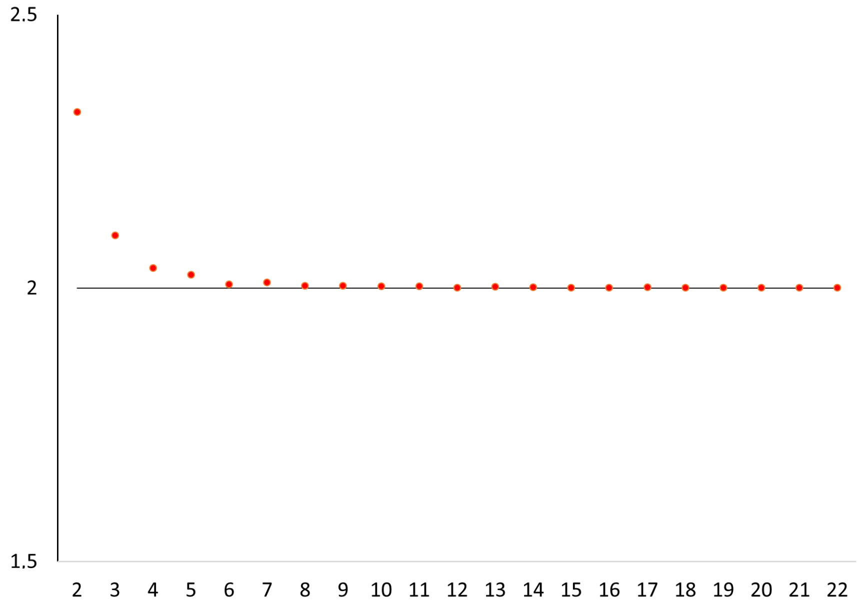

Figure 2. Visualization of $\log_P H$.

Theorem 3. *If p_n is the n-th prime number, $H_n = p_1^1 + p_2^2 + p_3^3 + \cdots + p_n^n$, $B_n = p_1 \cdot p_2 \cdot p_3 \cdots p_n$, we have*

$$\lfloor \log_{B_n} H_n \rfloor = 1 \tag{3}$$

Proof. We will use the previous ideas. We know that prime numbers are infinite [4] and that they differ from at least 1 unit (2 and 3 are the only examples), 2 units (for example, 3 and 5) and so on. Let us take some prime numbers greater than 2 and follow the previous steps.

$$H_1 = 2^1 = 2^1 \cdot (1 + 0)$$
$$H_2 = 2^1 + 3^2 = 3^2 \cdot (1 + \frac{2^1}{3^2}) \approx 3^2 \cdot (1 + 0.222)$$
$$H_3 = 2^1 + 3^2 + 5^3 = 5^3 \cdot (1 + \frac{2^1}{5^3} + \frac{3^2}{5^3}) \approx 5^3 \cdot (1 + 0.088)$$
$$H_4 = 2^1 + 3^2 + 5^3 + 7^4 = 7^4 \cdot (1 + \frac{2^1}{7^4} + \frac{3^2}{7^4} + \frac{5^3}{7^4}) \approx 7^4 \cdot (1 + 0.0566).$$

Note that the value inside the parentheses (we will indicate by z) decreases and approaches 1 as we make n large, with $z = 1.222...$ it is the largest among all. This is easy to see since the ratio between any prime number and its successor prime is always in the range $]0, 1[$. Furthermore, in reason, the antecedent is raised to a power lower than the power of the consequent. In other words:

$$0 < \frac{p_{n-1}^{n-1}}{p_n^n} = \frac{\left(\frac{p_{n-1}}{p_n}\right)^{n-1}}{p_n} < 1.$$

But if the ratio between any prime number and the successor prime is at $]0, 1[$, it is easy to conclude that the ratio between any other prime number $p_k (k \in [1, n])$ and the power p_n^n will also be in $]0, 1[$. Thus, we can express H_n as being:

$$H_n = p_1^1 + p_2^2 + p_3^3 + \cdots + p_n^n = p_n^n \cdot \big(\frac{p_1^1}{p_n^n} + \frac{p_2^2}{p_n^n} + \frac{p_3^3}{p_n^n} + \cdots + 1\big).$$

If $z = \frac{p_1^1}{p_n^n} + \frac{p_2^2}{p_n^n} + \frac{p_3^3}{p_n^n} + \cdots + 1$, we have $1 < z < B_n$, with $z \to 1$ if $n \to \infty$. So:

$$\lfloor \log_{B_n} H_n \rfloor = \left\lfloor \log_{B_n} p_n^n \cdot \left(\frac{p_1^1}{p_n^n} + \frac{p_2^2}{p_n^n} + \frac{p_3^3}{p_n^n} + \cdots + 1\right)\right\rfloor = \lfloor \log_{B_n} p_n^n + \log_{B_n} z \rfloor \tag{4}$$

In the floor function we have:

$$\lfloor x + y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor, & \text{if } 0 \leq \{x\} + \{y\} < 1 \\ \lfloor x \rfloor + \lfloor y \rfloor + 1, & \text{if } 1 \leq \{x\} + \{y\} < 2 \end{cases}. \tag{5}$$

We saw at the beginning of this proof that the decimal part of z decreases as n grows, with the largest $\{z\} = 0.2222...$ occurring with H_2. So, $\{\log_6 1.2222\} = 0.111996$ and $\{\log_{B_2} p_2^2\} = \{\log_6 9\} = 0.22629...$. We would then have $0 \leq 0.2222 + 0.22629 < 1 \to 0 \leq 0.44849 < 1$. The maximum of $\{\log_{B_n} {p_n}^n\}$ occurs when $n = 5$. In this case, we have $\{z\} = 0.002017$ and $\{\log_{B_5} p_5^5\} = \{\log_{2310} 161051\} = 0.54803$, where we get that $0 \leq 0.550047 < 1$.

When we used the largest $\{z\}$ and the largest $\{\log_{B_n} p_n^n\}$ we obtained values in the range $0 \leq \{x\} + \{y\} < 1$. So it is easy to deduce that all other cases will also be in this range. Thus, in the case of $\lfloor \log_{B_n} H_n \rfloor = \lfloor \log_{B_n} p_n^n + \log_{B_n} z \rfloor$ and remembering that $\lfloor \log_{B_n} z \rfloor = 0$, already than $1 < z < B_n$, we will use the first condition ($\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$), which results in:

$$\begin{aligned} \lfloor \log_{B_n} H_n \rfloor &= \lfloor \log_{B_n} p_n^n \rfloor + \lfloor \log_{B_n} z \rfloor \\ &= \lfloor n \cdot \log_{B_n} p_n \rfloor \\ &= \lfloor n \cdot \log_{p_1 \cdot p_2 \cdots p_n} p_n \rfloor \\ &= \left\lfloor \frac{n}{\log_{p_n} p_1 + \log_{p_n} p_2 + \cdots + 1} \right\rfloor. \end{aligned}$$

We can see that $1 \leq \log_{p_n} p_1 + \log_{p_n} p_2 + \cdots + 1 < n$, which results in:

$$1 \leq \frac{n}{\log_{p_n} p_1 + \log_{p_n} p_2 + \cdots + 1} < n.$$

Between 1 and n there can be several integers. For what

$$\left\lfloor \frac{n}{\log_{p_n} p_1 + \log_{p_n} p_2 + \cdots + 1} \right\rfloor \geq 2.$$

We should have

$$\log_{p_n} p_1 + \log_{p_n} p_2 + \cdots + 1 \leq \frac{n}{2} \to p_1 \cdot p_2 \cdots p_n \leq p_n^{\frac{n}{2}}.$$

If $n = 1$, we have $p_1 \leq p_1^{\frac{1}{2}} \to 1 \leq \frac{1}{2}$, that is false. Therefore, we see that (4) cannot be greater than 2. So the largest integer that satisfies (4) is 1, that is:

$$\lfloor \log_{B_n} H_n \rfloor = \left\lfloor \frac{n}{\log_{p_n} p_1 + \log_{p_n} p_2 + \cdots + 1} \right\rfloor = 1. \tag{6}$$

This completes the proof. $\square$

That can be viewed graphically in Figure 3.

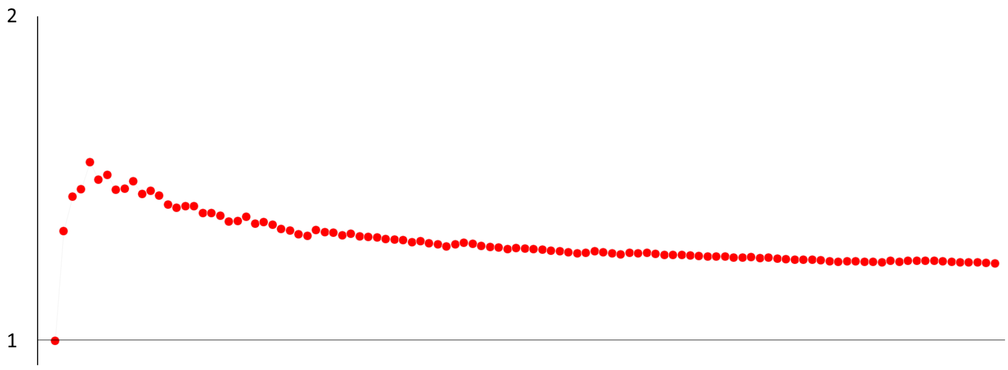

Figure 3. Visualization of $\log_{B_n} H_n$.

We conclude that for the calculation of $\log_{B_n} H_n$ just use $n \cdot \log_{B_n} p_n$ as a good approximation and if $n \to \infty$ have $\log_{B_n} H_n \approx n \cdot \log_{B_n} p_n$, that is, $H_n \approx p_n^n$. Based on all of this, we state that:

$$\lim_{n\to\infty} \frac{p_1^1 + p_2^2 + \cdots + p_n^n}{p_n^n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{n}{1 + \log_{p_n}(p_1 \cdot p_2 \cdots p_{n-1})} \approx 1.$$

Theorem 4. *If $N \in \mathbb{N} > 1$ has D divisors $d(N) = (d_1, d_2, d_3, \ldots, d_D)$, then*

$$\log_N \prod_{i=1}^{D} d_i^i \approx \frac{2}{3} \cdot T_D, \tag{7}$$

where T_D is a termial of D with $T_D = \frac{D(D+1)}{2}$.

Example 2. If $N = 10, d(10) = (1, 2, 5, 10)$, then

$$\log_N \prod_{i=1}^{D} d_i^i = \log_{10}(1^1 \cdot 2^2 \cdot 5^3 \cdot 10^4) \approx 6.69,$$

$$\frac{2}{3} \cdot T_4 = \frac{2}{3} \cdot \frac{4(4+1)}{2} \approx 6.66.$$

Proof. If N has $D = 2$ divisors, then it is a prime number and it will be easy to see that $d(N) = (1, N)$ and therefore:

$$\log_N \prod_{i=1}^{2} d_i^i = \log_N(1^1 \cdot N^2) = 2 = \frac{2}{3} \cdot \frac{2(2+1)}{2}.$$

If N has $D = 3$ divisors, it will be equal to the square of a prime number a, that is, $d(N) = (1, a, a^2)$, where $N = a^2$ and therefore:

$$\log_N \prod_{i=1}^{3} d_i^i = \log_{a^2}(1^1 \cdot a^2 \cdot a^6) = 4 = \frac{2}{3} \cdot \frac{3(3+1)}{2}.$$

It is important to emphasize that every number with three divisors is square of a prime number but not every perfect square number has three divisors. Example, $N = 16 = 4^2$ but $d(16) = (1, 2, 4, 8, 16)$.

In general, if $N = a^n$, that is, a is a prime number, then the number N has $D = n+1$ divisors that are $d(N) = (1, a, a^2, a^3, \ldots, a^n)$ and therefore:

$$\begin{aligned}\log_N \prod_{i=1}^{n+1} d_i^i &= \log_{a^n} \left(1^1 \cdot a^2 \cdot a^6 \cdot a^{12} \cdot a^{20} \cdots a^{n(n+1)}\right) \\ &= \log_{a^n} a^{2+6+12+20+\cdots+n(n+1)} \\ &= \log_{a^n} a^{2\cdot\left(1+3+6+10+\cdots+\frac{n(n+1)}{2}\right)} \\ &= \frac{2\cdot\left(1+3+6+10+\cdots+\frac{n(n+1)}{2}\right)}{n}.\end{aligned}$$

We can see that $1+3+6+10+\cdots+\frac{n(n+1)}{2}$ is the sum of the n first termials (triangular numbers), that is:

$$1+3+6+10+\cdots+\frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}.$$

Therefore:

$$\log_N \prod_{i=1}^{n+1} d_i^i = \frac{2\cdot\frac{n(n+1)(n+2)}{6}}{n} = \frac{2}{3}\cdot\frac{(n+1)(n+1+1)}{2} = \frac{2}{3}\cdot T_{n+1}.$$

This completes the proof. $\square$

Although we have proved the theorem for the power of a prime number, it is emphasized that a rigorous and generic demonstration of this theorem is open. However, if we graph $\log_N \prod_{i=1}^{D} d_i^i$ and $\frac{2}{3}\cdot T_D$ for the numbers of 2 to 60 (Figure 4) we visually confirm that the graphs practically coincide (with tiny differences in some points), which indicates the possibility of this theorem being true.

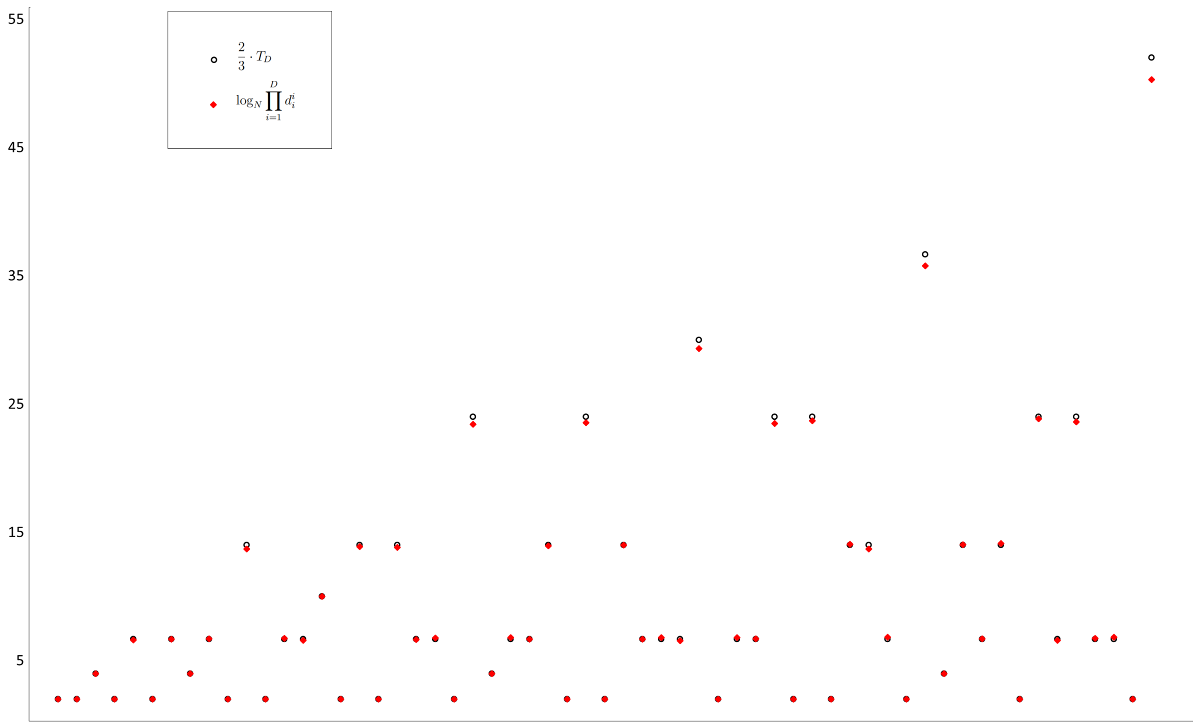

Figure 4. Visualization of $\log_N \prod_{i=1}^{D} d_i^i$ (red points) and $\frac{2}{3}\cdot T_D$ (black points) from 2 to 60.

These relationships may be useful for future work on how to determine the number of divisors for any number. So, if we know that a number has two divisors, it is evident that it is prime.

3 Conclusion

New relationships were analyzed involving natural number dividers that could serve as a tool for studying the number of these dividers. The relations have been extended to prime numbers and this may be useful in the analysis of the Riemann zeta function, cryptography and other areas involving the study of divisors.

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