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Some identities involving Chebyshev polynomials, Fibonacci polynomials and their derivatives

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Abstract: In this paper, we will derive the explicit formulae for Chebyshev polynomials of the third and fourth kind with odd and even indices using the combinatorial method. Similar results are also deduced for their r^{th} derivatives. Finally, some identities involving Chebyshev polynomials of the third and fourth kind with even and odd indices and Fibonacci polynomials with negative indices are obtained.

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1 Introduction

Fibonacci numbers are one of the most widely studied mathematical objects. The developments in the theory of Fibonacci numbers have led to the creation of an unparalleled niche in the arena of existing mathematical research. In addition to mathematics, these numbers find enormous applications in a wide range of fields, including physical and medical sciences, computer sciences,

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statistics, and graph theory. Due to their wide spectrum of applications in nature in general and mathematics in particular, the fibonacci numbers have attracted the interest of mathematicians and made their existence indispensable in the development of mathematical theory [4]. Several mathematicians have analysed Fibonacci sequences, studied their properties, and obtained various generalisations in different directions [3, 9]. One such direction of generalisation extends the Fibonacci sequences to the sequences of Fibonacci polynomials and Chebyshev polynomials.

Several authors have studied various properties of Fibonacci polynomials and Chebyshev polynomials of the first and second kind and their interactions and deduced many interesting results [7, 8, 10, 11], but very little work has been done in the case of Chebyshev polynomials of the third and fourth kind. In this paper, we will discuss the behaviour of Chebyshev polynomials of the third and fourth kind and their relationship with Fibonacci polynomials.

To start with, the Fibonacci numbers are defined by the second-order linear recursive relation as follows:

$$\mathfrak{F}_\alpha = \mathfrak{F}_{\alpha-1} + \mathfrak{F}_{\alpha-2}, \tag{1}$$

for all $\alpha \geq 0$ with $\mathfrak{F}_0 = 0$, $\mathfrak{F}_1 = 1$.

Similarly, Lucas numbers, another variant of Fibonacci numbers, are defined by the recursive relation as follows:

$$\mathfrak{L}_\alpha = \mathfrak{L}_{\alpha-1} + \mathfrak{L}_{\alpha-2}, \tag{2}$$

for all $\alpha \geq 0$ with $\mathfrak{L}_0 = 2$, $\mathfrak{L}_1 = 1$.

In one of its generalisations, the Fibonacci and Lucas numbers are extended to the Fibonacci and Lucas polynomials [11], which are defined recursively as:

$$\mathfrak{F}_\alpha(\zeta) = \zeta\mathfrak{F}_{\alpha-1}(\zeta) + \mathfrak{F}_{\alpha-2}(\zeta), \quad \mathfrak{F}_0(\zeta) = 0, \quad \mathfrak{F}_1(\zeta) = 1, \tag{3}$$

$$\mathfrak{L}_\alpha(\zeta) = \zeta\mathfrak{L}_{\alpha-1}(\zeta) + \mathfrak{L}_{\alpha-2}(\zeta), \quad \mathfrak{L}_0(\zeta) = 2, \quad \mathfrak{L}_1(\zeta) = \zeta, \tag{4}$$

for all integers, $\alpha \geq 1$.

Following this pattern, Chebyshev polynomials also exhibit similar recursive relations.Chebyshev polynomials appear as solutions of the Chebyshev differential equation occurring as a special case in the analysis of a Strum-Liouville boundary value problem [5]. For any integer $\alpha \geq 0$, the Chebyshev polynomials of first $(\mathfrak{T}_\alpha(\zeta))$, second $(\mathfrak{U}_\alpha(\zeta))$, third $(\mathfrak{V}_\alpha(\zeta))$, and fourth $(\mathfrak{W}_\alpha(\zeta))$ kind [2, 8] are defined as follows:

$$\mathfrak{T}_\alpha(\zeta) = \cos \alpha\theta, \tag{5}$$

$$\mathfrak{U}_\alpha(\zeta) = \frac{\sin(\alpha+1)\theta}{\sin\theta}, \tag{6}$$

$$\mathfrak{V}_\alpha(\zeta) = \frac{\cos(\alpha+\frac{1}{2})\theta}{\cos\frac{\theta}{2}}, \tag{7}$$

$$\mathfrak{W}_\alpha(\zeta) = \frac{\sin(\alpha+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}, \tag{8}$$

where $\zeta = \cos\theta$, for all $\zeta \in [-1, 1]$ and $\alpha \in N$. Using De Moivre's theorem, these Chebyshev polynomials [2], can be written recursively for any integer $\alpha > 1$, as follows:

$$\mathfrak{T}_\alpha(\zeta) = 2\zeta\mathfrak{T}_{\alpha-1}(\zeta) - \mathfrak{T}_{\alpha-2}(\zeta), \quad \mathfrak{T}_0(\zeta) = 1, \quad \mathfrak{T}_1(\zeta) = \zeta, \tag{9}$$

$$\mathfrak{U}_\alpha(\zeta) = 2\zeta\mathfrak{U}_{\alpha-1}(\zeta) - \mathfrak{U}_{\alpha-2}(\zeta), \quad \mathfrak{U}_0(\zeta) = 1, \quad \mathfrak{U}_1(\zeta) = 2\zeta, \tag{10}$$

$$\mathfrak{V}_\alpha(\zeta) = 2\zeta\mathfrak{V}_{\alpha-1}(\zeta) - \mathfrak{V}_{\alpha-2}(\zeta), \quad \mathfrak{V}_0(\zeta) = 1, \quad \mathfrak{V}_1(\zeta) = 2\zeta - 1, \tag{11}$$

$$\mathfrak{W}_\alpha(\zeta) = 2\zeta\mathfrak{W}_{\alpha-1}(\zeta) - \mathfrak{W}_{\alpha-2}(\zeta), \quad \mathfrak{W}_0(\zeta) = 1, \quad \mathfrak{W}_1(\zeta) = 2\zeta + 1.$$

Further algebraic manipulations of these recursive relations give rise to the following explicit formulae [1,7]

$$\mathfrak{F}_\alpha(\zeta) = \frac{1}{2^\alpha\sqrt{\zeta^2+4}}[(\zeta + \sqrt{\zeta^2+4})^\alpha - (\zeta - \sqrt{\zeta^2+4})^\alpha], \tag{12}$$

$$\mathfrak{T}_\alpha(\zeta) = \frac{1}{2}[(\zeta + \sqrt{\zeta^2-1})^\alpha + (\zeta - \sqrt{\zeta^2-1})^\alpha], \tag{13}$$

$$\mathfrak{U}_\alpha(\zeta) = \frac{1}{2\sqrt{\zeta^2-1}}[(\zeta + \sqrt{\zeta^2-1})^{\alpha+1} - (\zeta - \sqrt{\zeta^2-1})^{\alpha+1}], \tag{14}$$

for Fibonacci polynomials and Chebyshev polynomials of the first and second kind, respectively. A wide variety of literature is available on the Chebyshev polynomials of the first and second kind whereas limited work has been done on the Chebyshev polynomials of third and fourth kind which opens up an intriguing field for the prospective researchers.

Several authors have analysed the sequence of Chebyshev polynomials of the first and second kind, studied their properties, established various formulae for these polynomials, and deduced many interrelated identities and their linkages with other orthogonal polynomials [7, 8, 10, 11]. Wang and Zhang [10] studied these polynomials and deduced some interesting identities for the power sums of the derivatives of the Chebyshev polynomials of the first kind. In [2, 10], the authors have deduced the explicit relations for the Fibonacci polynomial and Chebyshev polynomials of the first and second kind as follows:

$$\mathfrak{F}_\alpha(\zeta) = \sum_{\beta=0}^{[\frac{\alpha-1}{2}]} \binom{\alpha-\beta-1}{\beta} (\zeta)^{\alpha-2\beta-1}, \tag{15}$$

$$\mathfrak{T}_\alpha(\zeta) = \frac{\alpha}{2}\sum_{\beta=0}^{[\frac{\alpha}{2}]}(-1)^\beta \frac{(\alpha-\beta-1)!}{(\alpha-2\beta)!\beta!}(2\zeta)^{\alpha-2\beta}, \tag{16}$$

$$\mathfrak{U}_\alpha(\zeta) = \sum_{\beta=0}^{[\frac{\alpha}{2}]}(-1)^\beta \frac{((\alpha-\beta)!}{\beta!(\alpha-2\beta)!}(2\zeta)^{\alpha-2\beta}. \tag{17}$$

Similar relations are obtained by Yang Li [6,7] for the Chebyshev polynomials of first and second kind as follows:

$$\mathfrak{T}_{2\alpha}(\zeta) = \sum_{\beta=0}^{\alpha+1}\sum_{\gamma=0}^{\alpha} \frac{2^{2\gamma}(2\alpha\beta-\alpha)(\alpha+\gamma-1)!}{(-1)^{\beta+\alpha-1}(\alpha-\gamma)!(\gamma+\beta)!(\gamma-\beta+1)!}\mathfrak{F}_{2\beta-1}(\zeta), \tag{18}$$

$$\mathfrak{T}_{2\alpha-1}(\zeta) = \sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{\alpha} \frac{2^{2\gamma-1}(2\alpha\beta-\beta)(\alpha+\gamma-2)!}{(-1)^{\beta+\alpha}(\alpha-\gamma)!(\gamma+\beta)!(\gamma-\beta)!}\mathfrak{F}_{2\beta}(\zeta), \tag{19}$$

$$\mathfrak{T}^{(2\mathfrak{r}-1)}{}_{2\alpha}(\zeta) = \sum_{\beta=1}^{\alpha-r+2}\sum_{\gamma=r}^{\alpha} \frac{(-1)^{\alpha-r-\gamma}2^{2\gamma+r-1}(2\beta-1)(2\alpha+1)(\alpha+\gamma)!}{(\alpha-\gamma)!(\beta+\gamma-r+1)!(\gamma-r-\beta+2)!}\mathfrak{F}_{2\beta-1}(\zeta), \tag{20}$$

$$\mathfrak{T}^{(2\mathfrak{r})}{}_{2\alpha}(\zeta) = \sum_{\beta=1}^{\alpha-r+1}\sum_{\gamma=r}^{\alpha} \frac{(-1)^{\alpha-r-\gamma+1}2^{2\gamma+r}(2\beta\alpha-\alpha)(\alpha+\gamma-1)!}{(\alpha-\gamma)!(\beta+\gamma-r)!(\gamma-r-\beta+1)!}\mathfrak{F}_{2\beta-1}(\zeta). \tag{21}$$

Similar results were obatained for Chebyshev polynomials of second kind.

A close perusal of the cited literature suggests a fair possibility for the deduction of similar identities for the Chebyshev polynomials of the third and fourth kinds. So, here in this paper, our main focus will be on the derivation of similar explicit relations for the third and fourth kinds of Chebyshev polynomials, their derivatives, and their relationship with Fibonacci polynomials, which will definitely strengthen the existing and prospective research work on Chebyshev polynomials.

2 Main results

In this section, we will discuss the main results of this paper, which depict explicit relations for the Chebyshev polynomials of the third and fourth kinds, their derivatives, and their relationship with Fibonacci polynomials with negative indices. To begin with, we will proceed as follows: From [2, 8] it can be easily seen that, for any positive integer α,

$$\mathfrak{V}_{\alpha}(\zeta) = \mathfrak{U}_{\alpha}(\zeta) - \mathfrak{U}_{\alpha-1}(\zeta), \tag{22}$$

$$\mathfrak{W}_{\alpha}(\zeta) = \mathfrak{U}_{\alpha}(\zeta) + \mathfrak{U}_{\alpha-1}(\zeta), \tag{23}$$

$$\mathfrak{W}_{\alpha}(\zeta) = (-1)^{\alpha}\mathfrak{V}_{\alpha}(-\zeta), \tag{24}$$

$$\mathfrak{F}_{\alpha}(-\zeta) = (-1)^{\alpha+1}\mathfrak{F}_{\alpha}(\zeta), \tag{25}$$

$$\mathfrak{F}_{-\alpha}(\zeta) = (-1)^{\alpha-1}\mathfrak{F}_{\alpha}(\zeta). \tag{26}$$

Firstly, we will establish the explicit formulae for the Chebyshev polynomials of the third and fourth kind.

Theorem 2.1. *For any positive integer α,*

$$\mathfrak{V}_{2\alpha}(\zeta) = (2\zeta)^{2\alpha} + \sum_{\beta=0}^{\alpha-1}(-1)^{\alpha-\beta}2^{2\beta+1}\binom{\alpha+\beta}{2\beta}\left[\frac{(\zeta)^{2\beta}}{2} + \frac{(\alpha-\beta)}{(2\beta+1)}(\zeta)^{2\beta+1}\right],$$

$$\mathfrak{V}_{2\alpha+1}(\zeta) = \sum_{\beta=0}^{\alpha}(-1)^{\alpha-\beta}2^{2\beta+1}\binom{\alpha+\beta}{2\beta}\left[\frac{(\alpha+\beta+1)}{(2\beta+1)}(\zeta)^{2\beta+1} - \frac{(\zeta)^{2\beta}}{2}\right].$$

Proof. From [7], for any positive integer α, we have

$$\mathfrak{T}_{2\alpha}(\zeta) = \sum_{\beta=0}^{\alpha} \frac{(-1)^{\alpha-\beta}2^{2\beta}\alpha}{(\beta+\alpha)}\binom{\alpha+\beta}{2\beta}(\zeta)^{2\beta}, \tag{27}$$

$$\mathfrak{T}_{2\beta+1}(\zeta)=\sum_{\beta=0}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta}(2\alpha+1)}{(\beta+\alpha+1)}\binom{\alpha+\beta+1}{2\beta+1}(\zeta)^{2\beta+1}. \tag{28}$$

Using the fact, $\mathfrak{T}'_{\alpha}(\zeta)=\alpha\mathfrak{U}_{\alpha-1}(\zeta)$ with (27) and (28), we have

$$\mathfrak{U}_{2\alpha}(\zeta)=\sum_{\beta=0}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta}(2\beta+1)}{(\beta+\alpha+1)}\binom{\alpha+\beta+1}{2\beta+1}(\zeta)^{2\beta}, \tag{29}$$

$$\mathfrak{U}_{2\alpha+1}(\zeta)=\sum_{\beta=0}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta+2}(\beta+1)}{(\beta+\alpha+2)}\binom{\alpha+\beta+2}{2\beta+2}(\zeta)^{2\beta+1}, \tag{30}$$

$$\mathfrak{U}_{2\alpha-1}(\zeta)=\sum_{\beta=0}^{\alpha-1}\frac{(-1)^{\alpha-\beta-1}2^{2\beta+2}(\beta+1)}{(\beta+\alpha+1)}\binom{\alpha+\beta+1}{2\beta+2}(\zeta)^{2\beta+1}. \tag{31}$$

Thus, using (22) with (29) and (31), we have

$$\begin{aligned}\mathfrak{V}_{2\alpha}(\zeta)&=\mathfrak{U}_{2\alpha}(\zeta)-\mathfrak{U}_{2\alpha-1}(\zeta)\\&=\sum_{\beta=0}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta}(2\beta+1)}{(\beta+\alpha+1)}\binom{\alpha+\beta+1}{2\beta+1}(\zeta)^{2\beta}\\&\quad-\sum_{\beta=0}^{\alpha-1}\frac{(-1)^{\alpha-\beta-1}2^{2\beta+2}(\beta+1)}{(\beta+\alpha+1)}\binom{\alpha+\beta+1}{2\beta+2}(\zeta)^{2\beta+1}\\&=\sum_{\beta=0}^{\alpha}(-1)^{\alpha-\beta}2^{2\beta}\binom{\alpha+\beta}{2\beta}(\zeta)^{2\beta}\\&\quad-\sum_{\beta=0}^{\alpha-1}\frac{(-1)^{\alpha-\beta-1}2^{2\beta+1}(\alpha-\beta)}{(2\beta+1)}\binom{\alpha+\beta}{2\beta}(\zeta)^{2\beta+1}\\&=(2\zeta)^{2\alpha}+\sum_{\beta=0}^{\alpha-1}(-1)^{\alpha-\beta}2^{2\beta}\binom{\alpha+\beta}{2\beta}(\zeta)^{2\beta}\\&\quad+\sum_{\beta=0}^{\alpha-1}\frac{(-1)^{\alpha-\beta}2^{2\beta+1}(\alpha-\beta)}{(2\beta+1)}\binom{\alpha+\beta}{2\beta}(\zeta)^{2\beta+1}\\&=(2\zeta)^{2\alpha}+\sum_{\beta=0}^{\alpha-1}(-1)^{\alpha-\beta}2^{2\beta+1}\binom{\alpha+\beta}{2\beta}\left[\frac{(\zeta)^{2\beta}}{2}+\frac{(\alpha-\beta)}{(2\beta+1)}(\zeta)^{(2\beta+1)}\right].\end{aligned}$$

Therefore,

$$\mathfrak{V}_{2\alpha}(\zeta)=(2\zeta)^{2\alpha}+\sum_{\beta=0}^{\alpha-1}(-1)^{\alpha-\beta}2^{2\beta+1}\binom{\alpha+\beta}{2\beta}\left[\frac{(\zeta)^{2\beta}}{2}+\frac{(\alpha-\beta)}{(2\beta+1)}(\zeta)^{(2\beta+1)}\right]. \tag{32}$$

Similarly, using (22) with (29) and (30) and proceeding as above, we have

$$\begin{aligned}\mathfrak{V}_{2\alpha+1}(\zeta)&=\mathfrak{U}_{2\alpha+1}(\zeta)-\mathfrak{U}_{2\alpha}(\zeta)\\&=\sum_{\beta=0}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta+2}(\beta+1)}{(\beta+\alpha+2)}\binom{\alpha+\beta+2}{2\beta+2}(\zeta)^{2\beta+1}\end{aligned}$$

$$
-\sum_{\beta=0}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta}(2\beta+1)}{(\beta+\alpha+1)}\binom{\alpha+\beta+1}{2\beta+1}(\zeta)^{2\beta}
$$

$$
=\sum_{\beta=0}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta+1}(\alpha+\beta+1)}{(2\beta+1)}\binom{\alpha+\beta}{2\beta}(\zeta)^{2\beta+1}-\sum_{\beta=0}^{\alpha}(-1)^{\alpha-\beta}2^{2\beta}\binom{\alpha+\beta}{2\beta}(\zeta)^{2\beta}
$$

$$
=\sum_{\beta=0}^{\alpha}(-1)^{\alpha-\beta}2^{2\beta+1}\binom{\alpha+\beta}{2\beta}\left[\frac{(\alpha+\beta+1)}{(2\beta+1)}(\zeta)^{2\beta+1}-\frac{(\zeta)^{2\beta}}{2}\right].
$$

Therefore,

$$
\mathfrak{V}_{2\alpha+1}(\zeta)=\sum_{\beta=0}^{\alpha}(-1)^{\alpha-\beta}2^{2\beta+1}\binom{\alpha+\beta}{2\beta}\left[\frac{(\alpha+\beta+1)}{(2\beta+1)}(\zeta)^{2\beta+1}-\frac{(\zeta)^{2\beta}}{2}\right]. \tag{33}
$$

This establishes the Theorem 2.1. □

Theorem 2.2. *For any positive integer α,*

$$
\mathfrak{W}_{2\alpha}(\zeta)=(2\zeta)^{2\alpha}+\sum_{\beta=0}^{\alpha-1}(-1)^{\alpha-\beta}2^{2\beta+1}\binom{\alpha+\beta}{2\beta}\left[\frac{(\zeta)^{2\beta}}{2}-\frac{(\alpha-\beta)}{(2\beta+1)}(\zeta)^{2\beta+1}\right],
$$

$$
\mathfrak{W}_{2\alpha+1}(\zeta)=\sum_{\beta=0}^{\alpha}(-1)^{\alpha-\beta}2^{2\beta+1}\binom{\alpha+\beta}{2\beta}\left[\frac{(\alpha+\beta+1)}{(2\beta+1)}(\zeta)^{2\beta+1}+\frac{(\zeta)^{2\beta}}{2}\right].
$$

Proof. Using (24) and (32), we have

$$
\mathfrak{W}_{2\alpha}(\zeta)=(-1)^{2\alpha}\mathfrak{V}_{2\alpha}(-\zeta)=\mathfrak{V}_{2\alpha}(-\zeta)
$$

$$
=(-2\zeta)^{2\alpha}+\sum_{\beta=0}^{\alpha-1}(-1)^{\alpha-\beta}2^{2\beta+1}\binom{\alpha+\beta}{2\beta}\left[\frac{(-\zeta)^{2\beta}}{2}+\frac{(\alpha-\beta)}{(2\beta+1)}(-\zeta)^{(2\beta+1)}\right]
$$

$$
=(2\zeta)^{2\alpha}+\sum_{\beta=0}^{\alpha-1}(-1)^{\alpha-\beta}2^{2\beta+1}\binom{\alpha+\beta}{2\beta}\left[\frac{(\zeta)^{2\beta}}{2}-\frac{(\alpha-\beta)}{(2\beta+1)}(\zeta)^{(2\beta+1)}\right].
$$

Therefore,

$$
\mathfrak{W}_{2\alpha}(\zeta)=(2\zeta)^{2\alpha}+\sum_{\beta=0}^{\alpha-1}(-1)^{\alpha-\beta}2^{2\beta+1}\binom{\alpha+\beta}{2\beta}\left[\frac{(\zeta)^{2\beta}}{2}-\frac{(\alpha-\beta)}{(2\beta+1)}(\zeta)^{(2\beta+1)}\right].
$$

Similarly, using (24) and (33), we have

$$
\mathfrak{W}_{2\alpha+1}(\zeta)=\sum_{\beta=0}^{\alpha}(-1)^{\alpha-\beta}2^{2\beta+1}\binom{\alpha+\beta}{2\beta}\left[\frac{(\zeta)^{2\beta}}{2}+\frac{(\alpha+\beta+1)}{(2\beta+1)}(\zeta)^{(2\beta+1)}\right].
$$

This establishes the Theorem 2.2. □

Thus, the Theorems 2.1 and 2.2 establish explicit relations for the Chebyshev polynomials of the third and fourth kinds. Next, we will proceed to deduce the derivative of these polynomials as follows:

Theorem 2.3. *For any positive integer α and r,*

$$\mathfrak{V}_{2\alpha}^{r}(\zeta)=\sum_{\beta=\lceil\frac{r}{2}\rceil}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta}(\alpha+\beta)!}{(\alpha-\beta)!(2\beta-r)!}\zeta^{2\beta-r}+\sum_{\beta=\lceil\frac{r-1}{2}\rceil}^{\alpha-1}\frac{(-1)^{\alpha-\beta}2^{2\beta+1}(\alpha+\beta)!}{(2\beta+1-r)!(\alpha-\beta-1)!}\zeta^{(2\beta+1)-r},$$

$$\mathfrak{V}_{2\alpha+1}^{r}(\zeta)=\sum_{\beta=\lceil\frac{r-1}{2}\rceil}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta+1}(\alpha+\beta+1)!}{(\alpha-\beta)!(2\beta+1-r)!}\zeta^{(2\beta+1)-r}-\sum_{\beta=\lceil\frac{r}{2}\rceil}^{\alpha-1}\frac{(-1)^{\alpha-\beta}2^{2\beta}(\alpha+\beta)!}{(2\beta-r)!(\alpha-\beta)!}\zeta^{2\beta-r},$$

where $\lceil\zeta\rceil$ is a ceiling function of ζ.

Proof. Diffrentiating (29), (30) and (31) with respect to ζ r-times, we have

$$\mathfrak{U}_{2\alpha}^{r}(\zeta)=\sum_{\beta=\lceil\frac{r}{2}\rceil}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta}(\alpha+\beta)!}{(\alpha-\beta)!(2\beta-r)!}\zeta^{2\beta-r}, \tag{34}$$

$$\mathfrak{U}_{2\alpha+1}^{r}(\zeta)=\sum_{\beta=\lceil\frac{r-1}{2}\rceil}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta+1}(\alpha+\beta+1)!}{(2\beta+1-r)!(\alpha-\beta)!}\zeta^{(2\beta+1)-r}, \tag{35}$$

$$\mathfrak{U}_{2\alpha-1}^{r}(\zeta)=\sum_{\beta=\lceil\frac{r-1}{2}\rceil}^{\alpha-1}\frac{(-1)^{\alpha-\beta-1}2^{2\beta+1}(\alpha+\beta)!}{(\alpha-\beta-1)!(2\beta+1-r)!}\zeta^{(2\beta+1)-r}. \tag{36}$$

Differentiating (22) with respect to ζ r-times and using (34) and (36), we have

$$\begin{aligned}\mathfrak{V}_{2\alpha}^{r}(\zeta)&=\mathfrak{U}_{2\alpha}^{r}(\zeta)-\mathfrak{U}_{2\alpha-1}^{r}(\zeta)\\&=\sum_{\beta=\lceil\frac{r}{2}\rceil}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta}(\alpha+\beta)!}{(2\beta-r)!(\alpha-\beta)!}\zeta^{2\beta-r}-\sum_{\beta=\lceil\frac{r-1}{2}\rceil}^{\alpha-1}\frac{(-1)^{\alpha-\beta-1}2^{2\beta+1}(\alpha+\beta)!}{(2\beta+1-r)!(\alpha-\beta-1)!}\zeta^{(2\beta+1)-r}.\end{aligned}$$

Therefore,

$$\mathfrak{V}_{2\alpha}^{r}(\zeta)=\sum_{\beta=\lceil\frac{r}{2}\rceil}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta}(\alpha+\beta)!}{(\alpha-\beta)!(2\beta-r)!}\zeta^{2\beta-r}+\sum_{\beta=\lceil\frac{r-1}{2}\rceil}^{\alpha-1}\frac{(-1)^{\alpha-\beta}2^{2\beta+1}(\alpha+\beta)!}{(\alpha-\beta-1)!(2\beta+1-r)!}\zeta^{(2\beta+1)-r}. \tag{37}$$

Again, differentiating (22) with respect to ζ r-times and using (34) and (35) , we have

$$\begin{aligned}\mathfrak{V}_{2\alpha+1}^{r}(\zeta)&=\mathfrak{U}_{2\alpha+1}^{r}(\zeta)-\mathfrak{U}_{2\alpha}^{r}(\zeta)\\&=\sum_{\beta=\lceil\frac{r-1}{2}\rceil}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta+1}(\alpha+\beta+1)!}{(2\beta+1-r)!(\alpha-\beta)!}\zeta^{(2\beta+1)-r}-\sum_{\beta=\lceil\frac{r}{2}\rceil}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta}(\alpha+\beta)!}{(2\beta-r)!(\alpha-\beta)!}\zeta^{2\beta-r}.\end{aligned}$$

Therefore,

$$\mathfrak{V}_{2\alpha+1}^{r}(\zeta)=\sum_{\beta=\lceil\frac{r-1}{2}\rceil}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta+1}(\alpha+\beta+1)!}{(2\beta+1-r)!(\alpha-\beta)!}\zeta^{(2\beta+1)-r}-\sum_{\beta=\lceil\frac{r}{2}\rceil}^{\alpha}\frac{(-1)^{\alpha-\beta}2^{2\beta}(\alpha+\beta)!}{(2\beta-r)!(\alpha-\beta)!}\zeta^{2\beta-r}. \tag{38}$$

This establishes the Theorem 2.3. □

Theorem 2.4. *For any positive integer α and r,*

$$\mathfrak{W}_{2\alpha}^{r}(\zeta) = \sum_{\beta=\lceil\frac{r}{2}\rceil}^{\alpha} \frac{(-1)^{\alpha-\beta}2^{2\beta}(\alpha+\beta)!}{(\alpha-\beta)!(2\beta-r)!}\zeta^{2\beta-r} - \sum_{\beta=\lceil\frac{r-1}{2}\rceil}^{\alpha-1} \frac{(-1)^{\alpha-\beta}2^{2\beta+1}(\alpha+\beta)!}{(\alpha-\beta-1)!(2\beta+1-r)!}\zeta^{(2\beta+1)-r},$$

$$\mathfrak{W}_{2\alpha+1}^{r}(\zeta) = \sum_{\beta=\lceil\frac{r-1}{2}\rceil}^{\alpha} \frac{(-1)^{\alpha-\beta}2^{2\beta+1}(\alpha+\beta+1)!}{(2\beta+1-r)!(\alpha-\beta)!}\zeta^{(2\beta+1)-r} + \sum_{\beta=\lceil\frac{r}{2}\rceil}^{\alpha} \frac{(-1)^{\alpha-\beta}2^{2\beta}(\alpha+\beta)!}{(2\beta-r)!(\alpha-\beta)!}\zeta^{2\beta-r},$$

where $\lceil\zeta\rceil$ is a ceiling function of ζ.

Proof. Differentiating (24) with respect to ζ r-times, and using (37), we have

$$\begin{aligned}
\mathfrak{W}_{2\alpha}^{r}(\zeta) &= (-1)^{r}\mathfrak{V}_{2\alpha}^{r}(-\zeta) \\
&= \sum_{\beta=\lceil\frac{r}{2}\rceil}^{\alpha} \frac{(-1)^{\alpha-\beta+r}2^{2\beta}(\alpha+\beta)!}{(\alpha-\beta)!(2\beta-r)!}(-\zeta)^{2\beta-r} \\
&\quad + \sum_{\beta=\lceil\frac{r-1}{2}\rceil}^{\alpha-1} \frac{(-1)^{\alpha-\beta+r}2^{2\beta+1}(\alpha+\beta)!}{(\alpha-\beta-1)!(2\beta+1-r)!}(-\zeta)^{(2\beta+1)-r} \\
&= \sum_{\beta=\lceil\frac{r}{2}\rceil}^{\alpha} \frac{(-1)^{\alpha-\beta+r}2^{2\beta}(\alpha+\beta)!}{(\alpha-\beta)!(2\beta-r)!}\zeta^{2\beta-r} \\
&\quad - \sum_{\beta=\lceil\frac{r-1}{2}\rceil}^{\alpha-1} \frac{(-1)^{\alpha-\beta+r}2^{2\beta+1}(\alpha+\beta)!}{(\alpha-\beta-1)!(2\beta+1-r)!}\zeta^{(2\beta+1)-r}.
\end{aligned}$$

Therefore,

$$\mathfrak{W}_{2\alpha}^{r}(\zeta) = \sum_{\beta=\lceil\frac{r}{2}\rceil}^{\alpha} \frac{(-1)^{\alpha-\beta+r}2^{2\beta}(\alpha+\beta)!}{(\alpha-\beta)!(2\beta-r)!}\zeta^{2\beta-r} - \sum_{\beta=\lceil\frac{r-1}{2}\rceil}^{\alpha-1} \frac{(-1)^{\alpha-\beta+r}2^{2\beta+1}(\alpha+\beta)!}{(\alpha-\beta-1)!(2\beta+1-r)!}\zeta^{(2\beta+1)-r}.$$

Again differentiating (24) with respect to ζ r-times, and using (38), we have

$$\mathfrak{W}_{2\alpha+1}^{r}(\zeta) = \sum_{\beta=\lceil\frac{r-1}{2}\rceil}^{\alpha} \frac{(-1)^{\alpha-\beta}2^{2\beta+1}(\alpha+\beta+1)!}{(2\beta+1-r)!(\alpha-\beta)!}\zeta^{(2\beta+1)-r} + \sum_{\beta=\lceil\frac{r}{2}\rceil}^{\alpha} \frac{(-1)^{\alpha-\beta}2^{2\beta}(\alpha+\beta)!}{(2\beta-r)!(\alpha-\beta)!}\zeta^{2\beta-r}.$$

This establishes the Theorem 2.4. □

The explicit formulae for the r^{th} derivative of the third and fourth kinds of Chebyshev polynomials with even and odd indices are thus established by Theorems 2.3 and 2.4.

Next, we will study the interaction between the Chebyshev polynomials of the third and fourth kind and the Fibonacci polynomials.

Theorem 2.5. *For any positive integer α and r,*

$$\begin{aligned}
\mathfrak{V}_{2\alpha}(\zeta) = &\sum_{\mu=1}^{\alpha+1}\sum_{\beta=0}^{\alpha} \frac{(-1)^{\mu+\alpha}2^{2\beta-1}(1-2\mu)(\alpha+\beta)!}{(\alpha-\beta)!(\beta+\mu)!(\beta-\mu+1)!} \times \mathfrak{F}_{-(2\mu-1)}(\zeta) \\
&+\sum_{\mu=1}^{\alpha}\sum_{\beta=0}^{\alpha-1} \frac{(-1)^{\mu+\alpha}2^{2\beta+2}\mu(\alpha+\beta-1)!}{(\beta-\mu+1)!(\beta+\mu+1)!(\alpha-\beta+1)!} \times \mathfrak{F}_{-2\mu}(\zeta),
\end{aligned}$$

$$\mathfrak{V}_{2\alpha+1}(\zeta) = \sum_{\mu=1}^{\alpha+1}\sum_{\beta=0}^{\alpha} \frac{(-1)^{\mu+\alpha}(\alpha+\beta)!}{(\beta-\mu+1)!}\left[\frac{2^{2\beta+2}\mu}{(\beta+\mu+1)!(\alpha-\beta+2)!}\times\mathfrak{F}_{-2\mu}(\zeta) - \frac{2^{2\beta-1}(1-2\mu)}{(\beta+\mu)!(\alpha-\beta)!}\times\mathfrak{F}_{-(2\mu-1)}(\zeta)\right].$$

Proof. For any positive integer α, we can deduce [7],

$$\mathfrak{U}_{2\alpha}(\zeta) = \sum_{\mu=1}^{+\infty} C_{2\alpha,\mu}\mathfrak{F}_{\mu}(\zeta),$$

$$\mathfrak{U}_{2\alpha-1}(\zeta) = \sum_{\mu=1}^{+\infty} C_{2\alpha-1,\mu}\mathfrak{F}_{\mu}(\zeta),$$

where

$$C_{2\alpha,\mu} = \begin{cases} \sum_{\beta=0}^{\alpha} \frac{2^{4\beta+1} i^{3\mu+2\alpha+1}\mu(\alpha+\beta)!}{(\alpha-\beta)!(2\beta+\mu+1)!!(2\beta-\mu+1)!!} & \text{if } \mu \text{ is odd} \\ 0 & \text{otherwise,} \end{cases}$$

$$C_{2\alpha-1,\mu} = \begin{cases} \sum_{\beta=0}^{\alpha} \frac{2^{4\beta+3} i^{3\mu+2\alpha}\mu(\alpha+\beta-1)!}{(\alpha-\beta-1)!(2\beta+\mu+2)!!(2\beta-\mu+2)!!} & \text{if } \mu \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Using this, from [7] , we have

$$\mathfrak{U}_{2\alpha}(\zeta) = \sum_{\mu=1}^{\alpha+1}\sum_{\beta=0}^{\alpha} \frac{(-1)^{\mu+\alpha}2^{2\beta-1}(1-2\mu)(\alpha+\beta)!}{(\alpha-\beta)!(\beta+\mu)!(\beta-\mu+1)!}\times\mathfrak{F}_{2\mu-1}(\zeta), \tag{39}$$

$$\mathfrak{U}_{2\alpha-1}(\zeta) = \sum_{\mu=1}^{\alpha}\sum_{\beta=0}^{\alpha-1} \frac{(-1)^{\mu+\alpha}2^{2\beta+2}\mu(\alpha+\beta-1)!}{(\beta-\mu+1)!(\beta+\mu+1)!(\alpha-\beta+1)!}\times\mathfrak{F}_{2\mu}(\zeta), \tag{40}$$

$$\mathfrak{U}_{2\alpha+1}(\zeta) = \sum_{\mu=1}^{\alpha+1}\sum_{\beta=0}^{\alpha} \frac{(-1)^{\mu+\alpha+1}2^{2\beta+2}\mu(\alpha+\beta)!}{(\beta-\mu+1)!(\beta+\mu+1)!(\alpha-\beta+2)!}\times\mathfrak{F}_{2\mu}(\zeta). \tag{41}$$

Using (22), (39) and (40) , we have

$$\mathfrak{V}_{2\alpha}(\zeta) = \sum_{\mu=1}^{\alpha+1}\sum_{\beta=0}^{\alpha} \frac{(-1)^{\mu+\alpha}2^{2\beta-1}(1-2\mu)(\alpha+\beta)!}{(\alpha-\beta)!(\beta+\mu)!(\beta-\mu+1)!}\times\mathfrak{F}_{2\mu-1}(\zeta) - \sum_{\mu=1}^{\alpha}\sum_{\beta=0}^{\alpha-1} \frac{(-1)^{\mu+\alpha}2^{2\beta+2}\mu(\alpha+\beta-1)!}{(\beta-\mu+1)!(\beta+\mu+1)!(\alpha-\beta+1)!}\times\mathfrak{F}_{2\mu}(\zeta).$$

Now, in view of (26), we have $\mathfrak{F}_{2\mu-1}(\zeta) = \mathfrak{F}_{-(2\mu-1)}(\zeta)$ and $\mathfrak{F}_{2\mu}(\zeta) = -\mathfrak{F}_{-2\mu}(\zeta)$, which in turn implies

$$\mathfrak{V}_{2\alpha}(\zeta) = \sum_{\mu=1}^{\alpha+1}\sum_{\beta=0}^{\alpha} \frac{(-1)^{\mu+\alpha}2^{2\beta-1}(1-2\mu)(\alpha+\beta)!}{(\beta+\mu)!(\beta-\mu+1)!(\alpha-\beta)!}\times\mathfrak{F}_{-(2\mu-1))}(\zeta) + \sum_{\mu=1}^{\alpha}\sum_{\beta=0}^{\alpha-1} \frac{(-1)^{\mu+\alpha}2^{2\beta+2}\mu(\alpha+\beta-1)!}{(\beta+\mu+1)!(\beta-\mu+1)!(\alpha-\beta+1)!}\times\mathfrak{F}_{-2\mu}(\zeta).$$

Therefore,

$$\mathfrak{V}_{2\alpha}(\zeta) = \sum_{\mu=1}^{\alpha+1}\sum_{\beta=0}^{\alpha} \frac{(-1)^{\mu+\alpha}2^{2\beta-1}(1-2\mu)(\alpha+\beta)!}{(\alpha-\beta)!(\beta+\mu)!(\beta-\mu+1)!} \times \mathfrak{F}_{-(2\mu-1)}(\zeta) + \sum_{\mu=1}^{\alpha}\sum_{\beta=0}^{\alpha-1} \frac{(-1)^{\mu+\alpha}2^{2\beta+2}\mu(\alpha+\beta-1)!}{(\beta+\mu+1)!(\beta-\mu+1)!(\alpha-\beta+1)!} \times \mathfrak{F}_{-2\mu}(\zeta). \tag{42}$$

Similarly, using (22), (39) and (41), we have

$$\mathfrak{V}_{2\alpha+1}(\zeta) = \sum_{\mu=1}^{\alpha+1}\sum_{\beta=0}^{\alpha} \frac{(-1)^{\mu+\alpha+1}2^{2\beta+2}\mu(\alpha+\beta)!}{(\beta-\mu+1)!(\beta+\mu+1)!(\alpha-\beta+2)!} \times \mathfrak{F}_{2\mu}(\zeta) - \sum_{\mu=1}^{\alpha+1}\sum_{\beta=0}^{\alpha} \frac{(-1)^{\mu+\alpha}2^{2\beta-1}(1-2\mu)(\alpha+\beta)!}{(\alpha-\beta)!(\beta+\mu)!(\beta-\mu+1)!} \times \mathfrak{F}_{2\mu-1}(\zeta).$$

Now, again, in view of (26), we have $\mathfrak{F}_{2\mu-1}(\zeta) = \mathfrak{F}_{-(2\mu-1)}(\zeta)$ and $\mathfrak{F}_{2\mu}(\zeta) = -\mathfrak{F}_{-2\mu}(\zeta)$, which in turn gives

$$\mathfrak{V}_{2\alpha+1}(\zeta) = \sum_{\mu=1}^{\alpha+1}\sum_{\beta=0}^{\alpha} \frac{(-1)^{\mu+\alpha}(\alpha+\beta)!}{(\beta-\mu+1)!} \left[\frac{2^{2\beta+2}\mu}{(\alpha-\beta+2)!(\beta+\mu+1)!} \times \mathfrak{F}_{-2\mu}(\zeta) - \frac{2^{2\beta-1}(1-2\mu)}{(\alpha-\beta)!(\beta+\mu)!} \times \mathfrak{F}_{-(2\mu-1)}(\zeta) \right]. \tag{43}$$

Thus, the Theorem 2.5 is established. □

Theorem 2.6. *For any positive integer α and r,*

$$\mathfrak{W}_{2\alpha}(\zeta) = \sum_{\mu=1}^{\alpha+1}\sum_{\beta=0}^{\alpha} \frac{(-1)^{\mu+\alpha}2^{2\beta-1}(1-2\mu)(\alpha+\beta)!}{(\alpha-\beta)!(\beta+\mu)!(\beta-\mu+1)!} \times \mathfrak{F}_{-(2\mu-1)}(\zeta) - \sum_{\mu=1}^{\alpha}\sum_{\beta=0}^{\alpha-1} \frac{(-1)^{\mu+\alpha}2^{2\beta+2}\mu(\alpha+\beta-1)!}{(\beta-\mu+1)!(\beta+\mu+1)!(\alpha-\beta+1)!} \times \mathfrak{F}_{-2\mu}(\zeta),$$

$$\mathfrak{W}_{2\alpha+1}(\zeta) = \sum_{\mu=1}^{\alpha+1}\sum_{\beta=0}^{\alpha} \frac{(-1)^{\mu+\alpha}(\alpha+\beta)!}{(\beta-\mu+1)!} \left[\frac{2^{2\beta+2}\mu}{(\alpha-\beta+2)!(\beta+\mu+1)!} \times \mathfrak{F}_{-2\mu}(\zeta) + \frac{2^{2\beta-1}(1-2\mu)}{(\alpha-\beta)!(\beta+\mu)!} \times \mathfrak{F}_{-(2\mu-1)}(\zeta) \right].$$

Proof. Using (42) and (24), we have

$$\begin{aligned}\mathfrak{W}_{2\alpha}(\zeta) &= (-1)^{2\alpha}\mathfrak{V}_{2\alpha}(-\zeta) = \mathfrak{V}_{2\alpha}(-\zeta) \\ &= \sum_{\mu=1}^{\alpha+1}\sum_{\beta=0}^{\alpha} \frac{(-1)^{\mu+\alpha}2^{2\beta-1}(1-2\mu)(\alpha+\beta)!}{(\beta+\mu)!(\beta-\mu+1)!(\alpha-\beta)!} \times \mathfrak{F}_{-(2\mu-1)}(-\zeta) \\ &\quad + \sum_{\mu=1}^{\alpha}\sum_{\beta=0}^{\alpha-1} \frac{(-1)^{\mu+\alpha}2^{2\beta+2}\mu(\alpha+\beta-1)!}{(\beta+\mu+1)!(\beta-\mu+1)!(\alpha-\beta+1)!} \times \mathfrak{F}_{-2\mu}(-\zeta).\end{aligned}$$

Now, in view of (25) we have $\mathfrak{F}_{-(2\mu-1)}(-\zeta) = \mathfrak{F}_{-(2\mu-1)}(\zeta)$ and $\mathfrak{F}_{-2\mu}(-\zeta) = -\mathfrak{F}_{-2\mu}(\zeta)$. Using these relations gives

$$\begin{aligned}\mathfrak{W}_{2\alpha}(\zeta) = &\sum_{\mu=1}^{\alpha+1}\sum_{\beta=0}^{\alpha} \frac{(-1)^{\mu+\alpha}2^{2\beta-1}(1-2\mu)(\alpha+\beta)!}{(\alpha-\beta)!(\beta+\mu)!(\beta-\mu+1)!} \times \mathfrak{F}_{-(2\mu-1)}(\zeta)\\ &-\sum_{\mu=1}^{\alpha}\sum_{\beta=0}^{\alpha-1} \frac{(-1)^{\mu+\alpha}2^{2\beta+2}\mu(\alpha+\beta-1)!}{(\alpha-\beta+1)!(\beta+\mu+1)!(\beta-\mu+1)!} \times \mathfrak{F}_{-2\mu}(\zeta).\end{aligned}$$

Again, using $\mathfrak{F}_{-(2\mu-1)}(-\zeta) = \mathfrak{F}_{-(2\mu-1)}(\zeta)$ and $\mathfrak{F}_{-2\mu}(-\zeta) = -\mathfrak{F}_{-2\mu}(\zeta)$ with equation (43) in (24) and proceeding as above, we have

$$\begin{aligned}\mathfrak{W}_{2\alpha+1}(\zeta) = \sum_{\mu=1}^{\alpha+1}\sum_{\beta=0}^{\alpha} \frac{(-1)^{\mu+\alpha}(\alpha+\beta)!}{(\beta-\mu+1)!} &\left[\frac{2^{2\beta+2}\mu}{(\beta+\mu+1)!(\alpha-\beta+2)!} \times \mathfrak{F}_{-2\mu}(\zeta)\right.\\ &\left.+ \frac{2^{2\beta-1}(1-2\mu)}{(\beta+\mu)!(\alpha-\beta)!} \times \mathfrak{F}_{-(2\mu-1)}(\zeta)\right].\end{aligned}$$

This establishes the Theorem 2.6. □

Thus, the Theorems 2.5 and 2.6 establish the relation between the third and fourth kinds of Chebyshev polynomials and Fibonacci polynomials with negative index.

3 Conclusion

In this paper, we considered the sequences of Chebyshev polynomials of the third and fourth kind and deduced their explicit formulae with even and odd indices. Similar relationships are obtained for their general r^{th} derivatives. At the end, their relationship with the Fibonacci polynomials with even and odd negative indices is obtained, which turns out to be an expression of Chebyshev polynomials of third and fourth kind with even and odd negative indices as a linear combination of Fibonacci polynomials with negative indices. These findings will definitely enrich and strengthen the existing literature on Chebyshev polynomials and their relationship with the Fibonacci and other similar and related orthogonal polynomials. This study is expected to add more depth to our understanding of the combinatorial and analytic properties of these Chebyshev polynomials and be instrumental in studying some general summation problems arising in both pure and applied mathematics involving these polynomials.

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