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Some new relations between $T(a_1, a_2, a_3, a_4, a_5; n)$ and $N(a_1, a_2, a_3, a_4, a_5; n)$

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Abstract: Let $N(a_1, a_2, a_3, a_4, a_5; n)$ and $T(a_1, a_2, a_3, a_4, a_5; n)$ count the representations of n as $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + a_5x_5^2$ and $a_1X_1(X_1 + 1)/2 + a_2X_2(X_2 + 1)/2 + a_3X_3(X_3 + 1)/2 + a_4X_4(X_4 + 1)/2 + a_5X_5(X_5 + 1)/2$, respectively, where a_1, a_2, a_3, a_4, a_5 are positive integers, x_1, x_2, x_3, x_4, x_5 are integers and $n, X_1, X_2, X_3, X_4, X_5$ are nonnegative integers. In this paper, we establish some new relations between $N(a_1, a_2, a_3, a_4, a_5; n)$ and $T(a_1, a_2, a_3, a_4, a_5; n)$. Also, we prove that $T(a_1, a_2, a_3, a_4, a_5; n)$ is a linear combination of $N(a_1, a_2, a_3, a_4, a_5; m)$ and $N(a_1, a_2, a_3, a_4, a_5; m/4)$, where $m = 8n + a_1 + a_2 + a_3 + a_4 + a_5$, for various values of a_1, a_2, a_3, a_4, a_5.

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1 Introduction

Let $\mathbb{N}^+$, $\mathbb{N}$ and $\mathbb{Z}$ denote the set of positive integers, the set of nonnegative integers and the set of integers, respectively. Let $\mathbb{Z}^5 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and $\mathbb{N}^5 = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. For $a_1, a_2, a_3, a_4, a_5 \in \mathbb{N}^+$ and $n \in \mathbb{N}$, define

$$N(a_1,a_2,a_3,a_4,a_5;n) := \left|\left\{(x_1,x_2,x_3,x_4,x_5)\in\mathbb{Z}^5 : a_1x_1^2+\cdots+a_5x_5^2=n\right\}\right|$$

and

$$T(a_1,a_2,a_3,a_4,a_5;n) := \left|\left\{(x_1,x_2,x_3,x_4,x_5)\in\mathbb{N}^5 : a_1\frac{x_1(x_1+1)}{2}+\cdots+a_5\frac{x_5(x_5+1)}{2}=n\right\}\right|,$$

where we take $N(a_1,a_2,a_3,a_4,a_5;0)=T(a_1,a_2,a_3,a_4,a_5;0)=1$.

Recently, numerous relations have been proved between sums of squares and sums of triangular numbers. The relations between sums of squares and sums of triangular numbers were first discovered by Bateman and Knopp in [5]. They proved that for $1\le k\le 7$,

$$r_k(8n+k)=2^k\left(1+\frac{k(k-1)(k-2)(k-3)}{48}\right)t_k(n),$$

where $r_k(n)$ and $t_k(n)$ denote the number of representations of n as the sum of k squares and the sum of k triangular numbers, respectively. In [2], Barrucand et al. used generating functions to rediscover the results of Bateman and Knopp. Later, Cooper and Hirschhorn [8] gave a bijective proof of these results. Afterward, Adiga et al. [1] provided a generalization of the result $r_k(8n+k)=c_kt_k(n)$, which is valid for $1\le k\le 7$. They introduced partitions of k as the index and this was a step forward. Later, Baruah et al. [3] extended these results to when k is a partition of 8. Their results, especially, [3, Theorem (1.4)], established a prototype for further research including the Theorems (3.1) and (3.2) of this paper. Thereafter, Wang and Sun [13, 14] proved several relations between sums of squares and sums of triangular numbers. In a series of papers [9–12], Sun proved many relations between sums of squares and sums of triangular numbers. Particularly, in [11], Sun proved some new relations between $N(a_1,a_2,a_3,a_4;n)$ and $T(a_1,a_2,a_3,a_4;n)$ and proposed 23 conjectures establishing the relations between $N(a_1,a_2,a_3,a_4;n)$ and $T(a_1,a_2,a_3,a_4;n)$. Five of these conjectures are proved by Yao [17] by using (p,k)-parametrization of theta functions and five conjectures follow as special cases in [16]. Sun himself proved three conjectures in [10] and three conjectures are proved by Xia and Zhong in [15] by utilizing theta function identities. The remaining seven conjectures are proved by Baruah et al. in [4]. They proved six conjectures by using Ramanujan's theta function identities and one is proved by elementary techniques. Recently, Cao and Lin [7] confirmed some conjectures of Sun [12] by using theta function identities.

In all of the above papers, numerous relations have been established between $N(a_1,a_2,a_3;n)$ and $T(a_1,a_2,a_3;n)$ and between $N(a_1,a_2,a_3,a_4;n)$ and $T(a_1,a_2,a_3,a_4;n)$. In this paper, we establish some new relations between $N(a_1,a_2,a_3,a_4,a_5;n)$ and $T(a_1,a_2,a_3,a_4,a_5;n)$ by using Ramanujan's theta function identities. Moreover, we prove that $T(a_1,a_2,a_3,a_4,a_5;n)$ is a linear combination of $N(a_1,a_2,a_3,a_4,a_5;m)$ and $N(a_1,a_2,a_3,a_4,a_5;m/4)$, where $m=8n+a_1+a_2+a_3+a_4+a_5$, for various values of a_1,a_2,a_3,a_4,a_5.

We organize the rest of the paper in the following way. In Section 2, we present the preliminary facts on Ramanujan's theta functions and useful lemmas. In the subsequent section, we prove the main results.

2 Preliminaries

Ramanujan's theta functions $\phi(q)$ and $\psi(q)$ are defined by

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

and

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

For $a_1,\ a_2,\ a_3,\ a_4,\ a_5 \in \mathbb{N}^+$, the generating functions of $N(a_1, a_2, a_3, a_4, a_5; n)$ and $T(a_1, a_2, a_3, a_4, a_5; n)$ are, respectively, given by

$$\sum_{n=0}^{\infty} N(a_1, a_2, a_3, a_4, a_5; n) q^n = \phi(q^{a_1})\phi(q^{a_2})\phi(q^{a_3})\phi(q^{a_4})\phi(q^{a_5}) \tag{1}$$

and

$$\sum_{n=0}^{\infty} T(a_1, a_2, a_3, a_4, a_5; n) q^n = \psi(q^{a_1})\psi(q^{a_2})\psi(q^{a_3})\psi(q^{a_4})\psi(q^{a_5}). \tag{2}$$

From [6, p. 36, entry 22 (i), (ii)], we have

$$\phi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \quad \psi(q) = \frac{f_2^2}{f_1}, \tag{3}$$

where, we use the standard notations

$$(a; q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n), \quad |q| < 1$$

and

$$f_k := (q^k; q^k)_\infty.$$

We require some well-known 2- and 3-dissections and several identities on Ramanujan's theta functions, which are listed in the following lemmas.

Lemma 2.1. *The following* 2*-dissections hold:*

$$\phi(q) = \phi(q^4) + 2q\psi(q^8), \tag{4}$$
$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2, \tag{5}$$
$$\phi(q)\psi(q^2) = \psi(q)^2, \tag{6}$$
$$\phi(q)\phi(q^3) = \phi(q^4)\phi(q^{12}) + 2q\psi(q^2)\psi(q^6) + 4q^4\psi(q^8)\psi(q^{24}), \tag{7}$$
$$\psi(q^3)\psi(q^5) = \psi(q^8)\phi(q^{60}) + q^3\psi(q^2)\psi(q^{30}) + q^{14}\phi(q^4)\psi(q^{120}), \tag{8}$$
$$\psi(q)\psi(q^{15}) = \psi(q^6)\psi(q^{10}) + q\phi(q^{20})\psi(q^{24}) + q^3\phi(q^{12})\psi(q^{40}). \tag{9}$$

Proof. The first identity follows from [6, p. 40, Entry 25 (i), (ii)]. The second identity follows from [6, p. 40, Entry 25 (v), (vi)]. The identity (6) is [6, p. 40, Entry 25 (iv)]. By using (4) in [6, p. 68, Eq. (36.2)] and setting $(\mu, \nu) = (2, 1)$, we readily arrive at (7) and (8) follows by setting $(\mu, \nu) = (4, 1)$ in [6, p. 69, Eq. (36.8)]. Finally, the identity (9) follows from [4, Lemma 2.5, Eq. (2.18)]. □

Lemma 2.2. *The following* 3*-dissections hold:*

$$\phi(q) = \frac{f_2^5}{f_1^2 f_4^2} = \frac{f_{18}^5}{f_9^2 f_{36}^2} + 2q\frac{f_6^2 f_9 f_{36}}{f_3 f_{12} f_{18}}, \tag{10}$$

$$\psi(q) = \frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}. \tag{11}$$

Proof. The above two identities follow from [6, p. 49, Corollary]. □

3 New relations between sums of squares and sums of triangular numbers

In this section, we prove five theorems. In these theorems, we prove some new relations between sums of five squares and sums of five triangular numbers. In Theorems (3.1) and (3.2), we prove that $T(a_1, a_2, a_3, a_4, a_5; n)$ can be represented as a linear combination of $N(a_1, a_2, a_3, a_4, a_5; m)$ and $N(a_1, a_2, a_3, a_4, a_5; m/4)$, where $m = 8n + a_1 + a_2 + a_3 + a_4 + a_5$, for certain values of a_1, a_2, a_3, a_4, a_5. The case $(1, 1, 1, 1, 4)$ of Theorem (3.1) and the case $(1, 1, 2, 2, 2)$ of Theorem (3.2) are the special cases of [3, Theorem (1.4)]. In Theorem (3.3), we prove some new relations between $T(a_1, a_2, a_3, a_4, a_5; n)$ and $N(a_1, a_2, a_3, a_4, a_5; m)$ for several values of a_1, a_2, a_3, a_4, a_5. The cases $(1, 1, 1, 1, 3), (1, 1, 1, 1, 2)$ and $(1, 1, 1, 2, 2)$ of Theorem (3.3) are due to Adiga et al. [1]. In Theorem (3.4), we prove a new relation between $T(3, 5, 12, 12, 12; 4n + 3)$ and $N(3, 5, 12, 12, 12; 8n + 17)$. In Theorem (3.5), we prove two new results about $T(3, 5, 12, 12, 12; 3n + 1)$ and $N(3, 5, 12, 12, 12; 3n + 1)$. The results in Theorems (3.4) and (3.5) are quite different from the results proved in the first three theorems as these results do not involve $m = 8n + a_1 + a_2 + a_3 + a_4 + a_5$.

Theorem 3.1. *If* $(a_1, a_2, a_3, a_4, a_5) = (1, 1, 1, 1, 4)$ *or* $(1, 1, 1, 1, 12)$*, then for* $n \in \mathbb{N}^+$*,*

$$\begin{aligned} T(a_1, a_2, a_3, a_4, a_5; n) = \frac{1}{48}(&N(a_1, a_2, a_3, a_4, a_5; 8n + a_1 + a_2 + a_3 + a_4 + a_5) \\ &- N(a_1, a_2, a_3, a_4, a_5; (8n + a_1 + a_2 + a_3 + a_4 + a_5)/4)). \end{aligned} \tag{12}$$

Proof. We only prove the case $(1, 1, 1, 1, 4)$ of (12), while the other case $(1, 1, 1, 1, 12)$ can be proved by similar method.

In view of (1) and (4), we have

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 1, 1, 1, 4; n)q^n &= \phi(q)^4\phi(q^4) \\ &= (\phi(q^4)^4 + 8q\phi(q^4)^3\psi(q^8) + 24q^2\phi(q^4)^2\psi(q^8)^2 + 32q^3\phi(q^4)\psi(q^8)^3 \\ &\quad + 16q^4\psi(q^8)^4) \times \phi(q^4). \end{aligned}$$

Extracting the terms involving q^{2n} and replacing q^2 with q, we have

$$\sum_{n=0}^{\infty} N(1,1,1,1,4;2n)q^n = (\phi(q^2)^4 + 24q\phi(q^2)^2\psi(q^4)^2 + 16q^2\psi(q^4)^4) \times \phi(q^2). \tag{13}$$

Again, extracting the terms involving q^{2n} and replacing q^2 with q and by using (4), we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,1,1,1,4;4n)q^n &= (\phi(q)^4 + 16q\psi(q^2)^4) \times \phi(q)\\ &= (\phi(q^4)^4 + 8q\phi(q^4)^3\psi(q^8) + 24q^2\phi(q^4)^2\psi(q^8)^2 + 32q^3\phi(q^4)\psi(q^8)^3\\ &\quad + 16q^4\psi(q^8)^4 + 16q\psi(q^2)^4) \times (\phi(q^4) + 2q\psi(q^8)).\end{aligned}$$

Extracting the terms involving q^{2n} and replacing q^2 with q and by using (13), we have

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,1,1,1,4;8n)q^n &= \phi(q^2)(\phi(q^2)^4 + 24q\phi(q^2)^2\psi(q^4)^2 + 16q^2\psi(q^4)^4)\\ &\quad + 16q\phi(q^2)\psi(q^4)^2(\phi(q^2)^2 + 4q\psi(q^4)^2) + 32q\psi(q)^4\psi(q^4)\\ &= \sum_{n=0}^{\infty} N(1,1,1,1,4;2n)q^n + 16q\phi(q)^2\phi(q^2)\psi(q^4)^2 + 32q\psi(q)^4\psi(q^4).\end{aligned}$$

Thus, in view of (2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,1,1,1,4;8n)q^n - \sum_{n=0}^{\infty} N(1,1,1,1,4;2n)q^n &= 48q\psi(q)^4\psi(q^4)\\ &= 48\sum_{n=0}^{\infty} T(1,1,1,1,4;n)q^{n+1},\end{aligned}$$

and hence,

$$N(1,1,1,1,4;8n+8) - N(1,1,1,1,4;2n+2) = 48T(1,1,1,1,4,n),$$

which completes the proof of the theorem. $\square$

Theorem 3.2. *If* $(a_1, a_2, a_3, a_4, a_5) = (1,1,2,2,2)$ *or* $(1,3,3,3,6)$ *and* $t = 80$ *or* 128*, respectively, then for* $n \in \mathbb{N}^+$*,*

$$\begin{aligned}T(a_1,a_2,a_3,a_4,a_5;n) &= \frac{1}{t}(N(a_1,a_2,a_3,a_4,a_5;8n + a_1 + a_2 + a_3 + a_4 + a_5)\\ &\quad - N(a_1,a_2,a_3,a_4,a_5;(8n + a_1 + a_2 + a_3 + a_4 + a_5)/4)).\end{aligned} \tag{14}$$

Proof. We only prove the case $(1,3,3,3,6)$ of (14), while the other case $(1,1,2,2,2)$ can be proved by similar method.

In view of (1) and (4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,3,3,3,6;n)q^n &= \phi(q)\phi(q^3)^3\phi(q^6) \\ &= \phi(q^6)(\phi(q^4) + 2q\psi(q^8))\\ &\quad \times (\phi(q^{12})^3 + 6q^3\phi(q^{12})^2\psi(q^{24}) + 12q^6\phi(q^{12})\psi(q^{24})^2 + 8q^9\psi(q^{24})^3).\end{aligned} \tag{15}$$

Extracting the terms involving q^{2n} and replacing q^2 with q and using (4), we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,3,3,3,6;2n)q^n &= \phi(q^3)(\phi(q^2)\phi(q^6)^3 + 12q^2\phi(q^6)^2\psi(q^4)\psi(q^{12}) \\ &\quad + 12q^3\phi(q^2)\phi(q^6)\psi(q^{12})^2 + 16q^5\psi(q^4)\psi(q^{12})^3) \\ &= (\phi(q^{12}) + 2q^3\psi(q^{24}))(\phi(q^2)\phi(q^6)^3 + 12q^2\phi(q^6)^2\psi(q^4)\psi(q^{12}) \\ &\quad + 12q^3\phi(q^2)\phi(q^6)\psi(q^{12})^2 + 16q^5\psi(q^4)\psi(q^{12})^3).\end{aligned}$$

Again, extracting the terms involving q^{2n} and replacing q^2 with q, we have

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,3,3,3,6;4n)q^n &= \phi(q)\phi(q^3)^3\phi(q^6) + 12q\phi(q^3)^2\phi(q^6)\psi(q^2)\psi(q^6) \\ &\quad + 24q^3\phi(q)\phi(q^3)\psi(q^6)^2\psi(q^{12}) + 32q^4\psi(q^2)\psi(q^6)^3\psi(q^{12}).\end{aligned}$$

In view of (5), (7) and (15), we have,

$$\begin{aligned}&\sum_{n=0}^{\infty}(N(1,3,3,3,6;4n) - N(1,3,3,3,6;n))q^n \\ &\quad = 12q\phi(q^6)\psi(q^2)\psi(q^6)(\phi(q^6)^2 + 4q^3\psi(q^{12})^2) + 32q^4\psi(q^2)\psi(q^6)^3\psi(q^{12}) \\ &\qquad + 24q^3\psi(q^6)^2\psi(q^{12})(\phi(q^4)\phi(q^{12}) + 2q\psi(q^2)\psi(q^6) + 4q^4\psi(q^8)\psi(q^{24})),\end{aligned}$$

From which we further extract

$$\begin{aligned}&\sum_{n=0}^{\infty}(N(1,3,3,3,6;8n) - N(1,3,3,3,6;2n))q^n \\ &\quad = 32q^2\psi(q)\psi(q^3)^3\psi(q^6) + 48q^2\phi(q^3)\psi(q)\psi(q^3)\psi(q^6)^2 + 48q^2\psi(q)\psi(q^3)^3\psi(q^6) \\ &\quad = 80q^2\psi(q)\psi(q^3)^3\psi(q^6) + 48q^2\psi(q)\psi(q^3)^3\psi(q^6),\end{aligned}$$

where the second equality is due to (6).

By using (2), we arrive at

$$\sum_{n=0}^{\infty}(N(1,3,3,3,6;8n) - N(1,3,3,3,6;2n))q^n = 128\sum_{n=0}^{\infty} T(1,3,3,3,6;n)q^{n+2},$$

and hence,

$$N(1,3,3,3,6;8n+16) - N(1,3,3,3,6;2n+4) = 128T(1,3,3,3,6;n),$$

which completes the proof of the theorem. □

Theorem 3.3. *If* $(a_1,a_2,a_3,a_4,a_5) = (1,1,1,1,3)$ *or* $(1,3,3,3,3)$ *or* $(1,1,1,1,2)$ *or* $(1,1,1,2,2)$ *and* $t = 112$ *or* 112 *or* 144 *or* 128*, respectively, then for* $n \in \mathbb{N}^+$*,*

$$T(a_1,a_2,a_3,a_4,a_5;n) = \frac{1}{t}N(a_1,a_2,a_3,a_4,a_5;8n + a_1 + a_2 + a_3 + a_4 + a_5). \tag{16}$$

Proof. We only prove the case $(1,1,1,2,2)$ of (16), while the other cases can be proved by similar method.

In view of (1) and (4), we have

$$\sum_{n=0}^{\infty} N(1,1,1,2,2;n)q^n = \phi(q)^3\phi(q^2)^2$$
$$= (\phi(q^4)^3 + 6q\phi(q^4)^2\psi(q^8) + 12q^2\phi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3)\phi(q^2)^2.$$

Extracting the terms involving q^{2n+1}, dividing with q and replacing q^2 with q and then using (5), we have

$$\sum_{n=0}^{\infty} N(1,1,1,2,2;2n+1)q^n = (6\phi(q^2)^2\psi(q^4) + 8q\psi(q^4)^3)\phi(q)^2$$
$$= (6\phi(q^2)^2\psi(q^4) + 8q\psi(q^4)^3)(\phi(q^2)^2 + 4q\psi(q^4)^2).$$

Again, extracting the terms involving q^{2n+1}, dividing with q and replacing q^2 with q, we have

$$\sum_{n=0}^{\infty} N(1,1,1,2,2;4n+3)q^n = 32\phi(q)^2\psi(q^2)^3$$
$$= 32\psi(q^2)^3(\phi(q^2)^2 + 4q\psi(q^4)^2).$$

From which we extract

$$\sum_{n=0}^{\infty} N(1,1,1,2,2;8n+7)q^n = 128\psi(q)^3\psi(q^2)^2$$
$$= 128\sum_{n=0}^{\infty} T(1,1,1,2,2;n)q^n,$$

and hence,

$$N(1,1,1,2,2;8n+7) = 128T(1,1,1,2,2;n),$$

which completes the proof. □

Theorem 3.4. *Let $n \in \mathbb{N}^+$. Then*

$$T(3,5,12,12,12;4n+3) = \frac{1}{16}N(3,5,12,12,12;8n+17). \tag{17}$$

Proof. In view of (1) and (4), we have

$$\sum_{n=0}^{\infty} N(3,5,12,12,12;n)q^n = \phi(q^3)\phi(q^5)\phi(q^{12})^3$$
$$= (\phi(q^{12}) + 2q^3\psi(q^{24}))(\phi(q^{20}) + 2q^5\psi(q^{40}))\phi(q^{12})^3.$$

If we extract those terms involving q^{2n+1}, then divide with q and replace q^2 with q, we have

$$\sum_{n=0}^{\infty} N(3,5,12,12,12;2n+1)q^n = (2q\phi(q^{10})\psi(q^{12}) + 2q^2\phi(q^6)\psi(q^{20}))\phi(q^6)^3.$$

Again, extracting the terms involving q^{2n+1}, dividing with q and replacing q^2 with q and then using (4), we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty} N(3,5,12,12,12;4n+1)q^n &= 2q\phi(q^3)^4\psi(q^{10}) \\ &= 2q\psi(q^{10})(\phi(q^{12})^4 + 8q^3\phi(q^{12})^3\psi(q^{24}) + 24q^6\phi(q^{12})^2\psi(q^{24})^2 \\ &\quad + 32q^9\phi(q^{12})\psi(q^{24})^3 + 16q^{12}\psi(q^{24})^4),\end{aligned}$$

From which we further extract

$$\begin{aligned}\sum_{n=0}^{\infty} N(3,5,12,12,12;8n+1)q^n &= 16q^2\phi(q^6)^3\psi(q^5)\psi(q^{12}) + 64q^5\phi(q^6)\psi(q^5)\psi(q^{12})^3 \\ &= 16q^2\phi(q^3)^2\phi(q^6)\psi(q^5)\psi(q^{12}) \\ &= 16q^2\psi(q^3)^4\psi(q^5),\end{aligned}$$

where the last equality is due to (6).

Thus,

$$\sum_{n=0}^{\infty} N(3,5,12,12,12;8n+1)q^n = 16q^2\psi(q^3)^4\psi(q^5). \tag{18}$$

On the other hand, using (2) and (8), we have,

$$\begin{aligned}\sum_{n=0}^{\infty} T(3,5,12,12,12;n)q^n &= \psi(q^3)\psi(q^5)\psi(q^{12})^3 \\ &= \psi(q^{12})^3(\psi(q^8)\phi(q^{60}) + q^3\psi(q^2)\psi(q^{30}) + q^{14}\phi(q^4)\psi(q^{120})).\end{aligned}$$

Now, using (9), we have

$$\begin{aligned}\sum_{n=0}^{\infty} T(3,5,12,12,12;2n+1)q^n &= q\psi(q)\psi(q^6)^3\psi(q^{15}) \\ &= q\psi(q^6)^3(\psi(q^6)\psi(q^{10}) + q\phi(q^{20})\psi(q^{24}) + q^3\phi(q^{12})\psi(q^{40})),\end{aligned}$$

So,

$$\sum_{n=0}^{\infty} T(3,5,12,12,12;4n+3)q^n = \psi(q^3)^4\psi(q^5). \tag{19}$$

Hence, in view of (18) and (19), we have

$$\sum_{n=0}^{\infty} N(3,5,12,12,12;8n+1)q^n = 16\sum_{n=0}^{\infty} T(3,5,12,12,12;4n+3)q^{n+2},$$

So,

$$N(3,5,12,12,12;8n+17) = 16T(3,5,12,12,12;4n+3),$$

which completes the proof. $\square$

Theorem 3.5. *Let $n \in \mathbb{N}^+$. Then*

$$T(3,5,12,12,12;3n+1) = N(3,5,12,12,12;3n+1) = 0. \tag{20}$$

Proof. In view of (1) and (10), we have

$$\begin{aligned}\sum_{n=0}^{\infty} N(3,5,12,12,12;n)q^n &= \phi(q^3)\phi(q^5)\phi(q^{12})^3 \\ &= \phi(q^3)\phi(q^{12})^3\left(\frac{f_{90}^5}{f_{45}^2 f_{180}^2} + 2q^5\frac{f_{30}^2 f_{45} f_{180}}{f_{15} f_{60} f_{90}}\right).\end{aligned}$$

If we extract those terms involving q^{3n+1}, then divide with q and replace q^3 with q, we have

$$N(3,5,12,12,12;3n+1) = 0. \tag{21}$$

Also, in view of (2) and (11), we have

$$\begin{aligned}\sum_{n=0}^{\infty} T(3,5,12,12,12;n)q^n &= \psi(q^3)\psi(q^5)\psi(q^{12})^3 \\ &= \psi(q^3)\psi(q^{12})^3\left(\frac{f_{30} f_{45}^2}{f_{15} f_{90}} + q^5\frac{f_{90}^2}{f_{45}}\right).\end{aligned}$$

If we extract those terms involving q^{3n+1}, then divide with q and replace q^3 with q, we have

$$T(3,5,12,12,12;3n+1) = 0. \tag{22}$$

(21) and (22) completes the proof. □

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