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Representations of positive integers as sums of arithmetic progressions, I

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Abstract: This is the first part of a two-part paper. Our paper was motivated by two classical papers: A paper of Sir Charles Wheatstone published in 1844 on representing certain powers of an integer as sums of arithmetic progressions and a paper of J. J. Sylvester published in 1882 for determining the number of ways a positive integer can be represented as the sum of a sequence of consecutive integers. There have been many attempts to extend Sylvester Theorem to the number of representations for an integer as the sums of different types of sequences, including sums of certain arithmetic progressions. In this part of the paper, we will make yet one more extension: We will describe a procedure for computing the number of ways a positive integer can be represented as the sums of all possible arithmetic progressions, together with an example to illustrate how this procedure can be carried out. In the process of doing this, we will also give an extension of Wheatstone's work. In the second part of the paper, we will continue on the problems initiated

by Wheatstone by studying certain relationships among the representations for different powers of an integer as sums of arithmetic progressions.
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1 Introduction

Motivated by the well-known fact that the square of a natural number can be represented as the sum of consecutive odd integers, Sir Charles Wheatstone, a fellow of the Royal Society of London and the inventor of the Wheatstone's bridge for measuring electrical resistance, published a paper [12] in 1844, showing various ways of representing the power N^k of a positive integer N as sums of arithmetic progressions for $k \leq 4$. In 1882, J. J. Sylvester stated in [11, Section 17, pp. 265-266] that the number of ways a positive integer N may be represented as the sum of consecutive positive integers is equal to the number of odd factors of N that exceed 1 (see also [6, Vol. 2, Chapter 3, p. 139]). This result is known as the *Sylvester Theorem* (see [2]). A special consequence of this theorem is that a power of 2 cannot be represented as the sum of any sequence of consecutive positive integers. In this two-part paper, we will deal with the problems initiated in both Wheatstone's and Sylvester's work in the same framework of arithmetic progressions, and extend both of their results.

Since the outset of the 20th century, many authors tried to extend Sylvester Theorem in different ways. For instance, Thomas E. Mason in his paper of 1912 [9], allowed the consecutive integers to include zero and negative terms. In 1930, Laurens E. Bush studied the problem of representing a positive integer as sums of arithmetic progressions of a given common difference, for both progressions of positive terms and for progressions including zero and negative terms (see [4]). In particular, he extended Sylvester Theorem on the impossibility of representing a power of 2 as the sum of any arithmetic progressions of an odd common difference ([4, the corollary on top of p. 356]). For more recent publications see [1–3, 5, 8], and [10]. However, in all these extensions, the authors considered, as Bush did, the arithmetic progressions of a *given* common difference d. In Part I of this paper, we will try to build up a procedure for finding the number of ways a positive integer N can be represented as the sum of an arithmetic progression of positive terms of *any* common difference $d \geq 1$. Our results require somewhat different methods. We believe that our extension might be more in line with Sylvester's original intention, since almost all the 80 pages of his paper [11] were devoted to the problem of partitioning of positive integers. To investigate the ways a positive integer can be partitioned into terms of arithmetic progressions, we should consider arithmetic progressions of all possible common differences.

To prepare for our investigations, we will first determine in Section 2, for a given positive integer N, the values of r and d for which N can be represented as the sum of an arithmetic progression consisting of r terms that has d as its common difference (Theorem 2.1). With this theorem, we will extend Wheatstone's work in Section 3 (see Theorems 3.1, 3.2 and 3.3) and lay the foundation for our general procedure for determining the number of ways a positive integer N

can be represented as the sums of different arithmetic progressions.The general procedure will be described in Section 5, and an example for illustrating the procedure will be given in Section 5. In part II of this paper [7], we will continue on the problems initiated by Wheatstone, by studying certain relationships among the representations of different powers of an integer.

2 Generating the sum of an arithmetic progression

As indicated in the introductory section, the problem of representing an integer as sums of arithmetic progressions has a long history. Some of our results in this section, though appear in a new form, may have already been covered by, or can be proved easily from, the existing publications. However, we will still give a self-contained account here since it would be easier for the reader if all the relevant facts are collected in one place.

It is also because the results of the existing publications are not stated in a form suitable for our purpose: We need to look for the sums of arithmetic progressions for which the parities of the number of the terms r and the common difference d are important. The existing publications are either too general or not going far enough.

In the following, by a *representation* of an integer N, we mean a representation of N as the sum of an arithmetic progression. We will let $S(a, r, d)$ to represent the sum of an arithmetic progression, beginning with the term $a \geq 1$, consisting of r terms, and with a common difference d . We will require $d > 0$ and $r > 1$ to rule out the trivial cases. We will call a pair of positive factors r and s of N *complementary* if $rs = N$.

Theorem 2.1. *Let N be a positive integer. The existence of a representation $N = S(a, r, d)$, depending on whether $r|N$, can be characterized by exactly one of the following two cases:*

1). $r|N$. In this case, either d is even or r is odd and $\frac{1}{2}(r-1)d < s$, where s is the complementary factor of r in N. Conversely, if $N = rs$ for some integers r and s such that $r > 1$ and $\frac{1}{2}(r-1)d < s$, then $N = S(a, r, d)$ for some integer $a \geq 1$. Furthermore, in this case, the first term a of the representation $N = S(a, r, d)$ is $a = s - \frac{1}{2}(r-1)d$.

2). $r \nmid N$. In this case, d is odd and r is even. Write $r = 2r_0$, then $r_0|N$. Let s_0 be the complementary factor of r_0, then s_0 is an odd integer with $s_0 > (2r_0 - 1)d$. Conversely, if r_0 and s_0 are a pair of complementary factors of N satisfying the following conditions: a) $r = 2r_0$ does not divide N and b) s_0 is an odd integer satisfying $s_0 > (2r_0 - 1)d$, then $N = S(a, r, d)$ for some integer $a \geq 1$. In this case, the first term of the representation $N = S(a, r, d)$ is $a = \frac{1}{2}[s_0 - (2r_0 - 1)d]$.

Proof. We first show that if $N = S(a, r, d)$ for some positive integers a, r, and d, then depending on whether $r|N$ or $r \nmid N$, there are two possibilities as described in the theorem. Suppose $N = S(a, r, d)$, i.e.,

$$\begin{aligned} N &= a + (a + d) + \cdots + [a + (r-1)d] \\ &= ra + \frac{1}{2}r(r-1)d \\ &= r[a + \frac{1}{2}(r-1)d]. \end{aligned}$$

Note that $r|N$ if and only if $[a + \frac{1}{2}(r-1)d]$ is an integer. Since a is an integer, $r|N$ if and only if either r is odd or d is even, and $r \nmid N$ if and only if r is even and d is odd.

In the case when r is odd or d is even, the complementary factor s of r is given by $s = [a + \frac{1}{2}(r-1)d]$. Since $a \geq 1$, we must have $\frac{1}{2}(r-1)d < s$.

Conversely, let $r > 1$ be a factor of N with s as its complementary factor, then for any integer $d \geq 1$ such that a) either r is odd or d is even and b) $\frac{1}{2}(r-1)d < s$. We will show that there exists a representation of $N = S(a, r, d)$ with the given integers r and d where a is some integer greater than or equal to 1. Since by the condition a), $\frac{1}{2}(r-1)d$ is a positive integer. By condition b), $a = s - \frac{1}{2}(r-1)d$ is also a positive integer. Then

$$\begin{aligned} a + (a+d) + (a+2d) + \cdots + (a + (r-1)d) &= ra + \frac{r(r-1)}{2}d \\ &= r[a + \frac{1}{2}(r-1)d] \\ &= r[s - \frac{1}{2}(r-1)d + \frac{1}{2}(r-1)d] \\ &= rs \\ &= N. \end{aligned}$$

Thus, $N = S(a, r, d)$. In this case, the first term of $S(a, r, d)$ is $a = s - \frac{1}{2}(r-1)d$.

Now, consider the case when $N = S(a, r, d)$ for some positive integers a, r and d such that $r \nmid N$. From the proof above, r is even and d is odd. In this case, let $r = 2r_0$ for some positive integer r_0, and let M and $M + d$ be the two middle terms in the sum $S(a, r, d)$. These two middle terms add up to $2M + d$. The two terms closest to these two middle terms, $M - d$ and $M + 2d$, again have a sum of $2M + d$. If we collect all the terms of the sum in pairs, one preceding the terms already considered and one succeeding them, there are r_0 such pairs, each of which has a sum of $2M + d$. Thus, $N = r_0(2M + d)$, and hence, $r_0|N$, with its complementary factor $s_0 = (2M + d)$. From this, we conclude that s_0 is odd and $M = \frac{1}{2}(s_0 - d)$. Since M is the r_0th term in the progression, $M = a + (r_0 - 1)d$ or $a = M - (r_0 - 1)d$. Since $a > 0$, we must have $M > (r_0 - 1)d$. Now substitute $\frac{1}{2}(s_0 - d)$ for M in this inequality and also in the equality for a and simplify, we will get both $(2r_0 - 1)d < s_0$ and $a = \frac{1}{2}[s_0 - (2r_0 - 1)d]$.

Conversely, let positive integers $r = 2r_0$ and d be given such that d is odd and $N = r_0 s_0$, where s_0 is an odd integer and $(2r_0 - 1)d < s_0$. Note that in this case, $r \nmid N$. This is because that since $N = r_0 s_0 = r(s_0/2)$ but s_0 is an odd integer. Let $a = \frac{1}{2}[s_0 - (2r_0 - 1)d]$. Note that a is a positive integer and then

$$\begin{aligned} a + (a+d) + (a+2d) + \cdots + (a + (r-1)d) &= ra + \frac{r(r-1)}{2}d \\ &= r\left[a + \tfrac{1}{2}(r-1)d\right] \\ &= 2r_0\left[\tfrac{1}{2}(s_0 - (2r_0 - 1)d) + \tfrac{1}{2}(2r_0 - 1)d\right] \\ &= r_0 s_0 \\ &= N. \end{aligned}$$

Again, $N = S(a, r, d)$, and in this case, $a = \frac{1}{2}[s_0 - (2r_0 - 1)d]$. $\square$

As a corollary of this theorem, we will give a stronger form of Bush's extension of the forbidden representations for the powers of 2, not only for odd common differences, but also for odd numbers of terms. This result will be extended further in our Corollary 3.1, which will impose more restrictions on the arithmetic progressions whose sums can be a power of 2.

Corollary 2.1. *None of the representations for a power 2^k, with $k \geq 1$, can have an odd common difference d or can consists of an odd number of terms.*

Proof. Consider a power 2^k with $k \geq 1$. If $2^k = S(a, r, d)$, we claim that r and d must both be even. First note that r cannot be an odd integer. This is because by Theorem 2.1, if r is odd, then $r|2^k$. Since we do not allow the trivial case of $r = 1$, $r = 2^j$ for some positive integer j, and thus, r is even. Say, $r = 2r_0$. Now, if d is odd, by the proof of Theorem 2.1, $2^k = r_0 s_0$, for some odd integer $s_0 > (2r_0 - 1)d$. This is impossible. We conclude that d is even. □

3 Representing the power of a prime as the sum of an arithmetic progression

For a positive integer N, we will let $\#(N, r)$ be the number of ways that N may have a representation consisting of r terms, and let $\#(N)$ be the sum of $\#(N, r)$ for all possible $r > 1$. In this section, we will first show how Wheatstone's work in [12] can be extended. We will then compute, for a given power p^k of a prime p, the possible values of r for which p^k can be represented as the sum of an arithmetic progression consisting of r terms. We will then determine both $\#(p^k, r)$ for all such r and $\#(p^k)$. These computations can also be considered as extensions of Wheatstone's work, and they will also be used in building our general procedures.

The main result in [12] is the observation that for a positive integer N, any power N^k, $k \geq 2$, can always be represented as the sum of an arithmetic progression consisting of N terms. However, instead a proof, Wheatstone only showed some examples for $k \leq 4$: with 2 examples for $k = 2$ and 4, and 3 examples for $k = 3$ (there is a second example for $k = 2$, but it involves fractions). No mentioning of the number of ways this can be done. In our next theorem, we will prove Wheatstone's Theorem, together with a computation for the number of ways this can be done and for the construction method for these representations.

Theorem 3.1. *Let N and k be any two integers ≥ 2. N^k always has a representation consisting of N terms. In fact, $\#(N^k, N) = \left\lfloor \frac{2N^{k-1}}{N-1} \right\rfloor$. Each integer $1, 2, \ldots, \left\lfloor \frac{2N^{k-1}}{N-1} \right\rfloor$ gives rise to a value of d for such an expression, and for each such d, the arithmetic progression can be constructed by letting the initial term a be $a = N^{k-1} - \frac{1}{2}(N-1)d$.*

Proof. Let N and k be two integers specified by the theorem. Since $N|N^k$, the cofactor for N is $s = N^{k-1}$. By the equivalence

$$\frac{1}{2}(N-1)d < N^{k-1} \Leftrightarrow d < \frac{2N^{k-1}}{N-1}$$

and the fact that d is an integer, we may claim that, by Theorem 2.1, there exists a representation of N^K consisting of N terms if and only if $d \leq \left\lfloor \frac{2N^{k-1}}{N-1} \right\rfloor$. But for an N and $k \geq 2$, it is certainly true that $\left\lfloor \frac{2N^{k-1}}{N-1} \right\rfloor \geq 1$. Thus, for each $d = 1, 2, \ldots, \left\lfloor \frac{2N^{k-1}}{N-1} \right\rfloor$, there is a representation of N^k of N terms. By Theorem 2.1 again, for each such d, the initial term of the representation is given by $a = N^{k-1} - \frac{1}{2}(N-1)d$. The arithmetic progression can easily be constructed by keeping adding d to each successive term beginning with a until all the N terms of the progression are obtained. □

We now return to our computations of $\#(p^k, r)$ and $\#(p^k)$. First note that 1 and 2 cannot be represented as a sum of any arithmetic progression consisting of more than one term, and thus, $\#(N) = 0$ for $N = 1$ or 2. The following lemma shows that any integer $N \geq 3$ can always be represented as such a sum.

Lemma 3.1. *Any integer $N \geq 3$ always has a representation consisting of two terms, and $\#(N, 2) = \left\lfloor \frac{1}{2}(N-1) \right\rfloor$. Furthermore, the difference d between these two terms is of the same parity as that of N: If N is odd, d may take the values of $1, 3, \ldots, N-2$. If N is even, d may take the values of $2, 4, \ldots, N-2$.*

Proof. Let an integer $N \geq 3$ be given. N has a representation consisting of two terms if and only if $N = a + (a + d)$, where a is the first term of the progression and d is the common difference. This condition is equivalent to $d = N - 2a$ for some integer $a \geq 1$. From this, we conclude that a necessary and sufficient condition for d to be an integer ≥ 1 is $a < N/2$. Thus, a can be any integer 1, 2, …, $\left\lfloor \frac{1}{2}(N-1) \right\rfloor$ and $d = N - 2a$. Consequently, $\#(N, 2) = \left\lfloor \frac{1}{2}(N-1) \right\rfloor$.

Since $d = N - 2a$, the common difference d is of the same parity as that of N. Thus, if N is odd, d may take the values of 1, 3, …, $N-2$. If N is even, d may take the values of 2, 4, …, $N-2$. □

We might observe in passing that since we do not allow $r = 1$, for any positive integer $N \geq 3$, the shortest length r for $N = S(a, r, d)$ is $r = 2$. We now study the problem of determining the number of ways the power of a prime can be represented as sums of arithmetic progressions. We now begin with the prime $p = 2$. As noted above that 2 cannot be represented as a sum of any arithmetic progression consisting of more than one term. We now consider the representations $2^k = S(a, r, d)$ for $k > 1$.

Theorem 3.2. *For $k > 1$, $2^k = S(a, r, d)$ if and only if $r = 2^j$ for some integer j, with $1 \leq j \leq \lfloor k/2 \rfloor$. For each such j, let l_j be the integer such that $0 \leq l_j < j$ and $k \equiv l_j \pmod{j}$. There are $\frac{2^{k-j} - 2^{l_j}}{2^j - 1}$ different ways for $2^k = S(a, 2^j, d)$. Each of these representations corresponds to an even integer $d = 2d_0$ with d_0 being one of the integers $1, 2, \ldots, \frac{2^{k-j} - 2^{l_j}}{2^j - 1}$, and the initial term of the representation for such a d_0 is given by $a = 2^{k-j} - (2^j - 1)d_0$. The arithmetic progression can then be found by adding repeatedly $d (= 2d_0)$ to a until we obtain all the 2^j terms of the progression. In particular,*

$$\#(2^k) = \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{2^{k-j} - 2^{l_j}}{2^j - 1}. \tag{1}$$

Furthermore, the longest r in the representation of $2^k = S(a, r, d)$ is for $r = 2^{\lfloor k/2 \rfloor}$.

Proof. Consider a power 2^k with $k \geq 2$. If $2^k = S(a,r,d)$, by Corollary 2.1, d must be even. Write $d = 2d_0$ for some positive integer d_0. By Theorem 2.1, $r|2^k$. Since $r > 1$, $r = 2^j$ for some positive integer j, and $s = 2^{k-j}$, where s is the complementary factor of r in 2^k. The condition $\frac{1}{2}(r-1)d < s$ now becomes $(2^j - 1)d_0 < 2^{k-j}$. In particular, $(2^j - 1) < 2^{k-j}$. This implies that $1 \leq j \leq \lfloor k/2 \rfloor$. Now, let l_j be the integer such that $0 \leq l_j < j$ and $k \equiv l_j \pmod{j}$, then $0 \leq l_j < j \leq \lfloor k/2 \rfloor$. From $(2^j - 1)d_0 < 2^{k-j}$, we have

$$d_0 < \frac{2^{k-j}}{2^j - 1} = \frac{2^{k-j} - 2^{l_j} + 2^{l_j}}{2^j - 1} = 2^{l_j}\left(\frac{2^{k-j-l_j} - 1}{2^j - 1}\right) + \frac{2^{l_j}}{2^j - 1}. \tag{2}$$

We contend that the first term, $2^{l_j}\left(\frac{2^{k-j-l_j}-1}{2^j-1}\right)$, is an integer and the second term, $\frac{2^{l_j}}{2^j-1}$, is less than or equal to 1. This is because that since $k \equiv l_j \pmod{j}$ and $j \leq \lfloor k/2 \rfloor$, $k - j - l_j$ is a positive multiple of j, and thus, $2^j - 1$ is a factor of $2^{k-j-l_j} - 1$. As for the term $\frac{2^{l_j}}{2^j-1}$, we first consider the case that $l_j = 0$. In this case, $\frac{2^{l_j}}{2^j-1} = \frac{1}{2^j-1} \leq 1$. Now, assume that $l_j > 0$ and $2^{l_j} > 1$. Since $j > l_j$, $2^{j-l_j} - 1 \geq 1$ and

$$2^j - 2^{l_j} = 2^{l_j}(2^{j-l_j} - 1) \geq 2^{l_j} > 1, \text{or, } 2^j - 1 > 2^{l_j}. \tag{3}$$

Thus, $0 < \frac{2^{l_j}}{2^j-1} < 1$ when $l_j > 0$. Now, since both d_0 and $2^{l_j}\left(\frac{2^{k-j-l_j}-1}{2^j-1}\right)$ are integers, and $0 < \frac{2^{l_j}}{2^j-1} \leq 1$, from the inequality (2), we may conclude that $d_0 \leq 2^{l_j}\left(\frac{2^{k-j-l_j}-1}{2^j-1}\right) = \frac{2^{k-j}-2^{l_j}}{2^j-1}$.

In fact, it is not difficult to show that

$$\frac{1}{2}(r-1)d < s \Leftrightarrow d_0 \leq \frac{2^{k-j} - 2^{l_j}}{2^j - 1}. \tag{4}$$

According to Theorem 2.1, $2^k = S(a,r,d)$ for some $r = 2^j$ if and only if d_0 is less than or equal to $\frac{2^{k-j}-2^{l_j}}{2^j-1}$, and each of these values of d_0 will give rise to a different representation of $2^k = S(a, 2^j, 2d_0)$. Thus, there are exactly $\frac{2^{k-j}-2^{l_j}}{2^j-1}$ different ways to write $2^k = S(a, 2^j, d)$ with this j, or $\#(2^k, 2^j) = \frac{2^{k-j}-2^{l_j}}{2^j-1}$ for $1 \leq j \leq \lfloor k/2 \rfloor$. The formula for the first term a of the progression follows from Theorem 2.1. Collecting $\#(2^k, 2^j)$ for all such j, we have the Formula (1). This finishes the proof. □

Corollary 3.1. *Let $k \geq 2$ be an integer. If $2^k = S(a,r,d)$ for some arithmetic progression, then both r and d are even. Furthermore, all the term of $S(a,r,d)$ are of the same parity, and these terms are even if and only if the common difference $d \equiv 0\ (mod\, 4)$.*

Proof. By Corollary 2.1, if $k \geq 2$ and $2^k = S(a,r,d)$, then both r and d are even. All terms of an arithmetic progression with an even d must be of the same parity. Thus, if $2^k = S(a,r,d)$, all the terms of $S(a,r,d)$ are all even or are all odd, depending on whether its first term a is even or odd. But $a = 2^{k-j} - (2^j - 1)d_0$. Thus, a is even if and only if d_0 is even. But $d = 2d_0$. Consequently, all the terms of $S(a,r,d)$ are even if and only if $d \equiv 0 (mod\, 4)$. □

Remark 3.1. *Our Theorem 3.2, together with Corollary 3.1, may be considered as an extension of Sylvester Theorem for the powers of 2 since they specify not only what kind of arithmetic progressions can have a sum which is a power of 2, but also how many of them can there be for a given power of 2. We now consider the case of $p^k = S(a,r,d)$, and our next theorem can be considered as an extension of Sylvester Theorem for powers of an odd prime.*

Theorem 3.3. *Let p be an odd prime and k an integer ≥ 1. Depending on whether $r|p^k$ or not, there are two types of representations for $p^k = S(a, r, d)$:*

1. *$\underline{r|p^k}$. The only way for $p^k = S(a, r, d)$ with $r|p^k$ is when $k > 1$ and $r = p^j$ for some j such that $1 \leq j \leq \lfloor k/2 \rfloor$. For each such j, let l_j be the integer such that $0 \leq l_j < j$ and $k \equiv l_j \,(\mathrm{mod}\ j)$. There are $2\left(\frac{p^{k-j}-p^{l_j}}{p^j-1}\right)$ many ways for $p^k = S(a, p^j, d)$. Each of the integers $1, 2, 3, \ldots, 2\left(\frac{p^{k-j}-p^{l_j}}{p^j-1}\right)$ gives rise to a value of d for a different arithmetic progression whose sum equals to p^k, and the initial term a of the progression for this d is given by $a = p^{k-j} - \frac{1}{2}(p^j - 1)d$.*
2. *$\underline{r \nmid p^k}$. In this case, $r = 2p^j$ for some j with $0 \leq j < \lfloor k/2 \rfloor$. Each odd integer $\leq \left\lfloor \frac{p^{k-j}-1}{2p^j-1} \right\rfloor$ gives rise to a value of d for a distinct representation $p^k = S(a, 2p^j, d)$. These are the only possible values for d for this $r = 2p^j$. For each of these representations, the initial term a is given by $a = \frac{1}{2}(p^{k-j} - (2p^j - 1)d)$. Thus, $\#(p^k, 2p^j) = \left\lfloor \frac{1}{2}\left(\frac{p^{k-j}-1}{2p^j-1} + 1\right) \right\rfloor$, where $0 \leq j < \lfloor k/2 \rfloor$.*

Collecting all the terms above, we have

$$\#(p^k) = \left\lfloor \frac{1}{2}(p^k - 1) \right\rfloor + \sum_{j=1}^{\lfloor k/2 \rfloor} \left(2\left(\frac{p^{k-j} - p^{l_j}}{p^j - 1}\right) + \left\lfloor \frac{1}{2}\left(\frac{p^{k-j} - 1}{2p^j - 1} + 1\right) \right\rfloor \right). \tag{5}$$

where the summation is zero if $k = 1$.

Proof. 1. Consider a power p^k for an odd prime p and an integer $r > 1$ such that $r|p^k$. Then $r = p^j$ for some $j \geq 1$, we claim that if $p^k = S(a, r, d)$ for such an r, then $k > 1$ and $r = p^j$ for an integer j with $1 \leq j \leq \lfloor k/2 \rfloor$. This is because that if s be the complementary factor of r in p^k, then $s = p^{k-j}$, and by Theorem 2.1, $s > \frac{1}{2}(r-1)d$. Thus, $2p^{k-j} > (p^j - 1)$, or $2p^{k-j} \geq p^j$, or $p^{k-2j} \geq \frac{1}{2}$. Since p is an odd prime, we must have $2j \leq k$ or $j \leq \lfloor k/2 \rfloor$. In particular, $k > 1$ and $1 \leq j \leq \lfloor k/2 \rfloor$.

Now, let j be an integer such that $1 \leq j \leq \lfloor k/2 \rfloor$. For $r = p^j$ and $s = p^{k-j}$. $p^k = S(a, r, d)$ if and only if $\frac{1}{2}(r-1)d < s$, or $(p^j - 1)d < 2p^{k-j}$. Let l_j be the integer such that $0 \leq l_j < j$ and $k \equiv l_j \,(\mathrm{mod}\ j)$. We have

$$d < \frac{2p^{k-j}}{p^j - 1} = \frac{2(p^{k-j} - p^{l_j} + p^{l_j})}{p^j - 1} = 2p^{l_j}\left(\frac{p^{k-j-l_j} - 1}{p^j - 1}\right) + \frac{2p^{l_j}}{p^j - 1}. \tag{6}$$

As in the proof of Theorem 3.2, $2p^{l_j}\left(\frac{p^{k-j-l_j}-1}{p^j-1}\right)$ is an integer. We now show that $0 < \frac{2p^{l_j}}{p^j-1} \leq 1$. Since p is an odd prime and $0 \leq l_j < j$, we have $p^j - 2p^{l_j} = p^{l_j}(p^{j-l_j} - 2) \geq 1$ or $p^j - 1 \geq 2p^{l_j}$. This shows that $\frac{2p^{l_j}}{p^j-1} \leq 1$. From (6), we conclude that $d \leq 2p^{l_j}\left(\frac{p^{k-j-l_j}-1}{p^j-1}\right)$. It is not difficult to show that $d \leq 2p^{l_j}\left(\frac{p^{k-j-l_j}-1}{p^j-1}\right)$ is equivalent to the condition $\frac{1}{2}(r-1)d < s$. Thus, there are exactly $2p^{l_j}\left(\frac{p^{k-j-l_j}-1}{p^j-1}\right)$, or $2\left(\frac{p^{k-j}-p^{l_j}}{p^j-1}\right)$, many ways for $p^k = S(a, p^j, d)$ with each of the integers $1, 2, \ldots, 2\left(\frac{p^{k-j}-p^{l_j}}{p^j-1}\right)$ giving rise to a value of d for a different arithmetic progression, and the corresponding initial term a of the arithmetic progression for this d is $a = p^{k-j} - \frac{1}{2}(p^j - 1)d$. This finishes the proof of Part 1 of the theorem.

2. Now assume that r is an integer such that $r \nmid p^k$. If $p^k = S(a, r, d)$ for such an r, then by Theorem 2.1, $r = 2r_0$ for some positive factor r_0 of p^k and d is an odd positive integer. Thus, $r_0 = p^j$ for some integer j with $0 \leq j \leq k$, and its complementary factor is given by $s_0 = p^{k-j}$. By Theorem 2.1 again, $N = S(a, r, d)$ for $r = 2p^j$ if and only if $(2r_0 - 1)d < s_0$, or $(2p^j - 1)d < p^{k-j}$. From this we may conclude that $j < \lfloor k/2 \rfloor$. Thus, for each such j, there is a representation for $p^k = S(a, 2p^j, d)$ for each positive odd integer d satisfying
$$d < \frac{p^{k-j}}{2p^j - 1} \text{ or } d \leq \left\lfloor \frac{p^{k-j} - 1}{2p^j - 1} \right\rfloor .$$
Furthermore, for each such d, the beginning term $a = \frac{1}{2}[p^{k-j} - (2p^j - 1)d]$. Note that the number of the odd integers less than or equal to $\left\lfloor \frac{p^{k-j}-1}{2p^j-1} \right\rfloor$ is given by $\left\lfloor \frac{1}{2} \left(\frac{p^{k-j}-1}{2p^j-1} + 1 \right) \right\rfloor$. Hence,
$$\#(p^k, 2p^j) = \left\lfloor \frac{1}{2} \left(\frac{p^{k-j} - 1}{2p^j - 1} + 1 \right) \right\rfloor \text{ for each } j \text{ with } \ 0 \leq j < \lfloor k/2 \rfloor .$$
Note that when $j = 0$, the above formula becomes $\#(p^k, 2) = \lfloor \frac{1}{2}(p^k - 1) \rfloor$, which agrees with the value given by Lemma 3.1.

The above two cases exhaust all the possible values of r. We can collect all the terms from the above two parts and get a formula for $\#(p^k)$. Note that the formula in Part 1 is for $1 \leq j \leq \lfloor k/2 \rfloor$, but for that in Part 2 is for $0 \leq j < \lfloor k/2 \rfloor$. We may combine the two formulas under the same summation sign for j ranging from 1 to $\lfloor k/2 \rfloor$ by leaving out the term in Part 2 for $j = 0$ from the summation, and noting that the formula in Part 2 for $j = \lfloor k/2 \rfloor$ is automatically zero. □

4 Representing a positive integer as sums of arithmetic progressions. I: A procedure

A similar procedure for computing the number of representations for a general positive integer N can also be established. This procedure is based on the prime factors of N since if $N = S(a, r, d)$, either r is a factor of N, or $r = 2r_0$ and r_0 is a factor of N. Given a positive integer N, we will factor N into a product of powers of primes: $N = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n}$. We will show how to compute $\#(N, r)$ for all the possible values of r, for which either r is a product of the powers of the prime factors of N, or $r = 2r_0$ and r_0 is a product of the powers of the prime factors of N. Collecting all such $\#(N, r)$, we will get the number $\#(N)$.

We start with the cases for $\#(N, r)$ when $r = {p_i}^j$ or $r = 2{p_i}^j$, for each prime factor p_i of N (Theorem 4.1 and 4.2). We will then build a computational procedure for r or r_0 being the product of powers of two or more prime factors of N. This allows us to find $\#(N, r)$ for all the possible values of r. Since the procedure is similar when r or r_0 is the product of the powers of two or more prime factors of N, we will describe, as the typical case, the product of powers of three distinct prime factors of N (Theorem 4.3).

We now begin with powers of a single prime factor. As noted before, 1 and 2 cannot be written as the sum of any arithmetic progression. We now determine whether an integer $N \geq 3$ can be written as a sum $S(a, r, d)$ when r is a power of 2, and if so, how many ways can this be done.

Theorem 4.1. *Let $N \geq 3$ be an integer. Depending on whether N is odd or even, we consider two cases:*

1. *If N is odd, the only way for $N = S(a, 2^j, d)$ for a positive integer j is when $j = 1$ and $\#(N, 2) = \frac{1}{2}(N-1)$.*
2. *If N is even, say $N = 2^k N_0$, for some positive integers k and N_0 such that N_0 is odd, we need to consider two further cases:*
   - *A) For each integer j such that $1 \leq j \leq k$, there are representations $N = S(a, 2^j, d)$ if and only if $d(= 2d_0)$ is even and $\left\lfloor \frac{2^{k-j}N_0-1}{2^j-1} \right\rfloor \geq 1$. In such a case, each positive integer from 1 to $\left\lfloor \frac{2^{k-j}N_0-1}{2^j-1} \right\rfloor$ gives rise to a value of d_0 for a representation $N = S(a, 2^j, 2d_0)$, and*
   $$\#(N, 2^j) = \left\lfloor \frac{2^{k-j}N_0 - 1}{2^j - 1} \right\rfloor \text{ for each } 1 \leq j \leq k. \tag{7}$$
   For each d_0 specified above, the beginning term $a = 2^{k-j}N_0 - (2^j - 1)d_0$. The progression itself can then be found from these a and $d(= 2d_0)$.
   - *B) $N = S(a, 2^j, d)$ can also happen when $j > k$. For this to happen, we must have $j = k + 1$ and $\left\lfloor \frac{N_0-1}{2^{k+1}-1} \right\rfloor \geq 1$. In such a case, each odd positive integer from 1 to $\left\lfloor \frac{N_0-1}{2^{k+1}-1} \right\rfloor$ gives rise to a value of d for a different representation of N, and*
   $$\#(N, 2^{k+1}) = \left\lfloor \frac{1}{2} \left(\frac{N_0 - 1}{2^{k+1} - 1} + 1 \right) \right\rfloor. \tag{8}$$
   For each of these arithmetic progressions, $a = \frac{1}{2}\left(N_0 - (2^j - 1)d\right)$.

Proof. 1. Let $N \geq 3$ be an odd integer and $r = 2^j$ for a positive integer j. We first show that $N = S(a, r, d)$ then $j = 1$. This is because that since N is odd, $r \nmid N$. The only possibility for $N = S(a, r, d)$ is, by Theorem 2.1, for $r = 2r_0$ and $r_0|N$. But $r_0 = \frac{1}{2}r = 2^{j-1}$. Thus, $r_0|N$ implies that $j = 1$, and $r = 2$. The rest follows from Lemma 3.1.

2. Let $N \geq 3$ be an even integer. Write $N = 2^k N_0$ for some odd integer N_0. We now consider the case that $N = S(a, r, d)$ for $r = 2^j$ for some positive integer j.
   - A) Suppose that $1 \leq j \leq k$. In this case $r = 2^j$ divides N. Since r is even, by Theorem 2.1, if $N = S(a, r, d)$, then d must be even. Write $d = 2d_0$ for some positive integer d_0. By Theorem 2.1 again, $N = S(a, 2^j, 2d_0)$ if and only if $(2^j - 1)d_0 < s$, where $s = 2^{k-j}N_0$ is the complementary factor of 2^j in N. Thus, there is a representation $N = S(a, 2^j, 2d_0)$ for each integer d_0 satisfying the inequality $d_0 < \frac{s}{2^j-1}$, or $d_0 < \frac{2^{k-j}N_0}{2^j-1}$. In particular, $\left\lfloor \frac{2^{k-j}N_0-1}{2^j-1} \right\rfloor \geq 1$. In such a case, we may conclude that each positive integer from 1 to $\left\lfloor \frac{2^{k-j}N_0-1}{2^j-1} \right\rfloor$ gives rise to a value of d_0 for a representation of N, and Formula (7) follows. The formula $a = 2^{k-j}N_0 - (2^j - 1)d_0$ also follows from Theorem 2.1.

     Note that we do not need to worry about whether the condition $\left\lfloor \frac{2^{k-j}N_0-1}{2^j-1} \right\rfloor \geq 1$ in (7), for if it is not satisfied, this term in (7) would automatically be zero. We also note that when $j = 1$, $\left\lfloor \frac{2^{k-j}N_0-1}{2^j-1} \right\rfloor = \lfloor 2^{k-1}N_0 - 1 \rfloor = \lfloor \frac{N}{2} - 1 \rfloor$. Since $N = 2^k N_0$ is even,

$N-1$ is odd, and $\lfloor \frac{1}{2}(N-1) \rfloor = \lfloor \frac{1}{2}(N-2) \rfloor = \lfloor \frac{N}{2} - 1 \rfloor$. Thus, Formula (7) reduces to $\#(N, 2) = \left\lfloor \frac{1}{2}(N-1) \right\rfloor$, which agrees with Lemma 3.1. This finishes the proof of 2(A).

B) Consider the case when $j > k$. In this case $r = 2^j$ does not divide N. By Theorem 2.1, we must have $r = 2r_0$ and $r_0 = 2^{j-1}$ is a factor of $N = 2^k N_0$. Consequently, we must have $j = k+1$. Since $r \nmid N$, by Theorem 2.1, $N = S(a, r, d)$ if and only if d is odd and $(2r_0 - 1)d < s_0$, where $s_0 = N_0$ is the complementary factor of r_0 in N. Thus, for each odd integer d satisfying the inequality $d < \frac{N_0}{2^{k+1}-1}$, there is a representation of $N = S(a, r, d)$. Thus, we must require $\left\lfloor \frac{N_0 - 1}{2^{k+1}-1} \right\rfloor \geq 1$. If this is the case, each odd integer from 1 to $\left\lfloor \frac{N_0 - 1}{2^{k+1}-1} \right\rfloor$ gives rise to a value of d for an arithmetic progression, and Formula (8) follows.

The rest of the assertions in (B) are then immediate. Again, we do not need to worry about the condition $\left\lfloor \frac{N_0 - 1}{2^{k+1}-1} \right\rfloor \geq 1$ for Formula (8), for if the condition is not satisfied, i.e., suppose

$$0 \leq \frac{N_0 - 1}{2^{k+1} - 1} < 1 \Rightarrow 1 \leq \frac{N_0 - 1}{2^{k+1} - 1} + 1 < 2.$$

Consequently, $\left\lfloor \frac{1}{2}\left(\frac{N_0 - 1}{2^{k+1}-1} + 1\right) \right\rfloor = 0$. This finishes the proof of the theorem. □

Theorem 4.2. *Consider a representation $N = S(a, r, d)$ for a positive integer $N = p^k N_0$ where p is an odd prime, and k and N_0 are positive integers such that N_0 is relatively prime to p.*

1. *The only case for $r = p^j$ for some integer j is for $1 \leq j \leq k$ and $\left\lfloor \frac{2p^{k-j} N_0 - 1}{p^j - 1} \right\rfloor \geq 1$. For such a j, $\#(N, p^j) = \left\lfloor \frac{2p^{k-j} N_0 - 1}{p^j - 1} \right\rfloor$. Each of the integers from 1 to $\left\lfloor \frac{2p^{k-j} N_0 - 1}{p^j - 1} \right\rfloor$ gives rise to a value of d for a different representation of N. For each such d, the initial term is given by $a = p^{k-j} N_0 - \frac{1}{2}(p^j - 1)d$. The progression itself can then be found from these a and d.*
2. *Now consider the case for $r = 2p^j$. For this part of the theorem, we may assume that N is an odd integer since the case N is even can be covered in the Part 1 of Theorem 4.3 (see also Remark 4.1) when r is a product of the powers of two or more primes, including 2.*

   In this case, $r = 2p^j$ for some integer j can happen if and only if $0 \leq j \leq k$ and $\left\lfloor \frac{p^{k-j} N_0 - 1}{2p^j - 1} \right\rfloor \geq 1$. If these conditions are satisfied, for each such j there will be a representation $N = S(a, r, d)$, with $r = 2p^j$, and $\#(N,\, 2p^j) = \left\lfloor \frac{1}{2}\left(\frac{p^{k-j} N_0 - 1}{2p^j - 1} + 1\right) \right\rfloor$. Each of the odd integers from 1 to $\left\lfloor \frac{p^{k-j} N_0 - 1}{2p^j - 1} \right\rfloor$ gives rise to a value of d for such a representation, and for each of these d, the beginning term $a = \frac{1}{2}[p^{k-j} N_0 - (2p^j - 1)d]$.

The above two cases exhaust all the possibilities for $p^k N_0 = S(a, r, d)$ for $r = p^j$ or $r = 2p^j$ for all possible integer j.

Proof. Consider a representation $N = S(a, r, d)$ for a positive integer $N = p^k N_0$, where p, k, and N_0 are as specified in the theorem.

1. If $r = p^j$ for some integer j. Since r is odd, $r | N$ by Theorem 2.1, and hence, $1 \leq j \leq k$. By Theorem 2.1 again, $N = S(a, r, d)$ for $r = p^j$ if and only if the complementary factor s of r satisfies $s > \frac{1}{2}(r-1)d$, or $2p^{k-j} N_0 > (p^j - 1)d$. This condition is satisfied if and only if d is a positive integer such that

$$d < \frac{2p^{k-j}N_0}{p^j - 1} \text{ or } d \leq \left\lfloor \frac{2p^{k-j}N_0 - 1}{p^j - 1} \right\rfloor.$$

In particular, $\left\lfloor \frac{2p^{k-j}N_0-1}{p^j-1} \right\rfloor \geq 1$. Thus, $N = S(a, p^j, d)$ for $d = 1, 2, \ldots, \left\lfloor \frac{2p^{k-j}N_0-1}{p^j-1} \right\rfloor$, and consequently,

$$\#(N, p^j) = \left\lfloor \frac{2p^{k-j}N_0 - 1}{p^j - 1} \right\rfloor.$$

Note that for certain values of j, k, p, and N_0 the inequality $2p^{k-j}N_0 > (p^j - 1)d$ is not satisfied for any positive integer d (for instance, when $p = 3, j = k = 2$ and $N_0 = 1$), but this does not invalidate the formula for $\#(N, p^j)$ since in such cases $(2p^{k-j}N_0 - 1) < (p^j - 1)$ and $\#(N, p^j)$ is zero, and thus, when $(2p^{k-j}N_0 - 1) < (p^j - 1)$, or when $s \leq \frac{1}{2}(r - 1)d$, there is no such representation for N. The formula for $\#(N)$ is still valid.

2. Now assume that N is an odd integer. If $r = 2p^j$, then $r \nmid N$. In this case, $r_0 = \frac{r}{2} = p^j$ does divide N, and hence, $0 \leq j \leq k$. The complementary factor for r_0 is then $s_0 = p^{k-j}N_0$. By Theorem 2.1 again, $N = S(a, r, d)$ for $r = 2p^j$ if and only if $(2r_0 - 1)d < s_0$, or $(2p^j - 1)d < p^{k-j}N_0$. Thus, there is a representation $N = S(a, 2p^j, d)$ for each positive odd integer d satisfying

$$d < \frac{p^{k-j}N_0}{2p^j - 1} \text{ or } d \leq \left\lfloor \frac{p^{k-j}N_0 - 1}{2p^j - 1} \right\rfloor.$$

In particular, $\left\lfloor \frac{p^{k-j}N_0-1}{2p^j-1} \right\rfloor \geq 1$. Again as before, the number of such representations is

$$\#(N, 2p^j) = \left\lfloor \frac{1}{2} \left(\frac{p^{k-j}N_0 - 1}{2p^j - 1} + 1 \right) \right\rfloor.$$

The rest of the theorem is immediate. □

Remark 4.1. *We now consider a general positive integer $N = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n} N_0$, where N_0 is relatively prime to all the prime factors p_i's. We will describe a way that N can be represented as sums of arithmetic progressions $S(a, r, d)$. The computations are all similar when r or r_0 is a product of powers of two or more prime factors of N. In the theorem, we will describe only a typical case: when $N = p^{k_1} q^{k_2} t^{k_3} N_0$, where p, q, t, are three distinct primes, all of which are relatively prime to N_0, and r or r_0 also involves these three primes with positive powers. The same procedure can be used when more, or fewer, number of prime factors of N are involved. This procedure allows us to compute the number $\#(N, r)$ for any factor r of N. By collecting all these $\#(N, r)$, we can then get $\#(N)$ itself.*

Theorem 4.3. *Consider a positive integer of the form $N = p^{k_1} q^{k_2} t^{k_3} N_0$, where p, q, t are three distinct primes, each $k_i > 0$, and N_0 is a positive integer relatively prime to p, q and t. Our procedure depends on whether N is even or odd:*

1. *N is an even integer. In this case, let $p = 2$ and write $N = 2^{k_1} q^{k_2} t^{k_3} N_0$. The number of ways for the representation $N = S(a, r, d)$, where r of the form $r = 2^{j_1} q^{j_2} t^{j_3}$, depends on two further cases: whether $r | N$ or $r \nmid N$:*

1A. If $r|N$, there is a representation for each even integer

$$d \leq \left\lfloor \frac{2^{k_1-j_1+1} q^{k_2-j_2} t^{k_3-j_3} N_0 - 1}{2^{j_1} q^{j_2} t^{j_3} - 1} \right\rfloor .$$

For each of such d, the first term a is $a = 2^{k_1-j_1} q^{k_2-j_2} t^{k_3-j_3} N_0 - \frac{1}{2}(2^{j_1} q^{j_2} t^{j_3} - 1)d$.

1B. If $r \nmid N$, then $j_1 = k_1 + 1$ and there is a representation for each odd integer

$$d \leq \left\lfloor \frac{q^{k_2-j_2} t^{k_3-j_3} N_0 - 1}{2^{k_1+1} q^{j_2} t^{j_3} - 1} \right\rfloor .$$

For each of such d, the first term is $a = \frac{1}{2}[q^{k_2-j_2} t^{k_3-j_3} N_0 - (2^{k_1+1} q^{j_2} t^{j_3} - 1)d]$.

Thus, the number of ways when N is an even integer and for r of the form $r = 2^{j_1} q^{j_2} t^{j_3}$ for some nonzero powers j_1, j_2 and j_3 is

$$\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \sum_{j_3=1}^{k_3} \left\lfloor \frac{1}{2} \left(\frac{2^{k_1-j_1+1} q^{k_2-j_2} t^{k_3-j_3} N_0 - 1}{2^{j_1} q^{j_2} t^{j_3} - 1} \right) \right\rfloor + \sum_{j_2=1}^{k_2} \sum_{j_3=1}^{k_3} \left\lfloor \frac{1}{2} \left(\frac{q^{k_2-j_2} t^{k_3-j_3} N_0 - 1}{2^{k_1+1} q^{j_2} t^{j_3} - 1} + 1 \right) \right\rfloor . \quad (9)$$

2. *N is still even, but $r = p^{j_1} q^{j_2} t^{j_3}$ is a product of powers of three odd primes. Since r is odd, $r|N$ and $N = p^{k_1} q^{k_2} t^{k_3} N_0$, where N_0 is even and $k_i \geq j_i$ for each $i = 1, 2$ or 3. In this case, there a representation for each integer*

$$d \leq \left\lfloor \frac{2p^{k_1-j_1} q^{k_2-j_2} t^{k_3-j_3} N_0 - 1}{p^{j_1} q^{j_2} t^{j_3} - 1} \right\rfloor .$$

For each of such d, the first term is $a = p^{k_1-j_1} q^{k_2-j_2} t^{k_3-j_3} N_0 - \frac{1}{2}\left(p^{j_1} q^{j_2} t^{j_3} - 1\right) d$, and consequently, for $r = p^{j_1} q^{j_2} t^{j_3}$,

$$\#(N, r) = \left\lfloor \frac{2p^{k_1-j_1} q^{k_2-j_2} t^{k_3-j_3} N_0 - 1}{p^{j_1} q^{j_2} t^{j_3} - 1} \right\rfloor . \quad (10)$$

3. *$N = p^{k_1} q^{k_2} t^{k_3} N_0$ is an odd integer. In this case, p, q, t and N_0 are all odd. There are again two possible ways for the representation $N = S(a, r, d)$ either for r of the form $r = p^{j_1} q^{j_2} t^{j_3}$ or for r of the form $r = 2p^{j_1} q^{j_2} t^{j_3}$, depending on whether $r|N$ or $r \nmid N$:*

3A) $r|N$. Then $r = p^{j_1} q^{j_2} t^{j_3}$, where $1 \leq j_i \leq k_i$ for each i = 1, 2 or 3. There is a representation for each integer

$$d \leq \left\lfloor \frac{2p^{k_1-j_1} q^{k_2-j_2} t^{k_3-j_3} N_0 - 1}{p^{j_1} q^{j_2} t^{j_3} - 1} \right\rfloor .$$

For each of such d, the first term is $a = p^{k_1-j_1} q^{k_2-j_2} t^{k_3-j_3} N_0 - \frac{1}{2}\left(p^{j_1} q^{j_2} t^{j_3} - 1\right) d$.

3B) $r \nmid N$. In this case, $r = 2p^{j_1} q^{j_2} t^{j_3}$, where $1 \leq j_i \leq k_i$ for each i. In this case, there is a representation for each odd integer

$$d \leq \left\lfloor \frac{p^{k_1-j_1} q^{k_2-j_2} t^{k_3-j_3} N_0 - 1}{2p^{j_1} q^{j_2} t^{j_3} - 1} \right\rfloor .$$

For each of such d, the first term is $a = \frac{1}{2}[p^{k_1-j_1} q^{k_2-j_2} t^{k_3-j_3} N_0 - (2p^{j_1} q^{j_2} t^{j_3} - 1)d]$.

Combining these two cases, we have for an odd integer $N = p^{k_1}q^{k_2}t^{k_3}N_0$ the number of possible representation for $r = p^{j_1}q^{j_2}t^{j_3}$, or of the form $r = 2p^{j_1}q^{j_2}t^{j_3}$, is given by

$$\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\sum_{j_3=1}^{k_3}\left(\left\lfloor\frac{2p^{k_1-j_1}q^{k_2-j_2}t^{k_3-j_3}N_0-1}{p^{j_1}q^{j_2}t^{j_3}-1}\right\rfloor+\left\lfloor\frac{1}{2}\left(\frac{p^{k_1-j_1}q^{k_2-j_2}t^{k_3-j_3}N_0-1}{2p^{j_1}q^{j_2}t^{j_3}-1}+1\right)\right\rfloor\right). \quad (11)$$

Proof. In each case, for a given r or r_0, we use the condition $\frac{1}{2}(r-1)d < s$ or $(2r_0-1)d < s_0$ to determine the allowable values for d for the number of possible representations for $N = S(a, r, d)$. In carrying out the computations, we need to consider the restrictions, as described in Theorem 2.1, that when $r|N$, then either r is odd or d is even, but when $r \nmid N$, r is even and d is odd. Finally, when we compute the number of ways, if d is an odd integer less than or equal to a given number we have to add an 1 before taking the floor function of $\frac{1}{2}$ of that number, but it the number is for d to be an even integer less than or equal to that number, we do not have to add an 1. The arguments are all similar to what we did before and will be skipped here. $\square$

5 Representing a positive integer as sums of arithmetic progressions. II: An example

To show how our method can be carried out, we now sketch a computation of $\#(N)$, for $N = 2^3 3^2 5^2 = 1800$. We will first compute $\#(N, r)$ for r or r_0 being the power of a single prime factor of N, and then, for r or r_0 being the product of powers for pairs of the prime factors of N, and so on, until all the factors of N are accounted for.

1. $\#(1800, p^j)$ for some positive integer j.

1A. First consider the case for $p = 2$ and $r = 2^j$ for some integer $j \geq 1$.

(a) Suppose $2^j|N$. In this case $1 \leq j \leq 3$. In the form of $N = 2^k N_0$ of Theorem 4.1-Part 2A, $N_0 = 225$. Thus, each integer from 1 to $\left\lfloor\frac{2^{k-j}N_0-1}{2^j-1}\right\rfloor$ gives rise to a value of d_0 for an arithmetic progression. These are the only possible ways for $N = S(a, 2^j, 2d_0)$ for $1 \leq j \leq k$. Hence, $\#(N, 2^j) = \left\lfloor\frac{2^{k-j}N_0-1}{2^j-1}\right\rfloor = \left\lfloor\frac{2^{3-j}225-1}{2^j-1}\right\rfloor$. For each of these d_0, the initial term of the progression is given by $a = 2^{k-j}N_0 - (2^j-1)d_0 = 2^{3-j}225 - (2^j-1)d_0$.

- $j = 1$. $\#(N, 2) = \lfloor 899 \rfloor = 899$. To find these progressions, we can let d_0 be any of the integers $1, 2, \ldots, 899$, and for each of the d_0, let $a = 2^2 \times 225 - d_0 = 900 - d_0$. The progression will then consists of the two terms a and $a + 2d_0$.
- $j = 2$. In this case, $\#(N, 2^2) = \left\lfloor\frac{2\times 225-1}{3}\right\rfloor = 149$. By a similar process, we can find all these 149 progressions: for each $d_0 = 1, 2, \ldots, 149$ we let $a = 2 \times 225 - 3d_0 = 450 - 3d_0$, then keep adding $d = 2d_0$ to a until we obtain all $2^2 = 4$ terms of the progression.
- $j = 3$. $\left\lfloor\frac{2^{k-j}N_0-1}{2^j-1}\right\rfloor = \left\lfloor\frac{225-1}{7}\right\rfloor = 32$. Thus, $d_0 = 1, 2, \ldots, 32$ and $\#(N, 2^2) = 32$.

(b) For $r = 2^j \nmid N$, then since $\frac{r}{2} = r_0$ divides N. We must have $r_0 = 2^3$ and $r = 2^j = 2^4$, and $s_0 = N_0 = 225$. By asserton 2(B) of Theorem 4.1, each odd positive integer from 1 to $\left\lfloor \frac{N_0-1}{2^j-1} \right\rfloor = \left\lfloor \frac{224}{15} \right\rfloor = 14$ gives rise to a value of d for a distinct arithmetic progression, consisting of $2r_0 = 16$ terms. Thus, $d = 1, 3, \ldots, 11, 13$, and $\#(N, 2^4) = 7$.

Combining all the cases, we have $\#(1800, 2^j) = 899 + 149 + 32 + 7 = 1087$. We stress again that if needed, we can construct any of these 1087 arithmetic progressions by using the values of r, the common difference d, and the first term a. This is also the case for any of the arithmetic progressions counted below.

1B. The case for $\#(1800, 3^j)$ is similar. If $r = 3^j$, then $3^j | N (= 2^3 3^2 5^2 = 3^2 N_0)$, where $N_0 = 200$ and $j = 1$ or 2. By Theorem 4.2, each of the integers from 1 to $\left\lfloor \frac{2p^{k-j}N_0-1}{p^j-1} \right\rfloor = \left\lfloor \frac{400 \times 3^{2-j}-1}{3^j-1} \right\rfloor$ gives rise to a value of d for a distinct arithmetic progression.

(a) For $j = 1$, $\#(1800, 3^1) = \left\lfloor \frac{1199}{2} \right\rfloor = 599$. There is a representation for $r = 3$ for each $d = 1,\ 2,\ 3, \ldots, 599$.

(b) For $j = 2$, $\#(1800, 3^2) = \left\lfloor \frac{399}{8} \right\rfloor = 49$. There is a representation for $r = 9$ for each $d = 1,\ 2, \ldots, 49$.

Combining the above two cases, we have that $\#(1800, 3^j) = 599 + 49 = 648$. Note that we need not compute $\#(N,\ 2p^j)$ described in Part (2) of Theorem 4.2 since here N_0 is not an odd integer and it is not the case that $r = 2p^j \nmid N$.

1C. The case for $\#(1800, 5^j)$ is also similar. If $r = 5^j$, then $5^j | N (= 2^3 3^2 5^2 = 5^2 N_0 = 5^2 \times 72)$, and $j = 1$ or 2. Each of the integers from 1 to $\left\lfloor \frac{144 \times 5^{2-j}-1}{5^j-1} \right\rfloor$ gives rise to a value of d for a different arithmetic progression.

(a) For $j = 1$, $\#(1800, 5) = \left\lfloor \frac{719}{4} \right\rfloor = 179$.

(b) For $j = 2$, $\#(1800, 5^2) = \left\lfloor \frac{143}{24} \right\rfloor = 5$.

Combining the above two cases, we have that $\#(1800, 5^j) = 179 + 5 = 184$.

2. We now consider $\#(1800, r)$ for $r = 2^{j_1} 3^{j_2} 5^{j_3}$, where at least two of j_1, j_2 and j_3 are greater than zero.

2A. For $j_1 = 0$, $r = 3^{j_2} 5^{j_3}$. This is the case described in Equation (10), except that we have only two prime powers instead of three. In this case $r|N$, and each of j_2 and j_3 can be 1 or 2. The complementary factor s of r will then be $s = 2^3 3^{2-j_2} 5^{2-j_3}$ and the condition $\frac{1}{2}(r-1)d < s$ or $d < \frac{2s}{r-1} = \frac{2^4 3^{2-j_2} 5^{2-j_3}}{3^{j_2} 5^{j_3} - 1}$. Thus, there is a representation for $r = 3^{j_2} 5^{j_3}$ for each value of $d = 1, 2, \ldots, \left\lfloor \frac{2^4 3^{2-j_2} 5^{2-j_3} - 1}{3^{j_2} 5^{j_3} - 1} \right\rfloor$, and for each such d, the corresponding initial term is $a = p^{k_1-j_1} q^{k_2-j_2} t^{k_3-j_3} N_0 - \frac{1}{2}\left(p^{j_1} q^{j_2} t^{j_3} - 1\right) d$.

(a) $(j_2, j_3) = (1, 1)$, $r = 3^{j_2} 5^{j_3} = 15$, and $\left\lfloor \frac{2^4 3^{2-j_2} 5^{2-j_3} - 1}{3^{j_2} 5^{j_3} - 1} \right\rfloor = \left\lfloor \frac{239}{14} \right\rfloor = 17$.

(b) $(j_2, j_3) = (2, 1)$, $r = 3^{j_2} 5^{j_3} = 45$, and $\left\lfloor \frac{2^4 3^{2-j_2} 5^{2-j_3} - 1}{3^{j_2} 5^{j_3} - 1} \right\rfloor = \left\lfloor \frac{80}{44} \right\rfloor = 1$.

(c) $\underline{(j_2, j_3) = (1,2), \text{ and } (j_2, j_3) = (2,2)}$; $2^4 3^{2-j_2} 5^{2-j_3} < 3^{j_2} 5^{j_3} - 1$, and hence, $\left\lfloor \frac{2^4 3^{2-j_2} 5^{2-j_3} - 1}{3^{j_2} 5^{j_3} - 1} \right\rfloor = 0$. There are no representations for these cases.

Combining all the cases in 2A, we have $\#(1800, 3^{j_2} 5^{j_3})= 17 + 1 = 18$.

In the following, we will consider the cases when $j_1 > 0$. Depending on whether $r = 2^{j_1} 3^{j_2} 5^{j_3}$ is a factor of $N = 2^3 3^2 5^2$ or not, there are two possibilities:

i) If $r|N$ then $1 \leq j_1 \leq 3,\ 0 \leq j_2 \leq 2$ and $0 \leq j_3 \leq 2$, but at least one of j_2 and j_3 is nonzero. In this case, r is an even integer, and by Theorem 4.3, d can only be an even integer less than or equal to $\left\lfloor \frac{2^{k_1 - j_1 + 1} q^{k_2 - j_2} t^{k_3 - j_3} N_0 - 1}{2^{j_1} q^{j_2} t^{j_3} - 1} \right\rfloor$.

ii). If $r \nmid N$, then $j_1 = k_1 + 1 = 4$, and d will be an odd integer less than or equal to $\left\lfloor \frac{q^{k_2 - j_2} t^{k_3 - j_3} N_0 - 1}{2^{k_1 + 1} q^{j_2} t^{j_3} - 1} \right\rfloor$.

2B. Consider the possibility i), when $r|N$. In this case $r = 2^{j_1} 3^{j_2} 5^{j_3}$ with $1 \leq j_1 \leq 3,\ 1 \leq j_2 \leq 2$ and $1 \leq j_3 \leq 2$, and there is a representation for each

$$d \leq \left\lfloor \frac{2^{k_1 - j_1 + 1} q^{k_2 - j_2} t^{k_3 - j_3} - 1}{2^{j_1} q^{j_2} t^{j_3} - 1} \right\rfloor = \left\lfloor \frac{2^{4 - j_1} 3^{2 - j_2} 5^{2 - j_3} - 1}{2^{j_1} 3^{j_2} 5^{j_3} - 1} \right\rfloor.$$

We now compute all the cases for $r|N$:

(a) $\underline{j_1 = 1}$

- $\underline{(j_2, j_3) = (1,0)}$. $r = 2 \times 3 = 6$, and d is an even integer $\leq \left\lfloor \frac{2^{4-j_1} 3^{2-j_2} 5^{2-j_3} - 1}{2^{j_1} 3^{j_2} 5^{j_3} - 1} \right\rfloor = \left\lfloor \frac{599}{5} \right\rfloor = 119$. Thus, $\#(1800, r) = \frac{118}{2} = 59$.
- $\underline{(j_2, j_3) = (0,1)}$. $r = 2 \times 5 = 10$, and d is an even integer $\leq \left\lfloor \frac{360-1}{9} \right\rfloor = 39$, or $d = 2, 4, \ldots, 38$. Thus, $\#(1800, r) = 19$.
- $\underline{(j_2, j_3) = (1,1)}$. $r = 2 \times 3 \times 5 = 30$, and d is an even integer $\leq \left\lfloor \frac{120-1}{29} \right\rfloor = 4$, or $d = 2$ or 4. Thus, $\#(1800, r) = 2$.
- $\underline{(j_2, j_3) = (2,0)}$. $r = 2 \times 9 = 18$ and d, is an even integer $\leq \left\lfloor \frac{200-1}{17} \right\rfloor = 11$, or $d = 2, 4, 6, 8, 10$. Thus, $\#(1800, r) = 5$.
- $\underline{(j_2, j_3) = (0,2),\ (1,2),\ (2,1), \text{ or } (2,2)}$. In each of these cases, $\frac{2^{4-j_1} 3^{2-j_2} 5^{2-j_3} - 1}{2^{j_1} 3^{j_2} 5^{j_3} - 1} < 2$, and $\left\lfloor \frac{2^{4-j_1} 3^{2-j_2} 5^{2-j_3} - 1}{2^{j_1} 3^{j_2} 5^{j_3} - 1} \right\rfloor \leq 1$. There cannot be any even integer for d in these cases, and consequently, and there are no representations in these cases.

Combining all the above for j_1 = 1, we have $\#(1800, r) = 59 + 19 + 2 + 5 = 85$.

(b) $\underline{j_1 = 2}$

- $\underline{(j_2, j_3) = (1,0)}$. $r = 2^2 \times 3 = 12$, and d is an even integer $\leq \left\lfloor \frac{2^{4-j_1} 3^{2-j_2} 5^{2-j_3} - 1}{2^{j_1} 3^{j_2} 5^{j_3} - 1} \right\rfloor = \left\lfloor \frac{299}{11} \right\rfloor = 27$, or $d = 2, 4, \ldots, 26$. Thus, $\#(1800, r) = 13$.
- $\underline{(j_2, j_3) = (0,1)}$. $r = 2^2 \times 5 = 20$, and d is an even integer $\leq \left\lfloor \frac{180-1}{19} \right\rfloor = 9$, or $d = 2, 4, 6, 8$. Thus, $\#(1800, r) = 4$.
- $\underline{(j_2, j_3) = (2,0)}$. $r = 2^2 \times 9 = 36$, and d is an even integer $\leq \left\lfloor \frac{100-1}{35} \right\rfloor = 2$, or $d = 2$. Thus, $\#(1800, r) = 1$.

- $\underline{(j_2, j_3) = (1,1),\ (0,2),\ (1,2),\ (2,1) \text{ or } (2,2)}$. $\left\lfloor \frac{2^{4-j_1}3^{2-j_2}5^{2-j_3}-1}{2^{j_1}3^{j_2}5^{j_3}-1} \right\rfloor = 0$. No representations in such cases.

Combining all the above for $j_1 = 2$, we have $\#(1800, r) = 13 + 4 + 1 = 18$.

(c) $\underline{j_1 = 3}$

- $\underline{(j_2, j_3) = (1, 0)}$. $r = 2^3 \times 3 = 24$ and d is an even integer $\leq \left\lfloor \frac{2^{4-j_1}3^{2-j_2}5^{2-j_3}-1}{2^{j_1}3^{j_2}5^{j_3}-1} \right\rfloor = \left\lfloor \frac{149}{23} \right\rfloor = 6$, or $d = 2, 4, 6$. Thus, $\#(1800, r) = 3$.
- $\underline{(j_2, j_3) = (0, 1)}$. $r = 2^3 \times 5 = 40$ and d is an even integer $\leq \left\lfloor \frac{90-1}{39} \right\rfloor = 2$ or $d = 2$. Thus, $\#(1800, r) = 1$.
- $\underline{(j_2, j_3) = (1,1),\ (0,2),\ (2,0),\ (1,2),\ (2,1) \text{ or } (2,2)}$. $\left\lfloor \frac{2^{4-j_1}3^{2-j_2}5^{2-j_3}-1}{2^{j_1}3^{j_2}5^{j_3}-1} \right\rfloor = 0$. No representations in such cases.

For $j_1 = 3$, $\#(1800, r) = 3 + 1 = 4$.

Summing up all the results for 2B for $j_1 = 1$, 2, and 3: $\#(1800, r) = 85 + 18 + 4 = 107$.

2C. Now, consider the possibility ii) when $r \nmid N$. Then $r = 2r_0$ and $r_0 | N (= 2^3 3^2 5^2)$. From these we may conclude that $j_1 = 4$ and $r = 2^4 3^{j_2} 5^{j_3}$ with $1 \leq j_1 \leq 3$, $1 \leq j_2 \leq 2$, and $s_0 = 3^{2-j_2} 5^{2-j_3}$. Furthermore, from Theorem 4.3, we know that there is a representation for each odd integer $d \leq \left\lfloor \frac{3^{2-j_2}5^{2-j_3}-1}{2^4 3^{j_2} 5^{j_3}-1} \right\rfloor$.

We can check quickly that the only way for $\#(1800, r) \neq 0$ in this case is for $r = 2^4 3^1 5^0 = 48$. All other combinations of $r = 2^4 3^{j_2} 5^{j_3}$ with $1 \leq j_1 \leq 3, 1 \leq j_2 \leq 2$, will make $\#(1800, r) = 0$. For the case that $r = 48$, $r_0 = 24$, $s_0 = 3 \times 5^2 = 75$, and

$$d = \left\lfloor \frac{3^{2-j_2}5^{2-j_3} - 1}{2^4 3^{j_2} 5^{j_3} - 1} \right\rfloor = \left\lfloor \frac{3 \times 5^2 - 1}{2^4 3^{j_2} 5^{j_3} - 1} \right\rfloor = \left\lfloor \frac{74}{47} \right\rfloor = 1.$$

Summing all the results of above for 1A, 1B, 1C, 2A, 2B, and 2C, we may conclude that

$$\#(1800) = 1087 + 648 + 184 + 18 + 107 + 1 = 2045.$$

This is the total number of ways that 1800 can be written as sums of arithmetic progressions.

We wish to emphasize again that for any of the 2045 ways of representing 1800 as sums of arithmetic progressions, it is easy to find the arithmetic progressions themselves. All we need is to find the first term a of the progression for the given r and d, using either Theorem 4.3 or one of the earlier theorems. then keep adding d's to a until all r terms of the progression are found. For instance, for the representation of 1800 as the sum of an arithmetic progression of 48 terms, since in this case $d = 1$, $r_0 = \frac{r}{2} = 24$, and $s_0 = \frac{1800}{24} = 75$, by Theorem 4.3, (or even Theorem 2.1), the first term of the progression is $a = \frac{1}{2}[s_0 - (2r_0 - 1)d] = \frac{1}{2}[75 - 47] = 14$, and the progression is

$$14 + 15 + \cdots + 60 + 61 = 1800.$$

6 Concluding remarks

J. J. Sylvester showed in his original paper [11] that the number of ways a positive integer N can be represented as sums of consecutive integers is equal to the number of odd factors of N that is greater than 1. We now see that the number of ways a positive integer N can be represented as sums of arithmetic progressions $S(a, r, d)$ is also closely related to the factors of N, but in a more complicated way: we need to find among the factors of N the admissible values for r or r_0 and use the conditions on r and r_0 to find these representations. With some patience, we may find all the possible representations for any positive integer N with our procedure. For instance, from our computations, we found that the integer 1800 can be written as sums of arithmetic progressions consisting of as few as 2 terms and as many as 48 terms, and our method also allows us to produce any of these arithmetic progressions.

In the second part of our paper, we will take up the problem initiated by Wheatstone again, by using as the main tool, an extension of a method recently introduced by Junaidu, Laradje and Umar, to study the relationships among various representations for different powers of an integer. Though these relationships provide us certain new insights for the representations studied in this part of the paper, each part of the paper can essentially be read independently from the other part.

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