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Representations of positive integers as sums of arithmetic progressions, II

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Abstract: As mentioned in the first part of this paper, our paper was motivated by two classical papers on the representations of integers as sums of arithmetic progressions. One of them is a paper by Sir Charles Wheatstone and the other is a paper by James Joseph Sylvester. Part I of the paper, though including some extensions of Wheatstone's work, was primarily devoted to extensions of Sylvester's Theorem. In this part of the paper, we will pay more attention on the problems initiated by of Wheatstone on the representations of powers of integers as sums of arithmetic progressions and the relationships among the representations for different powers of the integer. However, a large part in this portion of the paper will be devoted to the extension of a clever method recently introduced by S. B. Junaidu, A. Laradji, and A. Umar and the problems related to the extension. This is because that this extension, not only will be our main tool for study ing the relationships of the representations of different powers of an integer, but also seems

to be interesting in its own right. In the process of doing this, we need to use a few results from the first part of the paper. On the other hand, some of our results in this part will also provide certain new information on the problems studied in the first part. However, for readers who are interested primarily in the results of this part, we have repeated some basic facts from Part I of the paper so that the reader can read this part independently from the first part.

Keywords: Representation by arithmetic progressions, Junaidu–Laradji–Umar process, Inducement of representations, Complementary factors.

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1 Introduction

It is well-known that the square of an integer can be represented as the sum of a sequence of consecutive odd integers:

$$n^2 = 1 + 3 + \cdots + (2n - 1).$$

In a paper [6] published in 2010, S. B. Junaidu, A. Laradji, and A. Umar described a way of using this representation to induce a similar representation of any higher power of n. Their method can be explained by an example. For instance, $5^2 = 1 + 3 + 5 + 7 + 9$. Consider any higher power of 5, say 5^4.

$$\begin{aligned} 5^4 &= (5^4 - 5^2) + 5^2 \\ &= 5^2(5^2 - 1) + (1 + 3 + 5 + 7 + 9) \\ &= 5 \cdot 5 \cdot 24 + (1 + 3 + 5 + 7 + 9) \\ &= 5 \cdot 120 + (1 + 3 + 5 + 7 + 9) \\ &= (120 + 1) + (120 + 3) + (120 + 5) + (120 + 7) + (120 + 9). \end{aligned}$$

Thus, 5^4 can also be represented as a sum of 5 consecutive odd integers. This method can be used for any power (≥ 2) of a positive integer. It is an interesting method, but can all such representations for a higher power of 5 be induced this way? The answer is "No." This is because that, as shown by the above example, to induce a representation for higher power of 5 from that of 5^2, we need to add a certain number to each of the consecutive odd integers in order to get a sum for a larger number, and thus, any induced representation must start with an initial term greater than 1. But 5^4, being a square itself, can be represented as a sum of consecutive odd integers beginning with 1. Consequently, it cannot be induced from such a representation for 5^2. In general, how can we tell which representations for a higher power N^k are induced from a representation from that for N^2? There are also other related questions. Does this inducement work from some other power instead of a square? If so, from what power to what power? What about other kind of sums instead of consecutive odd numbers? What about consecutive even integers? Or something more general? To answer such questions, we will look at the problem in the framework of arithmetic progressions. In the following, by a *representation* for a positive integer N, we mean a way of writing N as the sum of an arithmetic progression. Two representations are said to be *similar*, if both consist of the same number of terms r and are for the same common difference d. We will find conditions under which a representation for a lower power of an integer can induce a similar representation for a higher power and conditions for a representation of a higher power which is induced from that of a lower power of the integer.

As we observed in Part I of this paper [4], the problem of representing the power of an integer seems an interesting problem in its own right, since as early as 1844, Sir Charles Wheatstone, a fellow of the Royal Society of London and a pioneer in developing telegraph, already published a paper [8], investigating various ways of representing a power n^k for $k \leq 4$ as sums of arithmetic progressions. It turns out the extension we developed from Junaidu, Laradji and Umar is an effective way of studying the representations of different power of an integer. In [4] we have made some extensions of Wheatstone's work. This part of the paper may be considered as a continuation of the first part. In fact, some results of this part also provide new information for the first part (see especially Theorem 3.2, Remark 3.3, and the observations in the last section of this paper). On the other hand, to carry out the study of this paper, we need to rely on certain results of the first part. However, for the convenience of readers who are interested in the results of this part of the paper only, we have repeated some basic definitions. There are only three places in this paper that depend on the first part. We have stated explicitly these results without proofs (Theorem 2.1, Remarks 5.1 and 5.2). If the reader is willing to accept these results, he/she can read this paper independently from the first part. For the past work, the reader might be interested in the essential ideas of the two classical papers of Carlitz [2] and Horadam [5], though they are not directly related to the results of this paper.

In the next section, we will collect some preliminaries that will be needed later. In Section 3, we will establish conditions, under which, a representation of a power of an integer can induce a similar representation for a higher power of the integer. In Section 4, we will consider the reverse problem: what conditions allow us to determine which representations of a higher power of an integer are induced from those for a lower power. In Section 5, we will apply the results of the preceding sections to study the relationships between the representations of two different powers of a prime. In our final section, we will make some observations, including how results of this paper can provide information for the first part.

2 Preliminaries

We first state a theorem of Part I that will be needed later. It describes certain criteria for an integer to have a representation. For its proof, see [4, Theorem 2.1]. In the following, if an integer $N = rs$, we will call r and s a pair of *complementary factors* in N. For any positive integers a, r and d, we will let $S(a, r, d)$ be the sum of the arithmetic progression beginning with the term $a \geq 1$, consisting of r terms, and having the common difference d. We will require $r > 1$ and $d > 0$ to rule out the trivial progressions.

Theorem 2.1. *Let N be a positive integer. The existence of a representation $N = S(a, r, d)$, depending on whether $r|N$, can be characterized by exactly one of the following two cases:*

1) *$r|N$. In this case, either d is even or r is odd and $\frac{1}{2}(r-1)d < s$, where s is the complementary factor of r in N. Conversely, if $N = rs$ for some integers r and s such that $r > 1$ and*

$\frac{1}{2}(r-1)d < s$, *then* $N = S(a, r, d)$ *for some integer* $a \geq 1$. *Furthermore, in this case, the first term* a *of the representation* $N = S(a, r, d)$ *is* $a = s - \frac{1}{2}(r-1)d$.

2) $r \nmid N$. *In this case,* d *is odd and* r *is even. Write* $r = 2r_0$, *then* $r_0|N$. *Let* s_0 *be the complementary factor of* r_0, *then* s_0 *is an odd integer with* $s_0 > (2r_0 - 1)d$. *Conversely, if* r_0 *and* s_0 *are a pair of complementary factors of* N *satisfying the following conditions: a)* $r = 2r_0$ *does not divide* N *and b)* s_0 *is an odd integer satisfying* $s_0 > (2r_0 - 1)d$, *then* $N = S(a, r, d)$ *for some integer* $a \geq 1$. *In this case, the first term of the representation* $N = S(a, r, d)$ *is* $a = \frac{1}{2}[s_0 - (2r_0 - 1)d]$.

To extend the method of Junaidu et al. to general arithmetic progressions, we first note that a straight forward extension may not work. For instance, $14 = 2+3+4+5$, a sum of 4 consecutive integers. We cannot use this to generate a sum of 4 consecutive integers for $14^2 (= 196)$. Since if

$$196 = a + (a+1) + (a+2) + (a+3) = 4a + 6,$$

a sum of 4 consecutive integers, then $190 = 4a$. But this is impossible since 190 is not a multiple of 4. Clearly, some conditions are necessary.

Lemma 2.1. *Let* m *and* n *be two positive integers with* $m < n$. *If* r *is an integer greater than* 1 *and* $r|(n-m)$, *say* $(n-m) = rq$, *then each representation* $m = S(a, r, d)$ *can always induce a representation* $n = S(a_1, r, d)$, *where* $a_1 = a + q$.

Proof. Since $r|(n-m)$, we may write $(n-m) = rq$, for some positive integer q. Then from the representation

$$\begin{aligned} m &= S(a, r, d) \\ &= a + (a+d) + \cdots + [a + (r-1)d], \end{aligned}$$

we may find a representation $n = S(a_1, r, d)$ as follows: let $a_1 = a + q$, then

$$\begin{aligned} n &= (n-m) + m \\ &= rq + a + (a+d) + \cdots + [a + (r-1)d] \\ &= (a+q) + [(a+q) + d] + \cdots + [(a+q) + (r-1)d] \\ &= a_1 + (a_1 + d) + \cdots + [a_1 + (r-1)d], \end{aligned}$$

a representation for n with $a_1 = a + q$. $\square$

Remark 2.1. *The process used in the lemma above agrees with that used by Junaidu, Laradji, and Umar. We will consider the inducement of representation for* m *to* n *through the relation* $n = (n-m) + m$ *an extension of the method of Junaidu et al. In fact,* $r|(n-m)$ *is not only sufficient, but can easily be proved to be necessary for such an inducement. Finally, in the above lemma, since* $a_1 = a + q$, *knowing one of these two representations, we can easily reproduce the other. This is because that both representations share the same* r *and* d. *The only information we need to produce the entire arithmetic progression is the initial term* a.

The following theorem provides a simple condition for determining whether a representation for a greater integer is induced from that of a smaller integer. It will be the main tool for us to study the inducement relationship between representations of two different integers.

Theorem 2.2. *Let m and n be two positive integers with $m<n$. A representation $m=S(a_1, r_1, d_1)$ induces the representation $n = S(a_2, r_2, d_2)$ if and only if the two representations are similar, i.e., if and only if $d_1 = d_2$ and $r_1 = r_2$. Furthermore, when $S(a_1, r_1, d_1)$ and $S(a_2, r_2, d_2)$ are similar, say $r_1 = r_2 = r$, then $n - m = r(a_2 - a_1)$.*

Proof. Suppose $m=S(a_1, r_1, d_1)$ and $n=S(a_2, r_2, d_2)$, are two representations with $d_1=d_2=d$ and $r_1 = r_2 = r$. Then for $i = 1$, or 2,

$$S(a_i, r, d) = a_i + (a_i + d) + \cdots + [a_i + (r-1)d] = ra_i + \frac{1}{2}(r-1)rd.$$

Thus,

$$n - m = S(a_2, r, d) - S(a_1, r, d) = r(a_2 - a_1).$$

Thus, if $d_1 = d_2$ and $r_1 = r_2$, then $r|(n-m)$, and by Lemma 2.1, the representation $m = S(a_1, r, d)$ induces a representation $n = S(a_0, r, d)$ for some positive integer a_0. Since both $S(a_2, r, d)$ and $S(a_0, r, d)$ have the same common difference d, the same number of terms r, and the same sum n, these two arithmetic progressions must be identical. Thus, we have proved that if $d_1 = d_2$ and $r_1 = r_2$, then $m = S(a_1, r_1, d_1)$ induces $n = S(a_2, r_2, d_2)$. The converse that if $m = S(a_1, r_1, d_1)$ induces the representation $n = S(a_2, r_2, d_2)$, then $d_1 = d_2$ and $r_1 = r_2$ are obvious since the process for one representation to induce another, as described in Lemma 2.1, does not change the values of either r or d. □

3 Inducing new representations from old

We now focus on the problems initiated by Wheatstone [8], representations for different powers of a positive integer.

Theorem 3.1. *Let n, j, k and r be positive integers such that n and $r > 1$ and $j < k$.*

1) *For an odd n, every representation $n^j = S(a, r, d)$ induces a representation for n^k.*
2) *For an even n, a representation $n^j = S(a, r, d)$ can induce a representation for n^k if and only if $r|n^j$.*

Proof. Let n, j, k and r be positive integers as specified in the theorem. Consider a representation $n^j = S(a, r, d)$ for an odd positive integer n. By Lemma 2.1, if $r|n^j$, the representation $n^j = S(a, r, d)$ can always induce a similar representation for any higher power n^k. Now assume that $r \nmid n$. By Theorem 2.1, the representation $n^j = S(a, r, d)$ must be of the kind where d is odd and r is even. Write $r = 2r_0$, then $r_0|n^j$. Also, since n is odd, $n^{k-j} - 1$ is even. Thus, $r = 2r_0$ and $n^k - n^j = n^j(n^{k-j} - 1)$ implies that $r|(n^k - n^j)$. By Lemma 2.1, the representation $n^j = S(a, r, d)$ can induce a similar representation for n^k. This proves the first assertion of the theorem.

Now, consider a representation $n^j = S(a,r,d)$ for an even positive integer n, say $n = 2^t n_0$ for some integer $t > 0$. If $r|n^j$, the representation $n^j = S(a,r,d)$ can induce a similar representation for any higher power n^k by Lemma 2.1. Now consider the case that $r \nmid n^j$. We claim that in this case, it is impossible for n^k to have a representation for the same r and d. This is because that since $r \nmid n^j$ and $n^j = S(a,r,d)$ for some integers a, r, and d, then by Theorem 2.1, r must be even and d is odd. Furthermore, if $r = 2r_0$, then $r_0|n^j$. Now suppose n^k also has a representation for the same r and d. Since r is even and d is odd, by Theorem 2.1 again, $r \nmid n^k$. But since $n^k = n^{k-j}n^j = (2^t n_0)^{k-j}n^j$ and $r = 2r_0$, clearly $r|n^k$. Contradiction. Thus, if $r \nmid n^j$, $n^j = S(a,r,d)$ cannot induce a representation for n^k. □

For integers n and r greater than 1, we will let $\#(n^k, r)$ be the number of ways n^k can be represented as the sum of an arithmetic progression consisting of r terms, and $\#(n^k)$ be the total number of ways n^k can be represented as the sum of an arithmetic progression for all possible r.

Corollary 3.1. *Let n, k and j be positive integers such that $j < k$. If n is odd, for each $r > 1$, $\#(n^k, r) > \#(n^j, r)$. If n is even, for each $r > 1$ such that $r|n^j$, $\#(n^k, r) > \#(n^j, r)$.*

Proof. This follows directly from Theorem 2.2 as soon as we can show that different representations of n^j induces different representations of n^k, but this is immediate since two arithmetic progressions of the same r, d, and sum must be the identical. □

Remark 3.1. *From the theorem above, we can conclude that the only representation $n^j = S(a,r,d)$ for a power of an integer $n > 1$ that does not induce a representation of a higher power of n are those for which n is even and $r \nmid n$, or equivalently, when n and r are both even and d is odd. In particular, the example $14 = 2+3+4+5$ cited in the first section is only a special case of this, since $14 = 2+3+4+5$ is a representation for which r and n are even but d is odd.*

Remark 3.2. *With the material developed so far, we can now answer some of the questions mentioned in the introductory section on the inducement process for sums of consecutive odd integers. First note that a sequence of consecutive odd integers is an arithmetic progression with a common difference $d = 2$ and any representation for a positive integer $M = S(a,r,2)$ must have an $r|M$. This is because that if $r \nmid M$, then by Theorem 2.1, r is even and d is odd. But for $d = 2$, this cannot happen.*

Now, consider the problem whether a representation for a power n^k, as a sum of r consecutive odd integers, is induced from a similar representation for a lower power n^j. All we need, by Theorem 2.2, is to see whether there exists a representation for n^j of the same r and $d = 2$. If such a representation exists, then its r must divide n^j as noted above. By Theorem 2.1 again, we can check the existence of such an r by finding the complementary factor s for this r in n^j and see whether it satisfies the inequality $s > \frac{1}{2}(r-1)d$. If so, such a representation fir n^j exists and it induces the one for n^k.

If n is a prime, we can say more, as shown by the following theorem. However, we will first observe that any power p^k of a prime p, with $k \geq 2$, cannot be the sum of a sequence of even integers. This is clear if p is an odd prime. Even if $p = 2$, 2^k cannot be a sum of consecutive even integers. This is because that if 2^k is such a sum, then 2^{k-1} would have been a sum of consecutive *integers, but as early as 1882, J. J. Sylvester already showed that this is impossible (see [7, Section 17, pp. 265–266], also [3, Vol. 2, Chapter 3, p. 139], and [1]). Thus, for representations for a power of prime, we may use $S(a, r, 2)$ for the sum of a sequence of odd positive integers.*

Theorem 3.2. *Let p be a prime and $k \geq 2$ be a positive integer.*

1. *A representation for p^k, as a sum of consecutive odd integers, is induced from a similar representation from p^2 if and only if the representation consists of p terms.*
2. *For an odd k, all the representations $p^k = S(a, r, 2)$ are induced from those for p^{k-1}.*
3. *For an even k, all except one, of the representations $p^k = S(a, r, 2)$ are induced from those for p^{k-1}. The only exception is the representation for p^k as the sum of the consecutive odd integers obtained when p^k is considered as a square.*
4. *p^k can be represented as a sum of consecutive odd integers in exactly $\lfloor k/2 \rfloor$ many ways.*

Proof. 1. As noted in the Remark 3.2, any representation for $p^2 = S(a, r, 2)$ can only have an r such that $r|p^2$. Since $r > 1$, $r = p$ or p^2. But $r = p^2$ cannot satisfy the condition $s > \frac{1}{2}(r-1)d$ required in Theorem 2.1. Thus, the only possible value for r is when $r = p$. On the other hand, p^2 does have a representation as the sum of consecutive odd integers consisting of $r = p$ terms, namely, the standard representation of a square as the sum of consecutive odd integers. Thus, by Theorem 2.2, a representation for p^k, as a sum of consecutive odd integers, is induced from a similar representation from p^2 if and only if the representation consists of $r = p$ terms.

2, 3. For the assertions 2 and 3, since $d = 2$ is even, we need only consider representations for which $r|p^k$. By Theorem 2.1, for this r to appear in a representation for p^k if and only if $s > \frac{1}{2}(r-1)d$. For the values of s, r and d, this condition becomes $p^{k-j} > p^j - 1$. Since both p^k and p^j are integers, this is equivalent to $p^{k-j} \geq p^j$, or $p^{k-2j} \geq 1$, or $j \leq \frac{k}{2}$.

If in the above argument, we replace k by $k-1$, we may conclude that $p^{k-1} = S(a', r, 2)$ for some integer a' and with $r = p^j$, if and only if $j \leq \frac{k-1}{2}$.

However, since j is an integer, for an odd integer k,

$$j \leq \frac{k}{2} \Leftrightarrow j \leq \frac{k-1}{2}.$$

Thus, for $d = 2$, r appears in a representation for p^k if and only if it appears in a representation for p^{k-1}. This proves the assertion 2.

Now, consider the case that k is even. Then $\frac{k}{2} = \frac{k-1}{2} + 1$. Thus, There is one more representation for p^k than for p^{k-1}. As we observed in the introductory section that any induced representation begins with an initial term greater than 1. Thus, the one representation for p^k that is not induced from the lower power is the representation when p^k is considered as a square. This proves the assertion 3.

4. As for the assertion 4, since by the proof of the first assertion, there is exactly one way to represent p^2 as a sum of consecutive odd integers, the representation consisting of p terms. From the assertions 2 and 3, each time the power of p increases from an even power to the next odd power there is no change in the number of such representations, but when the

power increase from an odd to an even one, the number of ways increase by one. Thus, for a general power p^k, the total number of way is given by the number of even integers $\leq k$, or $\lfloor k/2 \rfloor$. $\square$

Remark 3.3. *In the introductory section, we mentioned that the inducement relationships of this paper may provide new information for Part I of this paper. Theorem 3.2, especially its fourth assertion, is a good example.*

4 Which sums for a power are induced from that of a lower power?

In the last section, we have determined when a representation of a power n^j can induce a similar representation for a higher power n^k. We now determine which representations of n^k are induced from that of a lower power n^j.

Lemma 4.1. *Let $n \geq 2$ be an integer. Suppose j, h and k are three positive integers such that $j < h < k$. If a representation for $n^j = S(a, r, d)$ can induce a representation for n^k, it can do this in two different ways: either directly from n^j to n^k or from n^j through an intermediate power n^h, and then to n^k. These two representations are identical. In fact, for a power n^k, for $k \geq 2$ all the representations of n^k induced from a lower power n^j are among those induced from n^{k-1}.*

Proof. If a representation $n^j = S(a, r, d)$ can induce a representation for a higher power of n, by Theorem 3.1, either n is odd, or when n is even but $r|n^j$. These conditions still hold for any representation induced from $n^j = S(a, r, d)$ for a higher power, say n^h. Thus, the representation for n^h can itself induce a representation for a higher power n^k. The representation for n^k induced from n^j through n^h should be identical to that induced directly from n^j. This is because that both of the induced representations for n^k have the same values of d and r. Furthermore, they have the same sum n^k. Thus, they are identical. $\square$

Theorem 4.1. *Consider a representation $n^k = S(a, r, d)$, where n and k are both ≥ 2. There are two possibilities:*

1) *Either r is odd or d is even. $n^k = S(a, r, d)$ is induced from a representation for a lower power of n if and only if $r|n^{k-1}$ and the beginning term a in the representation $n^k = S(a, r, d)$ satisfy the inequality $a > (n-1)s_2$, where s_2 is the complementary factor of r in n^{k-1}.*
2) *r is even and d is odd. There are two further possibilities: a) n is even. In this case, none of the representations of n^k is induced from those of a lower power of n, and b) n is odd. In this case, let $r_0 = r/2$. The representation $n^k = S(a, r, d)$ is induced from a representation for a lower power of n if and only if $r_0|n^{k-1}$ and $a > \frac{1}{2}(n-1)s_2$, where $r_0 s_2 = n^{k-1}$.*

Proof. From Lemma 4.1 and Theorem 2.2, the representation $n^k = S(a, r, d)$ is induced from that for a lower power of n if and only if there is a representation for $n^{k-1} = S(a_1, r, d)$ with the same r and d. Such a representation $S(a_1, r, d)$, if exists, will induce the representation of $n^k = S(a, r, d)$.

Consider the case that either r is odd or d is even. Suppose the representation $n^k = S(a, r, d)$ is induced from a representation $n^{k-1} = S(a_1, r, d)$. Then since either r is odd or d is even, $r|n^{k-1}$ by Theorem 2.1, and hence, $r|n^k$. Let s_1 and s_2 be the complementary factors of r with respect to n^k and n^{k-1}, respectively. By Theorem 2.2,

$$rs_1 - rs_2 = n^k - n^{k-1} = r(a - a_1) \tag{1}$$

From $rs_1 = n^k$ and $rs_2 = n^{k-1}$, we have $s_1 = ns_2$. From this and Equation (1), we conclude that $a = (n-1)s_2 + a_1$. Since $a_1 \geq 1$, $a > (n-1)s_2$.

Conversely, if $r|n^{k-1}$ and the beginning term a of a representation $n^k = S(a, r, d)$ satisfy the condition $a > (n-1)s_2$, where s_2 is the complementary factor r in n^{k-1}. Write $a = (n-1)s_2 + a_1$, for some positive integer a_1. Then in the representation $n^k = S(a, r, d)$,

$$\begin{aligned}
n^k &= a + (a+d) + (a+2d) + \cdots + (a + (r-1)d) \\
&= ((n-1)s_2 + a_1) + [((n-1)s_2 + a_1) + d] + \cdots + [((n-1)s_2 + a_1) + (r-1)d)] \\
&= r(n-1)s_2 + [a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (r-1)d) \\
&= (n-1)n^{k-1} + [a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (r-1)d)] \\
&= n^k - n^{k-1} + [a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (r-1)d)].
\end{aligned}$$

From this we may conclude that $[a_1 + (a_1+d) + (a_1+2d) + \cdots + (a_1 + (r-1)d)]$ is a representation of n^{k-1}. Note that this representation will induce $n^k = S(a, r, d)$.

Now, consider the case that r is even and d is odd. First consider the case that n is even. In this case, none of the representations of n^k can be induced from those of a lower power of n because if there is any representation of n^{k-1} with these r and d, then this representation is one for which n and r are both even and d is odd. From Remark 3.1, we see that it cannot induce any representation for a higher power n^k.

Finally, consider the case that $r = 2r_0$ is even, d and n are both odd. If $n^k = S(a, r, d)$ is induced from a representation $n^{k-1} = S(a_1, r, d)$, then r_0 must divide both n^k and n^j. Suppose $r_0 s_1 = n^k$ and $r_0 s_2 = n^{k-1}$, we conclude that $s_1 = ns_2$. Similar to the above, we have

$$r_0 s_1 - r_0 s_2 = n^k - n^{k-1} = 2r_0(a - a_1). \tag{2}$$

From this and $s_1 = ns_2$, we conclude that $a = \frac{1}{2}(n-1)s_2 + a_1$, or $a > \frac{1}{2}(n-1)s_2$.

Conversely, if $r_0|n^{k-1}$ and if the beginning term a of a representation $n^k = S(a, 2r_0, d)$ satisfy the condition $a > \frac{1}{2}(n-1)s_2$, where s_2 is the complementary factor of r_0 in n^{k-1}. Let $a_1 = a - \frac{1}{2}(n-1)s_2$. Note that a_1 is a positive integer since n is an odd integer and $n > 1$. Thus, $a = \frac{1}{2}(n-1)s_2 + a_1$. Then in the representation $n^k = S(a, 2r_0, d)$,

$$\begin{aligned}
n^k &= a + (a+d) + (a+2d) + \cdots + (a + (2r_0 - 1)d) \\
&= (\tfrac{1}{2}(n-1)s_2 + a_1) + [(\tfrac{1}{2}(n-1)s_2 + a_1) + d] + \cdots + [(\tfrac{1}{2}(n-1)s_2 + a_1) + (2r_0 - 1)d)] \\
&= \tfrac{1}{2}(2r_0)(n-1)s_2 + [a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (2r_0 - 1)d)] \\
&= (n-1)n^{k-1} + [a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (r-1)d)] \\
&= n^k - n^{k-1} + [a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (r-1)d)].
\end{aligned}$$

From this we conclude that $[a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (r-1)d)]$ is a representation of n^{k-1} that gives rise to the representation $n^k = S(a, r, d)$. □

Remark 4.1. *If $n^k = S(a, r, d)$ is induced from a representation of n^{k-1}, we may actually construct this representation of n^{k-1}. This representation has the same r and d, and from the theorem above, we can easily find its initial term a_1. We will show in more detail how this is done in the examples in the next section.*

5 Powers of primes as sums of arithmetic progressions

In this section, we will apply our theory to compare the representations of different powers of an integer. In Part I of this paper, we have shown how to find all the representations for a given positive integer. We can now look at these representations for a given power of the integer and see which of them can induce a similar representation for a higher power of the integer and which are induced from a similar one for a lower power, and thus, compare the representations for the different powers of the integer. Since it is a little complicated to describe all the representations for the powers of a general positive integer, we will restrict our attention to representations of powers of a prime. The procedure is basically the same for a general positive integer (see the third observation in the last section).

Since our theorems specify different conditions for an even or odd integer, we will separate the case when the prime is 2 from the cases of odd primes. First consider the case when $n = 2^k$. Recall that $\#(2^k, r)$ is the number of representations $2^k = S(a, r, d)$ for a particular value of r, and $\#(2^k)$ is the total number of ways that 2^k can be represented as sums of arithmetic progressions.

Remark 5.1. *In Part I of the paper, we showed that there is a representations $2^k = S(a, r, d)$ if and only if $r = 2^j$ for some integer j, with $1 \le j \le \lfloor k/2 \rfloor$. For each such j, let l_j be the integer such that $0 \le l_j < j$ and $k \equiv l_j$ (mod j), and $\#(2^k, 2^j) = \frac{2^{k-j} - 2^{l_j}}{2^j - 1}$. Each of these representations corresponds to an even integer $d = 2d_0$ with d_0 being one of the integers $1, 2, \ldots, \frac{2^{k-j} - 2^{l_j}}{2^j - 1}$, and the initial term of the representation for such a d_0 is given by $a = 2^{k-j} - (2^j - 1)d_0$. The arithmetic progression can then be found by adding repeatedly $d(= 2d_0)$ to a until we obtain all the 2^j terms of the progression. In particular,*

$$\#(2^k) = \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{2^{k-j} - 2^{l_j}}{2^j - 1}. \tag{3}$$

Furthermore, the longest r in the representation of $2^k = S(a, r, d)$ is for $r = 2^{\lfloor k/2 \rfloor}$. For a proof, see [4, Theorem 3.2]. We repeat this result here for the convenience of readers who are interested in this part of the paper only.

Theorem 5.1. *Consider a power 2^k, where $k \ge 1$. All the representations in the summation of Formula (3) can induce a representation for 2^{k+1}. When k is even, none of the representations of the length $r = 2^{\frac{k}{2}}$ are induced from those of 2^{k-1}.*

For a general $k > 1$, regardless whether it is even or odd, for any j, with $1 \le j \le \lfloor k/2 \rfloor$, the representation $2^k = S(a, 2^j, 2d_0)$ is induced from a representation of 2^{k-1} if and only if d_0 satisfies the inequality $d_0 \le \frac{2^{k-j-1} - 2^{l'_1}}{2^j - 1}$, where l'_1 is the integer such that $0 \le l'_1 < j$ and $k - 1 \equiv l'_1$ $(\mathrm{mod}\ j)$.

Thus, among all the representations $2^k = S(a, 2^j, 2d_0)$*, the ones for which* $d_0 = 1, 2, \ldots, \frac{2^{k-j-1}-2^{l'_1}}{2^j-1}$ *are induced from that for* 2^{k-1}*, and for* $d_0 = \left(\frac{2^{k-j-1}-2^{l'_1}}{2^j-1} + 1\right), \ldots, \frac{2^{k-j}-2^{l_j}}{2^j-1}$ *are not induced from that for* 2^{k-1}

Proof. First note that all the representations in the Formula (3) are from the representations whose length r divides 2^k, and thus, by the second assertion of Theorem 3.1, all these representations can induce a representation for 2^{k+1}. Now, consider the case for an even k. We contend that none of the representations of length $r = 2^{\frac{k}{2}}$ are induced from any representation for 2^{k-1}. This is because, by Remark 5.1, any representation for 2^{k-1} can only have a length $r = 2^j$ with $j \leq \lfloor (k-1)/2 \rfloor$. Thus, there can not be any representation for 2^{k-1} of $r = 2^{\frac{k}{2}}$.

Now, consider the case for a general $k > 1$ and a representation $2^k = S(a, r, d)$ for $r = 2^j$ with $1 \leq j \leq \lfloor k/2 \rfloor$ and $d = 2d_0$ for some integer d_0. This representation is induced from one for 2^{k-1}, by Theorem 2.2, if and only if there is a representation $2^{k-1} = S(a, r, 2d_0)$ of the same r and d_0. But by Remark 5.1, such a representation for 2^{k-1} exists if and if this d_0 is an integer between 1 and $\frac{2^{k-j-1}-2^{l'_1}}{2^j-1}$, where l'_1 is the integer such that $0 \leq l'_1 < j$ and $k - 1 \equiv l'_1 \pmod{j}$. The final assertion then follows from this and Remark 5.1. □

Example 5.1. We now apply our results to the representations of $2^8 = 256$. By Theorem 5.1, $2^8 = S(a, r, d)$ can be true only for $r = 2^j$ for $j = 1, 2, 3$ and 4. We can compute the number of ways to represent 2^8 by Formula (3): First determine the value of l_j for each j. These can be determined easily since $8 \equiv 0 \pmod{1, 2, 4}$ and $8 \equiv 2 \pmod{3}$. Thus

$$\#(2^8) = \sum_{j=1}^{4} \frac{2^{k-j} - 2^{l_j}}{2^j - 1} = \frac{2^7 - 1}{2 - 1} + \frac{2^6 - 1}{2^2 - 1} + \frac{2^5 - 2^2}{2^3 - 1} + \frac{2^4 - 1}{2^4 - 1} = 127 + 21 + 4 + 1 = 153. \quad (4)$$

Since for each of these representation, $r | 2^8$, by the second assertion of Theorem 3.1, all these representations of 2^8 can induce a representation of 2^9.

A similar computation will show that $\#(2^7) = 63 + 10 + 1 = 74$. By the same reason, each of these 74 representations for 2^7 will induce a distinct representation for 2^8. Among the 153 representation of 2^8, there are exactly 74 of them that are induced from that of 2^7. In fact, we can determine which are these representations and which of them are induced from that of 2^7.

For instance, for the 4 representations of 2^8 for $r = 2^3$, or for $j = 3$, since $d_0 \leq \frac{2^{k-j}-2^{l_j}}{2^j-1} = 4$, we have $d_0 = 1, 2, 3, 4$. For each of these d_0, the first term of the progression is $a = 2^{k-j} - (2^j - 1)d_0$ $= 2^5 - 7d_0$. Thus, we have the following 4 representations of 2^8 as the sums of arithmetic progression consisting of 8 terms:

$$\begin{array}{lll}
\text{for } d = 2d_0 = 2 & a = 25: & 25 + 27 + 29 + 31 + 33 + 35 + 37 + 39 = 256 = 2^8, \\
\text{for } d = 2d_0 = 4 & a = 18: & 18 + 22 + 26 + 30 + 34 + 38 + 42 + 46 = 256 = 2^8, \\
\text{for } d = 2d_0 = 6 & a = 11: & 11 + 17 + 23 + 29 + 35 + 41 + 47 + 53 = 256 = 2^8, \\
\text{for } d = 2d_0 = 8 & a = 4: & 4 + 12 + 20 + 28 + 36 + 44 + 52 + 60 = 256 = 2^8.
\end{array}$$

On the other hand, since $\frac{2^{k-1-j}-2^{l_1}}{2^j-1} = \frac{2^4-2}{2^3-1} = 2$, among the four representations above, only the two representations for $d_0 = 1$ and 2 (or $d = 2$ and 4) are induced from those of 2^7. In fact, these two are induced, respectively, by

for $d = 2d_0 = 2 \quad a = 9: \quad 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 = 128 = 2^7,$
for $d = 2d_0 = 4 \quad a = 2: \quad 2 + 6 + 10 + 14 + 18 + 22 + 26 + 30 = 128 = 2^7.$

These are the only two representations for 2^7 consisting of $r = 8$ terms.

In Lemma 2.1, we observed that if the representation $m = S(a, r, d)$ induces a representation $n = S(a_1, r, d)$, then we have the following relations: if $q = a_1 - a$ then $n - m = rq$. We may now check this relation for the examples here. For instance, the difference for the first terms for $d = 2$ and $r = 2^3 = 8$ is

$$a_1 - a = 25 - 9 = 16 = q \text{ and } rq = 8 \times 16 = 128 = 2^8 - 2^7 = n - m.$$

This indeed confirms our observation.

Remark 5.2. *We now consider the powers p^k for an odd prime p. In the first part of the paper, we showed how to find all the representations $p^k = S(a, r, d)$ for such a power by dividing all the possibilities into two types depending on whether $r|p^k$ or not [4, Theorem 3.3]. Specifically:*

1) *$r|p^k$. The only way for $p^k = S(a, r, d)$ with $r|p^k$ is when $k > 1$ and $r = p^j$ for some j such that $1 \le j \le \lfloor k/2 \rfloor$. For each such j, let l_j be the integer such that $0 \le l_j < j$ and $k \equiv l_j$ (mod j). There are $2\left(\frac{p^{k-j}-p^{l_j}}{p^j-1}\right)$ different ways for $p^k = S(a, p^j, d)$. Each of the integers $1, 2, 3, \ldots, 2\left(\frac{p^{k-j}-p^{l_j}}{p^j-1}\right)$ gives rise to a value of d for a different arithmetic progression whose sum equals to p^k, and the initial term a of the progression for this d is given by $a = p^{k-j} - \frac{1}{2}(p^j - 1)d$.*
2) *$r \nmid p^k$. In this case, $r = 2p^j$ for some j with $0 \le j < \lfloor k/2 \rfloor$. Each odd integer $\le \left\lfloor \frac{p^{k-j}-1}{2p^j-1} \right\rfloor$ gives rise to a value of d for a distinct representation $p^k = S(a, 2p^j, d)$. These are the only possible values for d for this $r = 2p^j$. For each of these representations, the initial term a is given by $a = \frac{1}{2}(p^{k-j} - (2p^j - 1)d)$. Thus, $\#(p^k, 2p^j) = \left\lfloor \frac{1}{2}\left(\frac{p^{k-j}-1}{2p^j-1} + 1\right) \right\rfloor$, where $0 \le j < \lfloor k/2 \rfloor$.*

We are now ready to deal with the case for the powers of an odd prime.

Theorem 5.2. *Consider p^k for an odd prime p and for $k \ge 1$. Every representation for $p^k = S(a, r, d)$ can induce a representation for p^{k+1}. Now, consider the representations $p^k = S(a, r, d)$ for $k > 1$.*

1) *For the case $r|p^k$, or when $r = p^j$, with $1 \le j \le \lfloor k/2 \rfloor$, the representation $p^k = S(a, p^j, d)$ is induced from a representation for p^{k-1} if and only if $d \le 2\left(\frac{p^{k-1-j}-p^{l'_1}}{p^j-1}\right)$, where l'_1 is the integer such that $0 \le l'_1 < j$ and that $k - 1 \equiv l'_1$(mod j).*
2) *For the case $r \nmid p^k$, or for $r = 2p^j$, where $0 \le j \le \lfloor k/2 \rfloor$, the representation $p^k = S(a, 2p^j, d)$ is induced from a representation of p^{k-1} if and only if d is an odd integer $\le \left\lfloor \frac{p^{k-1-j}-1}{2p^j-1} \right\rfloor$.*

In each of the two cases, we can also determine which of the representations are induced from a lower power of p by specifying the value of d as in the case for $p = 2$.

Proof. Consider the power p^k for an odd prime p. By the first assertion of Theorem 3.1, every representation for $p^k = S(a, r, d)$ can induce a representation for p^{k+1}.

The proofs for both assertions follow from Remark 5.2 in a similar way that Theorem 5.1 follows from Remark 5.1, and will be omitted here. $\square$

Example 5.2. Consider $p^k = 7^5$. In this case, $\lfloor k/2 \rfloor = 2$.

<u>Case 1</u>. $r | p^k$.

In this case, $r = p^j$ with $j = 1$ or 2. From $5 \equiv l_j \pmod{j}$, we have $l_j = 0$, for $j = 1$, and $l_j = 1$, for $j = 2$. The number of ways for $7^5 = 16807$ to be represented as the sums of arithmetic progressions in this case is

$$\sum_{j=1}^{2} 2\left(\frac{7^{5-j} - 7^{l_j}}{7^j - 1}\right) = 2\left(\frac{7^4 - 1}{7 - 1}\right) + 2\left(\frac{7^3 - 7}{7^2 - 1}\right) = 800 + 14 = 814. \tag{5}$$

All these can induce a representation for 7^6. A similar computation will show that there are

$$\sum_{j=1}^{2} 2\left(\frac{7^{4-j} - 7^{l_j}}{7^j - 1}\right) = 2\left(\frac{7^3 - 1}{7 - 1}\right) + 2\left(\frac{7^2 - 1}{7^2 - 1}\right) = 686 + 2 = 688 \tag{6}$$

representations for 7^4. Of the 814 representations for 7^5 for an $r | 7^5$, or when r is odd, $814 - 688 = 126$ of them are not induced from a representation of 7^4. Of the 800 cases for $j = 1$, for instance, the values for $d = 1, 2, \ldots, 800$, and for each of these d, $a = 7^4 - \frac{1}{2}(7-1)d = 2401 - 3d$. The range for being induced from that for 7^4 is when $d \leq 2\left(\frac{p^{4-1} - p^0}{7-1}\right) = 114$. Thus, for $d = 1, 2, \ldots, 114$, the representation for 7^5 are induced from a similar representation for 7^4, but for $d = 115, 116, \ldots, 800$, the representation for 7^5 are not induced from any representation of 7^4. For instance,

$$\begin{array}{lll} \text{for } d = 1 & a = 2398: & 2398 + 2399 + 2400 + 2401 + 2402 + 2403 + 2404 = 7^5, \\ \text{for } d = 114 & a = 2059: & 2059 + 2173 + 2287 + 2401 + 2515 + 2629 + 2743 = 7^5, \end{array}$$

These are induced, respectively, by

$$\begin{array}{lll} \text{for } d = 1 & a = 340: & 340 + 341 + 342 + 343 + 344 + 345 + 346 = 7^4, \\ \text{for } d = 114 & a = 1: & 1 + 115 + 229 + 343 + 457 + 571 + 685 = 7^4, \end{array}$$

and these 114 representations are the only way that 7^4 can be represented as sums of arithmetic progressions consisting of 7 terms.

<u>Case 2</u>. $r \nmid p^k$.

In this case, each of the arithmetic progression consists of $r = 2p^j$ terms. For each $j = 0$, 1 or 2, the difference d is an odd integer less than $\left\lfloor \frac{7^{5-j}-1}{2 \times 7^j - 1} \right\rfloor$.

For $j = 0$, $\left\lfloor \frac{7^5-1}{2-1} \right\rfloor = 16806$, and $d = 1, 3, \ldots, 16805$. There are 8403 values for d.

For $j = 1$, $\left\lfloor \frac{7^4-1}{2 \times 7 - 1} \right\rfloor = 184$, and $d = 1, 3, \ldots, 183$. There are 92 values for d.

For $j = 2$, $\left\lfloor \frac{7^3-1}{2 \times 49 - 1} \right\rfloor = 3$, and $d = 1$ and 3. There are 2 values for d.

There are a total of 8497 different representations for 7^5 as sums of arithmetic progressions with $r \nmid 7^5$, or when r is even. All of these 8497 representations can induce a representation for 7^6. To find out how many among these 8497 representations are induce from representations for 7^4, we need to compute the corresponding representations for 7^4:

For $j = 0$, $\left\lfloor \frac{7^4-1}{2-1} \right\rfloor = 2400$, and $d = 1, 3, \ldots, 2399$. There are 1200 values for d.
For $j = 1$, $\left\lfloor \frac{7^3-1}{2\times7-1} \right\rfloor = 26$, and $d = 1, 3, \ldots, 25$. There are 13 values for d.
For $j = 2$, $\left\lfloor \frac{7^2-1}{2\times49-1} \right\rfloor = 0$, and there is no value for d in this case.

Thus, among the 8497 representations for 7^5, only $1200 + 13 = 1213$ of them are induced from representations for 7^4. Again, we may determine specifically which of the 8497 representations, are induce from those of 7^4. Among the 8403 representations of 7^5, with $r = 2$, (the ones with $d \leq 16805$), only those with $d \leq 2399$ are induced from a representation of 7^4. Among the 92 representations of 7^5 with $r = 14$, only those with $d \leq 25$ are induced from representations of 7^4. Finally, none of the representations of 7^5 with $r = 98$ are induced from any representations of 7^4.

Combining both $r =$ odd and $r =$ even cases, we have the following conclusions: There are $814+8497 = 9311$ different ways to represent 7^5 as sums of arithmetic progressions. The shortest of them consists of 2 terms and the longest consists of 98 terms. Each of these representations can induce a distinct representation for 7^6. Among these 9311 representations, only $688+1213 = 1901$ are induced from a representation of 7^4.

In Part I of the paper, we have extended Wheatstone's results in [8] by finding all the representations for the power of an integer. In this part, we have now shown that there are close relationships among the representations for different powers of the integer, at least when the integer is a prime, and the above example shows in detail how to find such relationships.

6 A few observations

In the introductory section of this paper, we wrote that the study of the inducement of representations can provide some new information to the first part of the paper. As we have mentioned, Theorem 3.2 is a good example for this. We now make a few more general observations, some of which may also show the kind of new information the inducement of representations can provide:

Observation 1. When one representation induces another, they have the same length r and of the same common difference d. These are among the main characteristics of an arithmetic progression. Thus, inducements of representations allows us to identify representations for two different powers of an integer to share these characteristics. For instance, since we know that it is possible to represent 7^5 as the sum of an arithmetic progression consisting of 7 terms with a common difference $d = 112$, every higher power of 7 can also be represented this way. Furthermore, since different representations will induce different representations, if we know, for example, there are 800 ways to represent 7^5 as a sum of arithmetic progression consisting of 7 terms, there are at least this many ways to represent any higher power of 7 as sums of arithmetic progressions consisting of 7 terms. In fact, each of these 800 representations for a higher power of 7 can be constructed easily using either Lemma 2.1 or Remark 2.1. On the other hand, if a representation is not induced from that of a lower power, say the representation 7^5 with $r = 7$ and $d = 115$, there can not be any such representation for 7^j for $j = 1,\ 2,\ 3$ or 4.

Observation 2. The necessary and sufficient conditions for a representation $n^k = S(a, r, d)$ to be induced from that for a lower power of n, as indicated by Theorem 4.1, are formulated in terms of the initial term a of the representation: they need to satisfy certain inequalities. However, when the integer is a prime, the conditions can be reformulated in terms of d, as we did in Theorems 5.1 and 5.2. The reason for this shift was because that for all the possible representations for a power p^k of a prime, the permissible values of d, unlike those of a, will always take the values of consecutive integers, or consecutive even or odd integers, and thus, easier for us to count how many they are. For instance, as we showed in Example 5.2, for the representations of 7^5 with $r = 7$, since $d \leq 2\left(\frac{7^3-1}{7-1}\right) = 800$, there are exactly 800 such representations with d taking the values, respectively, $1, 2, \ldots, 800$. Furthermore, we also showed that among these 800 representations, the condition for the representations that are induced from similar representations for 7^4 is for $d \leq 114$. We could then conclude immediately that for the 800 representations for 7^5, the first 114 representations are induced from a similar representation for a lower power of 7.

The value of d for a particular arithmetic progression is the size of the gaps between two successive terms in the progression. It is intuitively clear that when d increases, the gap may become too great for a given number to be its sum. For instance, as Example 5.2 shows, when the size of d increases to 115, its sum is big for 7^4. On the other hand, 7^5 can still be the sum of the progressions when d goes up to 800. This also explains that in Part I of the paper, when we tried to compute the number of ways an integer can be written as a sum of an arithmetic progressions, the numbers in all the theorems were invariably bounded by the sizes of d.

Observation 3. In this paper, we computed the inducements among the representations for the powers of a prime only. Can we do the same for a general integer? The answer is “Yes." The computations can be carried out as follows: Consider the representations for two integers M and N. To see which, if any, of the representations of M can induce, or is induced from, a representation of N all we need is to use the procedures described in the Part I and see whether the representations for M and for N have any representations that are of the same r and d. To make a specific computation, though the ideas are straightforward, the procedures are messy and the computations, tedious.

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