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The quaternion-type cyclic-Fibonacci sequences in groups

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Abstract: In this paper, we define the six different quaternion-type cyclic-Fibonacci sequences and present some properties, such as, the Cassini formula and generating function. Then, we study quaternion-type cyclic-Fibonacci sequences modulo m. Also we present the relationships between the lengths of periods of the quaternion-type cyclic-Fibonacci sequences of the first, second, third, fourth, fifth and sixth kinds modulo m and the generating matrices of these sequences. Finally, we introduce the quaternion-type cyclic-Fibonacci sequences in finite groups. We calculate the lengths of periods for these sequences of the generalized quaternion groups and obtain quaternion-type cyclic-Fibonacci orbits of the quaternion groups Q_8 and Q_{16} as applications of the results.
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1 Introduction

In [16], Sir William Rowan Hamilton defined the quaternions. Quaternions form a noncommutative, associative algebra over $\mathbb{R}$

$$H = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$$

where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$ are known as Hamilton's rules (see [16,27]).

It is well known that the Fibonacci sequence $\{F_n\}$ is defined by the following homogeneous linear recurrence relation:

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 2$, where $F_0 = 0$ and $F_1 = 1$. In [22], it can be obtained miscellaneous properties involving Fibonacci numbers. The initial work began with Fibonacci sequences in algebraic structures that Wall [28] investigated in cyclic groups. Number theoretic properties such as these get from homogeneous linear recurrence relations relevant to this subject have been researched recently by many authors; see for example, [2–15,17–21,23,25,26,29]. In [1], the author studied the complex-type Pell p-numbers modulo m and get the periods and the ranks of the complex-type Pell p-numbers modulo m. Deveci and Shannon [11] extended the theory to the quaternions. Lü and Wang demonstrated that the k-step Fibonacci sequence modulo m is simply periodic [24].

After a given point, a sequence is considered periodic if all it consists of is repeated iterations of a fixed subsequence. The number of elements in the shortest repeating subsequence determines the period of sequence. As an illustration, the sequence $e, f, g, h, i, f, g, h, i, f, g, h, i, \ldots$ is periodic and has a period of 4 following the first element e. If the first k components of a sequence form a repeating subsequence, the sequence is simply periodic with period k. The sequence $e, f, g, h, i, e, f, g, h, i, e, f, g, h, i, \ldots$, for instance, is merely periodic with period 5.

In Section 2, we define the six different quaternion-type cyclic-Fibonacci sequences and then present some properties, such as, the Cassini formulas, generating function. Also, we get the relationship between the Fibonacci sequence and the first three quaternion-type cyclic-Fibonacci numbers. In Section 3, we study quaternion-type cyclic-Fibonacci sequences modulo m and then, we give the relationships between the lengths of periods of the quaternion-type cyclic-Fibonacci sequences of the first, second, third, fourth, fifth and sixth kind modulo m and the generating matrices of these sequences. In Section 4, we introduce the quaternion-type cyclic-Fibonacci sequences in groups. After this, we calculate the quaternion Fibonacci lengths of generalized quaternion groups. Finally, we give a specific example for sequences of quaternion groups Q_8 and Q_{16}.

2 The quaternion-type cyclic-Fibonacci sequences

In this section, we will introduce six different quaternion-type cyclic-Fibonacci sequences for any positive integer number $n \geq 2$. Then, we will present miscellaneous properties of these sequences.

Definition 2.1. *Define the quaternion-type cyclic-Fibonacci sequences of the first, second, third, fourth, fifth and sixth kind, respectively:*

$$x_n^1 = \begin{cases} jx_{n-2}^1 + kx_{n-1}^1 & \textit{if } n \equiv 0\ (3), \\ ix_{n-2}^1 + jx_{n-1}^1 & \textit{if } n \equiv 1\ (3), \\ kx_{n-2}^1 + ix_{n-1}^1 & \textit{if } n \equiv 2\ (3), \end{cases} \qquad x_n^2 = \begin{cases} kx_{n-2}^2 + ix_{n-1}^2 & \textit{if } n \equiv 0\ (3), \\ jx_{n-2}^2 + kx_{n-1}^2 & \textit{if } n \equiv 1\ (3), \\ ix_{n-2}^2 + jx_{n-1}^2 & \textit{if } n \equiv 2\ (3), \end{cases}$$

$$x_n^3 = \begin{cases} ix_{n-2}^3 + jx_{n-1}^3 & \textit{if } n \equiv 0\ (3), \\ kx_{n-2}^3 + ix_{n-1}^3 & \textit{if } n \equiv 1\ (3), \\ jx_{n-2}^3 + kx_{n-1}^3 & \textit{if } n \equiv 2\ (3), \end{cases} \qquad x_n^4 = \begin{cases} jx_{n-2}^4 + kx_{n-1}^4 & \textit{if } n \equiv 0\ (3), \\ kx_{n-2}^4 + ix_{n-1}^4 & \textit{if } n \equiv 1\ (3), \\ ix_{n-2}^4 + jx_{n-1}^4 & \textit{if } n \equiv 2\ (3), \end{cases}$$

$$x_n^5 = \begin{cases} kx_{n-2}^5 + ix_{n-1}^5 & \textit{if } n \equiv 0\ (3), \\ ix_{n-2}^5 + jx_{n-1}^5 & \textit{if } n \equiv 1\ (3), \\ jx_{n-2}^5 + kx_{n-1}^5 & \textit{if } n \equiv 2\ (3), \end{cases} \qquad x_n^6 = \begin{cases} ix_{n-2}^6 + jx_{n-1}^6 & \textit{if } n \equiv 0\ (3), \\ jx_{n-2}^6 + kx_{n-1}^6 & \textit{if } n \equiv 1\ (3), \\ kx_{n-2}^6 + ix_{n-1}^6 & \textit{if } n \equiv 2\ (3), \end{cases}$$

the initial conditions for all type are $x_0^\tau = 0$ *and* $x_1^\tau = 1$ $(1 \le \tau \le 6)$.

Let the entries of the matrices A and B be the elements of the quaternion-type cyclic-Fibonacci sequences,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

then the following properties hold:

$(i).\ A \times B = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}.$

$(ii).\ \det A = a_{11}a_{22} - a_{12}a_{21}.$

$(iii).\ \det(A \cdot B) = \det A \cdot \det B.$

$(iv).\ A^n = A^{n-1} \times A \quad (n \in \mathbb{Z}^+).$

Since the multiplication of quaternions is not commutative, the above properties are given considering multiplicative order. Therefore, it is easy to see that

$$\det A \cdot \det B \neq \det B \cdot \det A$$

and

$$A^{n-1} \times A \neq A \times A^{n-1}.$$

In order to easy in our operations, we define $\epsilon(\eta)$ as follows:

$$\epsilon(\eta) = \begin{cases} j & \text{if } \eta \equiv 0\ (3), \\ k & \text{if } \eta \equiv 1\ (3), \\ i & \text{if } \eta \equiv 2\ (3), \end{cases} \tag{2.1}$$

where $\eta \in \mathbb{Z}^+$. We will give relation these sequences to the well-known classic Fibonacci sequence

$$x_n^\tau = \begin{cases} -(-1)^{\frac{n}{3}} F_n \epsilon(\tau+2) & \text{if } n \equiv 0\ (3), \\ (-1)^{\frac{n-1}{3}} F_n & \text{if } n \equiv 1\ (3), \\ (-1)^{\frac{n-2}{3}} F_n \epsilon(\tau+1) & \text{if } n \equiv 2\ (3), \end{cases}$$

where $\tau = 1, 2, 3$ and $\epsilon(\tau)$ is as defined in the Equation (2.1).

Now, we introduce matrices for the quaternion-type cyclic-Fibonacci sequences, similar to the Q-matrix for classic Fibonacci sequence. We can write for these sequences

$$G_\tau = \begin{bmatrix} -3 & -2\epsilon(\tau+2) \\ 2\epsilon(\tau+2) & -1 \end{bmatrix} \text{ for } \tau = 1, 2, 3, \tag{2.2}$$

and

$$G'_\tau = \begin{bmatrix} 1-\epsilon(\tau+1)+\epsilon(\tau) & \epsilon(\tau)+\epsilon(\tau-1) \\ -\epsilon(\tau+1)+\epsilon(\tau-1) & \epsilon(\tau-1) \end{bmatrix} \text{ for } \tau = 4, 5, 6. \tag{2.3}$$

By iterative operations on n, we find

$$(G_\tau)^n = \begin{bmatrix} x^\tau_{3n+1} & -x^\tau_{3n} \\ x^\tau_{3n} & x^\tau_{3n-1}\epsilon(\tau+1) \end{bmatrix} \text{ for } \tau = 1, 2, 3, \tag{2.4}$$

and

$$(G'_\tau)^n = \begin{bmatrix} x^\tau_{3n+1} & g_{12} \\ x^\tau_{3n} & g_{22} \end{bmatrix} \text{ for } \tau = 4, 5, 6, \tag{2.5}$$

where $n \geq 1$,

$$g_{12} = \sum_{s=0}^{n-1} x^\tau_{3s+1} \left(\epsilon(\tau+2)+\epsilon(\tau)\right) \left(\epsilon(\tau+2)\right)^{n-s-1},$$
$$g_{22} = \sum_{s=0}^{n-1} x^\tau_{3s} \left(\epsilon(\tau+2)+\epsilon(\tau)\right) \left(\epsilon(\tau+2)\right)^{n-s-1} + \left(\epsilon(\tau+2)\right)^{n}.$$

Now we obtain the Cassini formula for the quaternion-type cyclic-Fibonacci sequences. By using the determinant function and the Equations (2.2), (2.4), we have

$$x^\tau_{3n+1}x^\tau_{3n-1}\epsilon(\tau+1) + (x^\tau_{3n})^2 = (-1)^n \text{ for } \tau = 1, 2, 3. \tag{2.6}$$

By using the determinant function and the Equations (2.3), (2.5), we have

$$x^\tau_{3n+1}g_{22} - g_{12}x^\tau_{3n} = \left(1+2\epsilon(\tau+2)-2\epsilon(\tau)\right)^n \text{ for } \tau = 4, 5, 6. \tag{2.7}$$

Lemma 2.1. *We give the recurrence relations for the quaternion-type cyclic-Fibonacci sequences as follows:*

(i). $x^\tau_n = -4x^\tau_{n-3} + x^\tau_{n-6}$, $(\tau = 1, 2, 3)$.

(ii). $x^\tau_n = \left(1+2\epsilon(n+\tau-1)-\epsilon(n+\tau)\right)x^\tau_{n-3} - \epsilon(n+\tau)x^\tau_{n-6}$, $(\tau = 4, 5, 6)$.

Proof. (i). The proof will only be done for the case $\tau = 1$, the others are done similarly. By Definition 2.1, we get

$$\begin{cases} x^1_{3n} = kx^1_{3n-1} + jx^1_{3n-2}, \\ x^1_{3n+1} = jx^1_{3n} + ix^1_{3n-1}, \\ x^1_{3n+2} = ix^1_{3n+1} + kx^1_{3n}. \end{cases}$$

Thus, we have

$$\begin{aligned}x^1_{3n+2} &= ix^1_{3n+1} + kx^1_{3n}\\ &= 2kx^1_{3n} - x^1_{3n-1}\\ &= -x^1_{3n-1} + 2k\left(kx^1_{3n-1} + jx^1_{3n-2}\right)\\ &= -3x^1_{3n-1} + k2jx^1_{3n-2}.\end{aligned}$$

And then, since $2jx^1_{3n-2} = k\left(x^1_{3n-1} - x^1_{3n-4}\right)$, we obtain

$$x^1_{3n+2} = -4x^1_{3n-1} + x^1_{3n-4}. \tag{2.8}$$

Similarly, we can write

$$\begin{aligned}x^1_{3n+1} &= jx^1_{3n} + ix^1_{3n-1}\\ &= 2ix^1_{3n-1} - x^1_{3n-2}\\ &= -x^1_{3n-2} + 2i\left(ix^1_{3n-2} + kx^1_{3n-3}\right)\\ &= -3x^1_{3n-2} + i2kx^1_{3n-3}.\end{aligned}$$

And then, since $2kx^1_{3n-3} = i\left(x^1_{3n-2} - x^1_{3n-5}\right)$, we acquire

$$x^1_{3n+1} = -4x^1_{3n-2} + x^1_{3n-5}. \tag{2.9}$$

Similarly, we have

$$\begin{aligned}x^1_{3n} &= kx^1_{3n-1} + jx^1_{3n-2}\\ &= 2jx^1_{3n-2} - x^1_{3n-3}\\ &= -x^1_{3n-3} + 2j\left(jx^1_{3n-3} + ix^1_{3n-4}\right)\\ &= -3x^1_{3n-3} + j2ix^1_{3n-4}.\end{aligned}$$

And then, since $2ix^1_{3n-4} = j\left(x^1_{3n-3} - x^1_{3n-6}\right)$, we get

$$x^1_{3n} = -4x^1_{3n-3} + x^1_{3n-6}. \tag{2.10}$$

From the Equations (2.8), (2.9) and (2.10), we obtain $x^1_n = -4x^1_{n-3} + x^1_{n-6}$, as required.

(ii). The proof will only be done for the case $\tau = 4$, the others are done similarly. By Definition 2.1, we get

$$\begin{cases} x^4_{3n} = kx^4_{3n-1} + jx^4_{3n-2}\,,\\ x^4_{3n+1} = ix^4_{3n} + kx^4_{3n-1}\,,\\ x^4_{3n+2} = jx^4_{3n+1} + ix^4_{3n}\,.\end{cases}$$

Thus, we have

$$\begin{aligned}x^4_{3n+2} &= jx^4_{3n+1} + ix^4_{3n}\\ &= (i-k)x^4_{3n} + ix^4_{3n-1}\\ &= ix^4_{3n-1} + (i-k)\left(kx^4_{3n-1} + jx^4_{3n-2}\right)\\ &= (1+i-j)x^4_{3n-1} + (k+i)x^4_{3n-2}.\end{aligned}$$

And then, since $(k+i)x^4_{3n-2} = ix^4_{3n-1} - jx^4_{3n-4}$, we obtain

$$x^4_{3n+2} = (1+2i-j)x^4_{3n-1} - jx^4_{3n-4}. \tag{2.11}$$

Similarly, we can write

$$\begin{aligned}x_{3n+1}^4 &= ix_{3n}^4 + kx_{3n-1}^4 \\ &= (-j+k)x_{3n-1}^4 + kx_{3n-2}^4 \\ &= kx_{3n-2}^4 + (-j+k)\left(jx_{3n-2}^4 + ix_{3n-3}^4\right) \\ &= (1-i+k)x_{3n-2}^4 + (k+j)x_{3n-3}^4.\end{aligned}$$

And then, since $(k+j)x_{3n-3}^4 = kx_{3n-2}^4 - ix_{3n-5}^4$, we acquire

$$x_{3n+1}^4 = (1+2k-i)x_{3n-2}^4 - ix_{3n-5}^4. \tag{2.12}$$

Similarly, we have

$$\begin{aligned}x_{3n}^4 &= kx_{3n-1}^4 + jx_{3n-2}^4 \\ &= (-i+j)x_{3n-2}^4 + jx_{3n-3}^4 \\ &= jx_{3n-3}^4 + (-i+j)\left(ix_{3n-3}^4 + kx_{3n-4}^4\right) \\ &= (1+j-k)x_{3n-3}^4 + (i+j)x_{3n-4}^4.\end{aligned}$$

And then, since $(i+j)x_{3n-4}^4 = jx_{3n-3}^4 - kx_{3n-6}^4$, we get

$$x_{3n}^4 = (1+2j-k)x_{3n-3}^4 - kx_{3n-6}^4. \tag{2.13}$$

From the Equations (2.11), (2.12) and (2.13), we obtain

$$x_n^4 = (1+2\epsilon(n) - \epsilon(n+1))\, x_{n-3}^4 - \epsilon(n+1)x_{n-6}^4,$$

as required. □

In the following theorem, we develop the generating functions for the quaternion-type cyclic-Fibonacci sequences.

Theorem 2.1. *The generating functions of the* $\{x_n^\tau\}$ *are*

(i). $\sum\limits_{n=0}^{\infty} x_n^\tau t^n = \frac{t+\epsilon(\tau+1)t^2+2\epsilon(\tau+2)t^3+t^4-\epsilon(\tau+1)t^5}{1+4t^3-t^6}$, $(\tau = 1,2,3)$.

(ii). $\sum\limits_{n=0}^{\infty} x_n^\tau t^n = \frac{(-\epsilon(\tau+1)+\epsilon(\tau-1))t^3}{1-(1+2\epsilon(\tau-1)-\epsilon(\tau))t^3+\epsilon(\tau)t^6} + \frac{t-\epsilon(\tau)t^4}{1-(1+2\epsilon(\tau)-\epsilon(\tau+1))t^3+\epsilon(\tau+1)t^6}$
$+\frac{\epsilon(\tau+2)t^2+(-\epsilon(\tau+1)+2\epsilon(\tau+2)+2\epsilon(\tau))t^5}{1-(1+2\epsilon(\tau+1)-\epsilon(\tau+2))t^3+\epsilon(\tau+2)t^6}$, $(\tau = 4,5,6)$.

Proof. (i). Assume that $f(t)$ is the generating function of the $\{x_n^\tau\}$ for $\tau = 1,2,3$. Then we have

$$f(t) = \sum_{n=0}^{\infty} x_n^\tau t^n.$$

From Lemma 2.1, we obtain

$$\begin{aligned}f(t) &= x_0^\tau + x_1^\tau t + x_2^\tau t^2 + x_3^\tau t^3 + x_4^\tau t^4 + x_5^\tau t^5 + \sum_{n=6}^{\infty}\left(-4x_{n-3}^\tau + x_{n-6}^\tau\right)t^n \\ &= x_0^\tau + x_1^\tau t + x_2^\tau t^2 + x_3^\tau t^3 + x_4^\tau t^4 + x_5^\tau t^5 - 4\left(f(t) - x_0^\tau - x_1^\tau t - x_2^\tau t^2\right)t^3 + f(t)t^6.\end{aligned}$$

Now the rearrangement of the equation implies that

$$f(t) = \frac{x_1^\tau t + x_2^\tau t^2 + x_3^\tau t^3 + \left(x_4^\tau + 4x_1^\tau\right) t^4 + \left(x_5^\tau + 4x_2^\tau\right) t^5}{1 + 4t^3 - t^6},$$

which equals to the left-hand sides $\sum\limits_{n=0}^{\infty} x_n^\tau t^n$ in the Theorem.

(ii). The proof can be done similarly to (i). □

3 The quaternion-type cyclic-Fibonacci sequence modulo m

In this section, we study quaternion-type cyclic-Fibonacci sequences modulo m. Then, we give the relationships between the lengths of periods of the quaternion-type cyclic-Fibonacci sequences of the first, second, third, fourth, fifth and sixth kind modulo m and the generating matrices of these sequences.

Let f_n denote the n-th member of the Fibonacci sequences $f_0 = a$, $f_1 = b$, $f_{n+1} = f_n + f_{n-1}$ $(n \geq 1)$.

Theorem 3.1. (Wall, [28]) $f_n \pmod m$ *forms a simply periodic sequence. That is, the sequence is periodic and repeats by returning to its starting values.*

The length of the period of the ordinary Fibonacci sequence $\{F_n\}$ modulo m was denoted by $k(m)$.

If we take the least nonnegative residues and decrease the first, second, third, fourth, fifth, and sixth kinds of quaternion-type cyclic-Fibonacci sequences modulo m, we obtain the following recurrence sequences:

$$\{x_n^\tau(m)\} = \{x_1^\tau(m),\ x_2^\tau(m),\ \ldots,\ x_u^\tau(m),\ \ldots\}$$

for every integer $1 \leq \tau \leq 6$, where $x_u^\tau(m)$ is used to mean the u-th element of the τ-th quaternion-type cyclic-Fibonacci sequence when read modulo m. We observe here that the recurrence relations in the sequences $\{x_n^\tau(m)\}$ and $\{x_n^\tau\}$ are the same.

Theorem 3.2. *The sequences $\{x_n^\tau(m)\}$ are periodic and the lengths of their periods are divisible by* 3.

Proof. Let us consider the quaternion-type cyclic-Fibonacci sequence of the first kind $\{x_n^1\}$ as an example. Consider the set

$$Q = \{(q_1, q_2) \mid q_u\text{'s are quaternions } a_u + b_u i + c_u j + d_u k \text{ where} a_u, b_u, c_u \text{ and } d_u \\ \text{are integers such that } 0 \leq a_u, b_u,\ c_u,\ d_u \leq m-1 \text{ and } u \in \{1,2\}\}.$$

Suppose that the cardinality of the set Q is denoted by the notation $|Q|$. Since the set Q is finite, there are $|Q|$ distinct 2-tuples of the quaternion-type cyclic-Fibonacci sequences of the first kind $\{x_n^1\}$ modulo m. Thus, it is clear that at least one of these 2-tuples appears twice in the sequence

$\{x_n^1(m)\}$. Let $x_\alpha^1(m) \equiv x_\beta^1(m)$ and $x_{\alpha+1}^1(m) \equiv x_{\beta+1}^1(m)$. If $\beta - \alpha \equiv 0\,(\text{mod}3)$, then we get $x_{\alpha+2}^1(m) \equiv x_{\beta+2}^1(m)$, $x_{\alpha+3}^1(m) \equiv x_{\beta+3}^1(m)$, ... So, it is easy to see that the subsequence following this 2-tuple repeats; that is, $\{x_n^1(m)\}$ is a periodic sequence and the length of its period must be divisible by 3.

The proofs for the sequences $\{x_n^2\}$, $\{x_n^3\}$, $\{x_n^4\}$, $\{x_n^5\}$ and $\{x_n^6\}$ are similar to the above and are omitted. □

We next denote the lengths of periods of the sequences $\{x_n^\tau(m)\}$ by $l_{x_n^\tau}(m)$.

Consider the matrices

$$A_1 = \begin{bmatrix} i & k \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} k & j \\ 1 & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} j & i \\ 1 & 0 \end{bmatrix}.$$

Suppose that $G_1 = A_3A_2A_1$, $G_2 = A_2A_1A_3$, $G_3 = A_1A_3A_2$, $G_4' = A_1A_2A_3$, $G_5' = A_3A_1A_2$ and $G_6' = A_2A_3A_1$. Using the above, we define the following matrices:

$$(M_1)^n = \begin{cases} (G_1)^{\frac{n}{3}} & \text{if } n \equiv 0\ (3), \\ A_1\,(G_1)^{\frac{n-1}{3}} & \text{if } n \equiv 1\ (3), \\ A_2A_1\,(G_1)^{\frac{n-2}{3}} & \text{if } n \equiv 2\ (3), \end{cases} \qquad (M_2)^n = \begin{cases} (G_2)^{\frac{n}{3}} & \text{if } n \equiv 0\ (3), \\ A_3\,(G_2)^{\frac{n-1}{3}} & \text{if } n \equiv 1\ (3), \\ A_1A_3\,(G_2)^{\frac{n-2}{3}} & \text{if } n \equiv 2\ (3), \end{cases}$$

$$(M_3)^n = \begin{cases} (G_3)^{\frac{n}{3}} & \text{if } n \equiv 0\ (3), \\ A_2\,(G_3)^{\frac{n-1}{3}} & \text{if } n \equiv 1\ (3), \\ A_3A_2\,(G_3)^{\frac{n-2}{3}} & \text{if } n \equiv 2\ (3), \end{cases} \qquad (M_4)^n = \begin{cases} (G_4')^{\frac{n}{3}} & \text{if } n \equiv 0\ (3), \\ A_3\,(G_4')^{\frac{n-1}{3}} & \text{if } n \equiv 1\ (3), \\ A_2A_3\,(G_4')^{\frac{n-2}{3}} & \text{if } n \equiv 2\ (3), \end{cases}$$

$$(M_5)^n = \begin{cases} (G_5')^{\frac{n}{3}} & \text{if } n \equiv 0\ (3), \\ A_2\,(G_5')^{\frac{n-1}{3}} & \text{if } n \equiv 1\ (3), \\ A_1A_2\,(G_5')^{\frac{n-2}{3}} & \text{if } n \equiv 2\ (3), \end{cases} \qquad (M_6)^n = \begin{cases} (G_6')^{\frac{n}{3}} & \text{if } n \equiv 0\ (3), \\ A_1\,(G_6')^{\frac{n-1}{3}} & \text{if } n \equiv 1\ (3), \\ A_3A_1\,(G_6')^{\frac{n-2}{3}} & \text{if } n \equiv 2\ (3). \end{cases}$$

Then we get

$$(M_\tau)^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_{n+1}^\tau \\ x_n^\tau \end{pmatrix},$$

where τ is an integer such that $1 \leq \tau \leq 6$. Therefore, we immediately deduce that $l_{x_n^\tau}(m)$ is the smallest positive integer α such that $(M_\tau)^\alpha \equiv I(\text{mod } m)$ for every integer $1 \leq \tau \leq 6$.

4 The quaternion-type cyclic-Fibonacci sequence in groups

In this section, we will define six different quaternion-type cyclic-Fibonacci sequences in finite groups. Subsequently, we will examine the quaternion-type cyclic-Fibonacci orbits of the first, second, third, fourth, fifth and sixth kinds of the generalized quaternion group. Finally, we will give a specific example for sequences of quaternion groups Q_8 and Q_{16}.

Let G be a 2-generator group and let

$$X = \{(x_1, x_2) \in G \times G \mid \langle\{x_1, x_2\}\rangle = G\}.$$

We call (x_1, x_2) a generating pair for G.

Definition 4.1. *Let G be a 2-generator group. For the generating pair (x, y), we define the quaternion-type cyclic-Fibonacci orbits of the first, second, third, fourth, fifth and sixth kinds of G as follows, respectively:*

$$a_n^1 = \begin{cases} (a_{n-2}^1)^j (a_{n-1}^1)^k & \text{if } n \equiv 0\ (3), \\ (a_{n-2}^1)^i (a_{n-1}^1)^j & \text{if } n \equiv 1\ (3), \\ (a_{n-2}^1)^k (a_{n-1}^1)^i & \text{if } n \equiv 2\ (3), \end{cases} \qquad a_n^2 = \begin{cases} (a_{n-2}^2)^k (a_{n-1}^2)^i & \text{if } n \equiv 0\ (3), \\ (a_{n-2}^2)^j (a_{n-1}^2)^k & \text{if } n \equiv 1\ (3), \\ (a_{n-2}^2)^i (a_{n-1}^2)^j & \text{if } n \equiv 2\ (3), \end{cases}$$

$$a_n^3 = \begin{cases} (a_{n-2}^3)^i (a_{n-1}^3)^j & \text{if } n \equiv 0\ (3), \\ (a_{n-2}^3)^k (a_{n-1}^3)^i & \text{if } n \equiv 1\ (3), \\ (a_{n-2}^3)^j (a_{n-1}^3)^k & \text{if } n \equiv 2\ (3), \end{cases} \qquad a_n^4 = \begin{cases} (a_{n-2}^4)^j (a_{n-1}^4)^k & \text{if } n \equiv 0\ (3), \\ (a_{n-2}^4)^k (a_{n-1}^4)^i & \text{if } n \equiv 1\ (3), \\ (a_{n-2}^4)^i (a_{n-1}^4)^j & \text{if } n \equiv 2\ (3), \end{cases}$$

$$a_n^5 = \begin{cases} (a_{n-2}^5)^k (a_{n-1}^5)^i & \text{if } n \equiv 0\ (3), \\ (a_{n-2}^5)^i (a_{n-1}^5)^j & \text{if } n \equiv 1\ (3), \\ (a_{n-2}^5)^j (a_{n-1}^5)^k & \text{if } n \equiv 2\ (3), \end{cases} \qquad a_n^6 = \begin{cases} (a_{n-2}^6)^i (a_{n-1}^6)^j & \text{if } n \equiv 0\ (3), \\ (a_{n-2}^6)^j (a_{n-1}^6)^k & \text{if } n \equiv 1\ (3), \\ (a_{n-2}^6)^k (a_{n-1}^6)^i & \text{if } n \equiv 2\ (3), \end{cases}$$

for $n \geq 2$, with initial conditions $a_0^\tau = x$ and $a_1^\tau = y$ $(1 \leq \tau \leq 6)$, where the following conditions hold for every $x, y \in G$:

(i). *Let $q = a + bi + cj + dk$ such that a, b, c and d are integers and let e be the identity of G, then*

- $x^q = x^{a(mod|x|)+b(mod|x|)i+c(mod|x|)j+d(mod|x|)k} = x^{a(mod|x|)}x^{b(mod|x|)i}x^{c(mod|x|)j}x^{d(mod|x|)k}$.
- $(x^u)^a = (x^a)^u$, *where $u \in \{i, j, k\}$ and a is an integer.*
- $e^q = e$ *and* $x^{0+0i+0j+0k} = e$.

(ii). *Let $q_1 = a_1 + b_1 i + c_1 j + d_1 k$ and $q_2 = a_2 + b_2 i + c_2 j + d_2 k$ such that $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ are integers, then $(x^{q_1} x^{q_2})^{-1} = x^{-q_2} x^{-q_1}$.*

(iii). *If $xy \neq yx$, then $x^u y^u \neq y^u x^u$ for $u \in \{i, j, k\}$.*

(iv). $(xy)^u = y^u x^u$ *for $u \in \{i, j, k\}$.*

(v). $(x^{u_1} y^{u_2})^{u_3} = x^{u_3 u_1} y^{u_3 u_2}$, $(xy^{u_1})^{u_2} = x^{u_2} y^{u_2 u_1}$ *and* $(x^{u_1} y)^{u_2} = x^{u_2 u_1} y^{u_2}$ *for $u_1, u_2, u_3 \in \{i, j, k\}$ and so $(x^{u_1} y^{u_1})^{u_1} = x^{-1} y^{-1}$.*

(vi). *For $u_1, u_2 \in \{i, j, k\}$ such that $u_1 \neq u_2$, $x^{u_1} y^{u_2} = y^{u_2} x^{u_1}$, $xy^{u_1} = y^{u_1} x$, $x^{u_1} y = y x^{u_1}$ and so $(xy^{u_1})^{u_1} = x^{u_1} y^{-1}$ and $(x^{u_1} y)^{u_1} = x^{-1} y^{u_1}$.*

Let the notation $F_{(x,y)}^{q,\tau}(G)$ denote the τ-th quaternion-type cyclic-Fibonacci orbit of the group G for the generating pair (x, y). From the definition of the orbit $F_{(x,y)}^{q,\tau}(G)$ it is clear that the length of the period of this sequence in a finite group depend on the chosen generating pair and the order in which the assignments of x, y are made.

Theorem 4.1. *Let G be a 2-generator group. If G is finite, then the quaternion-type cyclic-Fibonacci orbits of the first, second, third, fourth, fifth and sixth kinds of G are periodic and the lengths of their periods are divisible by 3.*

Proof. Let us consider the 2nd quaternion-type cyclic-Fibonacci orbit of the group G. Consider the set

$$S=\Big\{(s_1)^{a_1(\mathrm{mod}|s_1|)+b_1(\mathrm{mod}|s_1|)i+c_1(\mathrm{mod}|s_1|)j+d_1(\mathrm{mod}|s_1|)k},(s_2)^{a_2(\mathrm{mod}|s_2|)+b_2(\mathrm{mod}|s_2|)i+c_2(\mathrm{mod}|s_2|)j+d_2(\mathrm{mod}|s_2|)k}: s_1,s_2\in G \text{ and } a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2,\in\mathbb{Z}\Big\}.$$

Since the group G is finite, S is a finite set. Then for any $u\geq 0$, there exists $v>u$ such that $a^2_u=a^2_v$ and $a^2_{u+1}=a^2_{v+1}$. If $v-u\equiv 0\,(\mathrm{mod}3)$, then we get $a^2_{u+2}=a^2_{v+2}$, $a^2_{u+3}=a^2_{v+3}$, $\ldots$. Because of the repeating, for all generating pairs, the sequence $F^{q,2}_{(x,y)}(G)$ is periodic and the length of its period must be divisible by 3.

The proofs for the orbits $F^{q,1}_{(x,y)}(G)$, $F^{q,3}_{(x,y)}(G)$, $F^{q,4}_{(x,y)}(G)$, $F^{q,5}_{(x,y)}(G)$ and $F^{q,6}_{(x,y)}(G)$ are similar to the above and are omitted. □

We next denote the lengths of the periods of the orbits $F^{q,\tau}_{(x,y)}(G)$ by $LF^{q,\tau}_{(x,y)}(G)$.

We shall now address the lengths of the periods of the orbits $F^{q,1}_{(x,y)}(Q_{2^{m+1}})$, $F^{q,2}_{(x,y)}(Q_{2^{m+1}})$, $F^{q,3}_{(x,y)}(Q_{2^{m+1}})$, $F^{q,4}_{(x,y)}(Q_{2^{m+1}})$, $F^{q,5}_{(x,y)}(Q_{2^{m+1}})$ and $F^{q,6}_{(x,y)}(Q_{2^{m+1}})$. It is well-known that the generalized quaternion group $Q_{2^{m+1}}$ of order 2^m is defined by the presentation

$$Q_{2^{m+1}}=\langle x,y\mid x^{2^m}=y^4=1,\ \ x^{2^{m-1}}=y^2,\ \ y^{-1}xy=x^{-1}\rangle.$$

Theorem 4.2. *For $m\geq 2$, $LF^{q,1}_{(x,y)}(Q_{2^{m+1}})=LF^{q,2}_{(x,y)}(Q_{2^{m+1}})=LF^{q,3}_{(x,y)}(Q_{2^{m+1}})=3.2^{m-1}$.*

Proof. By direct calculation, we obtain the orbits $F^{q,1}_{(x,y)}(Q_{2^{m+1}})$, $F^{q,2}_{(x,y)}(Q_{2^{m+1}})$ and $F^{q,3}_{(x,y)}(Q_{2^{m+1}})$ as follows, respectively. Firstly, the orbit $F^{q,1}_{(x,y)}(Q_{2^{m+1}})$ is

$$\begin{array}{llllll}
a^1_0=x, & a^1_1=y, & a^1_2=y^ix^k, & a^1_3=y^{2j}x^{-1}, & a^1_4=yx^{-2j}, & a^1_5=y^{-i}x^{-3k},\\
a^1_6=x^5, & a^1_7=yx^{8j}, & a^1_8=y^ix^{13k}, & a^1_9=y^{2j}x^{-21}, & a^1_{10}=yx^{-34j}, & a^1_{11}=y^{-i}x^{-55k},\\
a^1_{12}=x^{89}, & a^1_{13}=yx^{144j}, & a^1_{14}=y^ix^{233k}, & a^1_{15}=y^{2j}x^{-377}, & a^1_{16}=yx^{-610j}, & a^1_{17}=y^{-i}x^{-987k},
\end{array}$$

$$\cdots$$

$$\begin{array}{lll}
a^1_{6n}=x^{F_{6n-1}}, & a^1_{6n+1}=yx^{F_{6n}j}, & a^1_{6n+2}=y^ix^{F_{6n+1}k},\\
a^1_{6n+3}=y^{2j}x^{-F_{6n+2}}, & a^1_{6n+4}=yx^{-F_{6n+3}j}, & a^1_{6n+5}=y^{-i}x^{-F_{6n+4}k}.
\end{array}$$

Secondly, we take into account the orbit $F^{q,2}_{(x,y)}(Q_{2^{m+1}})$. We have the sequence

$$\begin{array}{llllll}
a^2_0=x, & a^2_1=y, & a^2_2=y^jx^i, & a^2_3=y^{2k}x^{-1}, & a^2_4=yx^{-2k}, & a^2_5=y^{-j}x^{-3i},\\
a^2_6=x^5, & a^2_7=yx^{8k}, & a^2_8=y^jx^{13i}, & a^2_9=y^{2k}x^{-21}, & a^2_{10}=yx^{-34k}, & a^2_{11}=y^{-j}x^{-55i},\\
a^2_{12}=x^{89}, & a^2_{13}=yx^{144k}, & a^2_{14}=y^jx^{233i}, & a^2_{15}=y^{2k}x^{-377}, & a^2_{16}=yx^{-610k}, & a^2_{17}=y^{-j}x^{-987i},
\end{array}$$

$$\cdots$$

$$\begin{array}{lll}
a^2_{6n}=x^{F_{6n-1}}, & a^2_{6n+1}=yx^{F_{6n}k}, & a^2_{6n+2}=y^jx^{F_{6n+1}i},\\
a^2_{6n+3}=y^{2k}x^{-F_{6n+2}}, & a^2_{6n+4}=yx^{-F_{6n+3}k}, & a^2_{6n+5}=y^{-j}x^{-F_{6n+4}i}.
\end{array}$$

Finally, we consider the 3rd quaternion-type cyclic-Fibonacci orbit of the generalized quaternion group $Q_{2^{m+1}}$ with respect to the generating pair (x,y), $F^{q,3}_{(x,y)}(Q_{2^{m+1}})$. Using a similar argument to the above, we obtain the following sequence:

$$
\begin{array}{llllll}
a_0^3 = x, & a_1^3 = y, & a_2^3 = y^k x^j, & a_3^3 = y^{2i} x^{-1}, & a_4^3 = y x^{-2i}, & a_5^3 = y^{-k} x^{-3j}, \\
a_6^3 = x^5, & a_7^3 = y x^{8i}, & a_8^3 = y^k x^{13j}, & a_9^3 = y^{2i} x^{-21}, & a_{10}^3 = y x^{-34i}, & a_{11}^3 = y^{-k} x^{-55j}, \\
a_{12}^3 = x^{89}, & a_{13}^3 = y x^{144i}, & a_{14}^3 = y^k x^{233j}, & a_{15}^3 = y^{2i} x^{-377}, & a_{16}^3 = y x^{-610i}, & a_{17}^3 = y^{-k} x^{-987j},
\end{array}
$$

$$\cdots$$

$$
\begin{array}{lll}
a_{6n}^3 = x^{F_{6n-1}}, & a_{6n+1}^3 = y x^{F_{6n} i}, & a_{6n+2}^3 = y^k x^{F_{6n+1} j}, \\
a_{6n+3}^3 = y^{2i} x^{-F_{6n+2}}, & a_{6n+4}^3 = y x^{-F_{6n+3} i}, & a_{6n+5}^3 = y^{-k} x^{-F_{6n+4} j},
\end{array}
$$

where F_n is the n-th term of the ordinary Fibonacci sequence $\{F_n\}$.

It is known that $k(2^m) = 2^{m-1}.3$; see [28] for proof. So we get that the lengths of the periods of the sequences $F^{q,1}_{(x,y)}(Q_{2^{m+1}})$, $F^{q,2}_{(x,y)}(Q_{2^{m+1}})$ and $F^{q,3}_{(x,y)}(Q_{2^{m+1}})$ are $\operatorname{lcm}[6, k(2^m)] = \operatorname{lcm}[6, 2^{m-1}.3] = 3.2^{m-1}$. □

Theorem 4.3. *For $m \geq 2$, $LF^{q,4}_{(x,y)}(Q_{2^{m+1}}) = LF^{q,5}_{(x,y)}(Q_{2^{m+1}}) = LF^{q,6}_{(x,y)}(Q_{2^{m+1}}) = 3.2^m$.*

Proof. We prove this by direct calculation. At first, let us consider the 4-th quaternion-type cyclic-Fibonacci orbit of the generalized quaternion group $Q_{2^{m+1}}$ with respect to the generating pair (x, y). The orbit $F^{q,4}_{(x,y)}(Q_{2^{m+1}})$ is in the following form:

$$
\begin{array}{llll}
a_0^4 = x, \quad a_1^4 = y, \quad a_2^4 = y^j x^i, & a_3^4 = y^{j-i} x^j, \quad a_4^4 = y^{1-i+k} x^{j+k}, & a_5^4 = y^{1+i+j+2k} x^{-1+i+k}, & \ldots, \\
a_{12}^4 = x^{1-4i-16j+4k}, & a_{13}^4 = y x^{4+4i-16j-16k}, & a_{14}^4 = y^j x^{20-15i-20k}, & \ldots, \\
a_{24}^4 = x^{-495+304i+360j-104k}, & a_{25}^4 = y x^{-632-512i+448j+192k}, & a_{26}^4 = y^j x^{-752-303i-528j+872k}, & \ldots,
\end{array}
$$

$$\cdots$$

$$a_{12n}^4 = x^{4nq_0+1}, \quad a_{12n+1}^4 = y x^{4nq_1}, \quad a_{12n+2}^4 = y^j x^{q_2}, \quad \ldots,$$

where $q_u \ (0 \leq u \leq 11)$ are quaternion numbers which represented in the form $q_u = a_u + b_u i + c_u j + d_u k \ (a_u, b_u, c_u, d_u \in \mathbb{R}, \ 0 \leq u \leq 11)$ such that $\gcd(q_0, q_1, q_2, \cdots, q_{11}) = 1$.
Secondly, we consider the orbit $F^{q,5}_{(x,y)}(Q_{2^{m+1}})$. We obtain the sequence

$$
\begin{array}{llll}
a_0^5 = x, \ a_1^5 = y, \ a_2^5 = y^k x^j, & a_3^5 = y^{k-j} x^k, \ a_4^5 = y^{1+i-j} x^{i+k}, & a_5^5 = y^{1+2i+j+k} x^{-1+i+j}, & \ldots, \\
a_{12}^5 = x^{1+4i-4j-16k}, & a_{13}^5 = y x^{4-16i+4j-16k}, & a_{14}^5 = y^k x^{20-20i-15j}, & \ldots, \\
a_{24}^5 = x^{-655+120i+264j-16k}, & a_{25}^5 = y x^{-760-160i-600j-64k}, & a_{26}^5 = y^k x^{-200+584i-815j-880k}, & \ldots,
\end{array}
$$

$$\cdots$$

$$a_{12n}^5 = x^{4nq'_0+1}, \quad a_{12n+1}^5 = y x^{4nq'_1}, \quad a_{12n+2}^5 = y^k x^{q'_2}, \ \ldots,$$

where $q'_u \ (0 \leq u \leq 11)$ are quaternion numbers which represented in the form $q'_u = a'_u + b'_u i + c'_u j + d'_u k \ \left(a'_u, b'_u, c'_u, d'_u \in \mathbb{R}, \ 0 \leq u \leq 11\right)$ such that $\gcd(q'_0, q'_1, q'_2, \cdots, q'_{11}) = 1$.

Finally, we present the 6-th quaternion-type cyclic-Fibonacci orbit of the generalized quaternion group $Q_{2^{m+1}}$ with respect to the generating pair (x, y), $F^{q,6}_{(x,y)}(Q_{2^{m+1}})$. The orbit $F^{6}_{(x,y)}(Q_{2^{m+1}})$ is as follows:

$$
\begin{array}{llll}
a_0^6 = x, \ a_1^6 = y, \ a_2^6 = y^i x^k, & a_3^6 = y^{i-k} x^i, & a_4^6 = y^{1+j-k} x^{i+j}, & a_5^6 = y^{1+i+2j+k} x^{-1+j+k}, \ldots, \\
a_{12}^6 = x^{1-16i+4j-4k}, & a_{13}^6 = y x^{4-16i-16j+4k}, & a_{14}^6 = y^i x^{20-20j-15k}, & \ldots, \\
a_{24}^6 = x^{-655+80i-264j+264k}, & a_{25}^6 = y x^{-632+448i-288j-632k}, & a_{26}^6 = y^i x^{-712-368i+712j-943k}, & \ldots, \\
& \cdots & &
\end{array}
$$

$$
a_{12n}^6 = x^{4nq_0'' + 1}, \quad a_{12n+1}^6 = y x^{4nq_1''}, \quad a_{12n+2}^6 = y^i x^{q_2''}, \quad \ldots,
$$

where q_u'' $(0 \leq u \leq 11)$ are quaternion numbers which represented in the form $q_u'' = a_u'' + b_u'' i + c_u'' j + d_u'' k$ $\left(a_u'', b_u'', c_u'', d_u'' \in \mathbb{R},\ 0 \leq u \leq 11\right)$ such that $\gcd(q_0'', q_1'', q_2'', \cdots, q_{11}'') = 1$.

So we need the smallest integer n such that $4n = 2^m k$ for $k \in \mathbb{N}$. Thus it is verified that the lengths of the periods of the sequences $F_{(x,y)}^{q,4}(Q_{2^{m+1}})$, $F_{(x,y)}^{q,5}(Q_{2^{m+1}})$ and $F_{(x,y)}^{q,6}(Q_{2^{m+1}})$ are 12.2^{m-2}. □

Now, for the generating pair (x, y), we give the quaternion-type cyclic-Fibonacci orbits of the quaternion groups Q_8 and Q_{16} presented by $Q_8 = \langle x, y \mid x^4 = 1,\ x^2 = y^2,\ y^{-1} x y = x^{-1} \rangle$ and $Q_{16} = \langle x, y \mid x^8 = 1,\ x^4 = y^2,\ y^{-1} x y = x^{-1} \rangle$, respectively.

Example 4.1.

(i). *The sequence* $F_{(x,y)}^{q,2}(Q_8)$ *is*

$$
\begin{array}{llllll}
a_0^2 = x, & a_1^2 = y, & a_2^2 = y^j x^i, & a_3^2 = y^{2k} x^{-1}, & a_4^2 = y^{1+2k}, & a_5^2 = y^{2i-j} x^{-i}, \\
a_6^2 = x, & a_7^2 = y, & a_8^2 = y^j x^i, & a_9^2 = y^{2k} x^{-1}, & a_{10}^2 = y^{1+2k}, & a_{11}^2 = y^{2i-j} x^{-i}, \\
a_{12}^2 = x, & a_{13}^2 = y, & a_{14}^2 = y^j x^i, & a_{15}^2 = y^{2k} x^{-1}, & a_{16}^2 = y^{1+2k}, & a_{17}^2 = y^{2i-j} x^{-i}, \ \ldots .
\end{array}
$$

So $LF_{(x,y)}^{q,2}(Q_8) = 6$.

(ii). *The sequence* $F_{(x,y)}^{q,4}(Q_8)$ *is*

$$
\begin{array}{llll}
a_0^4 = x, & a_1^4 = y, & a_2^4 = y^j x^i, & a_3^4 = y^{j-i} x^j, \\
a_4^4 = y^{1-i+k} x^{j+k}, & a_5^4 = y^{1+i+j+2k} x^{-1+i+k}, & a_6^4 = y^{2j+2k} x^{i+j-k}, & a_7^4 = y^{-i+j+3k}, \\
a_8^4 = y^{-1+3i-2j+3k} x^{-1+j+k}, & a_9^4 = y^{i+3j} x^{-1-i-k}, & a_{10}^4 = y^{3j+2k} x^{j-k}, & a_{11}^4 = y^{3k} x^j, \\
a_{12}^4 = x, & a_{13}^4 = y, & a_{14}^4 = y^j x^i, & a_{15}^4 = y^{j-i} x^j, \\
a_{16}^4 = y^{1-i+k} x^{j+k}, & a_{17}^4 = y^{1+i+j+2k} x^{-1+i+k}, & a_{18}^4 = y^{2j+2k} x^{i+j-k}, & a_{19}^4 = y^{-i+j+3k}, \\
a_{20}^4 = y^{-1+3i-2j+3k} x^{-1+j+k}, & a_{21}^4 = y^{i+3j} x^{-1-i-k}, & a_{22}^4 = y^{3j+2k} x^{j-k}, & a_{23}^4 = y^{3k} x^j, \ldots .
\end{array}
$$

So, $LF_{(x,y)}^{q,4}(Q_8) = 12$.

(iii). *The sequence* $F_{(x,y)}^{q,1}(Q_{16})$ *is*

$$
\begin{array}{llllll}
a_0^1 = x, & a_1^1 = y, & a_2^1 = y^i x^k, & a_3^1 = y^{2j} x^{-1}, & a_4^1 = y^{-3} x^{-2j}, & a_5^1 = y^{-i} x^{-3k}, \\
a_6^1 = x^5, & a_7^1 = y, & a_8^1 = y^i x^{5k}, & a_9^1 = y^{2j} x^3, & a_{10}^1 = y^{-3} x^{-2j}, & a_{11}^1 = y^{-5i} x^k, \\
a_{12}^1 = x, & a_{13}^1 = y, & a_{14}^1 = y^i x^k, & a_{15}^1 = y^{2j} x^{-1}, & a_{16}^1 = y^{-3} x^{-2j}, & a_{17}^1 = y^{-i} x^{-3k}, \\
a_{18}^1 = x^5, & a_{19}^1 = y, & a_{20}^1 = y^i x^{5k}, & a_{21}^1 = y^{2j} x^3, & a_{22}^1 = y^{-3} x^{-2j}, & a_{23}^1 = y^{-5i} x^k, \ \ldots,
\end{array}
$$

which implies that $LF_{(x,y)}^{q,1}(Q_{16}) = 12$.

(iv). *The sequence $F_{(x,y)}^{q,6}(Q_{16})$ is*

$$
\begin{array}{lll}
a_0^6 = x, & a_1^6 = y, & a_2^6 = y^i x^k, \\
a_3^6 = y^{i-k} x^i & a_4^6 = y^{1+j-k} x^{i+j}, & a_5^6 = y^{1+i+2j+k} x^{-1+j+k}, \\
a_6^6 = y^{-2+2i+2j} x^{-2+i-j+k}, & a_7^6 = y^{-2-i-3j-3k} x^{-2+2i-2k}, & a_8^6 = y^{1+j+k} x^{-3-i+3j-2k}, \\
a_9^6 = y^{-i+3k} x^{-5-4i-j+k}, & a_{10}^6 = y^i x^{-4-i-7j-4k}, & a_{11}^6 = y^{-j} x^{-3i-4k} \\
a_{12}^6 = x^{1+4j-4k}, & a_{13}^6 = y x^{4+4k}, & a_{14}^6 = y^i x^{4-4j-7k}, \\
a_{15}^6 = y^{i-k} x^{4-3i}, & a_{16}^6 = y^{1+j-k} x^{4-7i+j+4k}, & a_{17}^6 = y^{1+i+2j+k} x^{7+4i-7j+5k}, \\
a_{18}^6 = y^{-2+2i+2j} x^{6+i+3j-3k}, & a_{19}^6 = y^{-2-i+3j-3k} x^{2+2i+2k}, & a_{20}^6 = y^{1+j+k} x^{-7+7i+7j+6k}, \\
a_{21}^6 = y^{-i+3k} x^{-1-j+k}, & a_{22}^6 = y^i x^{7i+j}, & a_{23}^6 = y^{-j} x^{-7i}, \\
 & & \\
a_{24}^6 = x, & a_{25}^6 = y, & a_{26}^6 = y^i x^k, \\
a_{27}^6 = y^{i-k} x^i, & a_{28}^6 = y^{1+j-k} x^{i+j}, & a_{29}^6 = y^{1+i+2j+k} x^{-1+j+k}, \\
a_{30}^6 = y^{-2+2i+2j} x^{-2+i-j+k}, & a_{31}^6 = y^{-2-i-3j-3k} x^{-2+2i-2k}, & a_{32}^6 = y^{1+j+k} x^{-3-i+3j-2k}, \\
a_{33}^6 = y^{-i+3k} x^{-5-4i-j+k}, & a_{34}^6 = y^i x^{-4-i-7j-4k}, & a_{35}^6 = y^{-j} x^{-3i-4k} \\
a_{36}^6 = x^{1+4j-4k}, & a_{37}^6 = y x^{4+4k}, & a_{38}^6 = y^i x^{4-4j-7k}, \\
a_{39}^6 = y^{i-k} x^{4-3i}, & a_{40}^6 = y^{1+j-k} x^{4-7i+j+4k}, & a_{41}^6 = y^{1+i+2j+k} x^{7+4i-7j+5k}, \\
a_{42}^6 = y^{-2+2i+2j} x^{6+i+3j-3k}, & a_{43}^6 = y^{-2-i+3j-3k} x^{2+2i+2k}, & a_{44}^6 = y^{1+j+k} x^{-7+7i+7j+6k}, \\
a_{45}^6 = y^{-i+3k} x^{-1-j+k}, & a_{46}^6 = y^i x^{7i+j}, & a_{47}^6 = y^{-j} x^{-7i}, \qquad \ldots,
\end{array}
$$

which implies that $LF_{(x,y)}^{q,6}(Q_{16}) = 24$.

Conclusion

In this paper, we defined the quaternion-type cyclic-Fibonacci sequences and then we obtained the relationships among the elements of these sequences and the generating matrices of these sequences. Also, we gave the Cassini formula, generating functions of the quaternion-type cyclic-Fibonacci sequences.

Then, we studied the quaternion-type cyclic-Fibonacci sequences modulo m. Furthermore, we got the cyclic groups generated by reducing the multiplicative orders of the generating matrices and the auxiliary equations of these sequences modulo m and then, we investigated the orders of these cyclic groups. Moreover, using the terms of 2-generator groups which is called the quaternion-type cyclic-Fibonacci orbit, we redefined the quaternion-type cyclic-Fibonacci sequences. Also, these sequences in finite groups were examined in detail. With this study, we will gain a new perspective to the Fibonacci quaternions in the literature.

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