Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2023, Volume 29, Number 2, 195–203
DOI: 10.7546/nntdm.2023.29.2.195-203

A note on telephone numbers and their matrix generators

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Received: 5 September 2022 **Revised:** 15 March 2023
Accepted: 11 April 2023 **Online First:** 12 April 2023

Abstract: In the present work, we indicate some matrix properties that allow us to determine new relationships involving telephone numbers and some products that allow us to obtain telephone terms, based on second-order matrices.
Keywords: Matrix generators, Sequence, Telephone number.
2020 Mathematics Subject Classification: 11B37, 15B36.

1 Introduction

Telephone numbers were studied by the German mathematician Heinrich August Rothe (1773–1842) and involve an immediate application, considering the forms or ways of connecting telephone lines. Telephone numbers admit numerous interpretations [1,5,7]. Below we present the following definition.

Definition 1.1. *The sequence of the telephone numbers $\{T_n\}_{n\in\mathbb{N}}$ is defined by the following recurrence relation: $T_n = T_{n-1} + (n-1)T_{n-2}$, and with the initial values indicated by: $T_0 = 1$, $T_1 = 1$.*

From the previous recurrence, we can write the numerical list $1, 1, 2, 4, 10, 26, 76, 232, \ldots$ On the other hand, from the work indicated in [3], we find the following generalization.

Definition 1.2. *The k-telephone sequence $\{T_{k,n}\}_{n\in\mathbb{N}}$ is defined by the following recurrence relation: $T_{k,n} = kT_{k,n-1} + (n-1)T_{k,n-2}$, and with the initial values indicated by: $T_{k,0} = 1, T_{k,1} = k$.*

Similarly, we verify that

$$T_{k,2} = k^2 + 1, T_{k,3} = k^3 + 3k, T_{k,4} = k^4 + 6k^2 + 3, T_{k,5} = k^5 + 10k^3 + 15k,$$

etc.

Finally, based on [2] we found another generalized form that allows another interpretation for telephone numbers.

Definition 1.3. *The $\{T_{p,n}\}_{n\in\mathbb{N}}$ generalized telephone number sequence is defined by the following recurrence relation: $T_{kp,n} = T_{p,n-1} + p(n-1)T_{p,n-2}$, and with the initial values indicated by: $T_{p,0} = 1, T_{p,1} = 1$.*

We can find that $T_{p,2} = 1 + p, T_{p,3} = 1 + 3p, T_{p,4} = 3p^2 + 6p + 1, T_{p,5} = 15p^2 + 10p + 1$, etc.

From the previous Definitions 1.1, 1.2 and 1.3, we can easily verify the following elementary identities:

$$T_{n+1} = T_1 + \sum_{i=1}^{n} iT_{i-1},$$

$$T_{k,n+1} = T_{k,1} + \sum_{i=1}^{n} iT_{k,i-1},$$

$$T_{p,n+1} = T_{p,1} + p(n-1)\sum_{i=1}^{n} iT_{p,i-1}.$$

Furthermore, based on the previous Definitions 1.1, 1.2 and 1.3, we will introduce generating matrices of order 2 for the numbers indicated earlier.

2 Some properties with matrices

Let us consider the following matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1 & T_0 \\ 0 & 0 \end{bmatrix}$ and then we will consider the following product $\prod_{k=0}^{0} \begin{bmatrix} 1 & 1 \\ k & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1 & T_0 \\ 0 & 0 \end{bmatrix}$.

Then we can easily verify that: $\prod_{k=0}^{1} \begin{bmatrix} 1 & 1 \\ k & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_2 & T_1 \\ 0 & 0 \end{bmatrix}$ and that, when we consider the following case, it results in:

$$\prod_{k=0}^{2}\begin{bmatrix}1&1\\k&0\end{bmatrix}=\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}1&1\\2&0\end{bmatrix}=\begin{bmatrix}2&1\\0&0\end{bmatrix}\begin{bmatrix}1&1\\2&0\end{bmatrix}=\begin{bmatrix}4&2\\0&0\end{bmatrix}=\begin{bmatrix}T_3&T_2\\0&0\end{bmatrix}.$$

Or even that:

$$\prod_{k=0}^{3}\begin{bmatrix}1&1\\k&0\end{bmatrix}=\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}1&1\\2&0\end{bmatrix}\begin{bmatrix}1&1\\3&0\end{bmatrix}=\begin{bmatrix}10&4\\0&0\end{bmatrix}=\begin{bmatrix}T_4&T_3\\0&0\end{bmatrix}.$$

From the matrix multiplication indicated by $\prod_{i=0}^{n}\begin{bmatrix}k&1\\i&0\end{bmatrix}$, we verified some properties and theorems.

Next, let us consider that: $\prod_{i=0}^{0}\begin{bmatrix}k&1\\i&0\end{bmatrix}=\begin{bmatrix}k&1\\0&0\end{bmatrix}=\begin{bmatrix}T_{k,1}&T_{k,0}\\0&0\end{bmatrix}$ and we still have that $\prod_{i=0}^{1}\begin{bmatrix}k&1\\i&0\end{bmatrix}=\begin{bmatrix}k&1\\0&0\end{bmatrix}\begin{bmatrix}k&1\\1&0\end{bmatrix}=\begin{bmatrix}k^2+1&k\\0&0\end{bmatrix}=\begin{bmatrix}T_{k,2}&T_{k,1}\\0&0\end{bmatrix}$ occurs.

Then:

$$\prod_{i=0}^{2}\begin{bmatrix}k&1\\i&0\end{bmatrix}=\begin{bmatrix}k&1\\0&0\end{bmatrix}\begin{bmatrix}k&1\\1&0\end{bmatrix}\begin{bmatrix}k&1\\2&0\end{bmatrix}=\begin{bmatrix}k^2+1&k\\0&0\end{bmatrix}\begin{bmatrix}k&1\\2&0\end{bmatrix}$$
$$=\begin{bmatrix}k^3+3k&k^2+1\\0&0\end{bmatrix}=\begin{bmatrix}T_{k,3}&T_{k,2}\\0&0\end{bmatrix}.$$

And similarly, we see that:

$$\prod_{i=0}^{3}\begin{bmatrix}k&1\\i&0\end{bmatrix}=\begin{bmatrix}k&1\\0&0\end{bmatrix}\begin{bmatrix}k&1\\1&0\end{bmatrix}\begin{bmatrix}k&1\\2&0\end{bmatrix}\begin{bmatrix}k&1\\3&0\end{bmatrix}=\begin{bmatrix}k^3+3k&k^2+1\\0&0\end{bmatrix}\begin{bmatrix}k&1\\3&0\end{bmatrix}$$
$$=\begin{bmatrix}k^4+6k^2+3&k^3+3k\\0&0\end{bmatrix}=\begin{bmatrix}T_{k,4}&T_{k,3}\\0&0\end{bmatrix}.$$

From these cases, we will use the following product $\prod_{i=0}^{n}\begin{bmatrix}k&1\\i&0\end{bmatrix}$ to verify the theorems below.

Theorem 2.1. *Consider the numbers $\{T_n\}_{n\in\mathbb{N}}$. So, for every positive integer $n\geq 0$, the relation $\prod_{i=0}^{n}\begin{bmatrix}1&1\\k&0\end{bmatrix}=\begin{bmatrix}T_{n+1}&T_n\\0&0\end{bmatrix}$ is valid.*

Proof. By mathematical induction, we will assume that $\prod_{k=0}^{n}\begin{bmatrix}1&1\\k&0\end{bmatrix}=\begin{bmatrix}T_{n+1}&T_n\\0&0\end{bmatrix}$ and then we have $\prod_{k=0}^{n+1}\begin{bmatrix}1&1\\k&0\end{bmatrix}=\prod_{k=0}^{n}\begin{bmatrix}1&1\\k&0\end{bmatrix}\begin{bmatrix}1&1\\n+1&0\end{bmatrix}=\begin{bmatrix}T_{n+1}&T_n\\0&0\end{bmatrix}\begin{bmatrix}1&1\\n+1&0\end{bmatrix}=\begin{bmatrix}T_{n+1}+(n+1)T_n&T_n\\0&0\end{bmatrix}$ $=\begin{bmatrix}T_{n+2}&T_{n+1}\\0&0\end{bmatrix}$ for every positive integer $n\geq 0$. □

Theorem 2.2. *Consider the numbers* $\{T_{k,n}\}_{n\in\mathbb{N}}$*. For every positive integer* $n \geq 0$*, the following relation holds:*

$$\prod_{i=0}^{n}\begin{bmatrix} k & 1 \\ i & 0 \end{bmatrix} = \begin{bmatrix} T_{k,n+1} & T_{k,n} \\ 0 & 0 \end{bmatrix}.$$

Proof. By mathematical induction, we will assume that: $\prod_{i=0}^{n}\begin{bmatrix} k & 1 \\ i & 0 \end{bmatrix} = \begin{bmatrix} T_{k,n+1} & T_{k,n} \\ 0 & 0 \end{bmatrix}$.

For $n+1$, we have:

$$\prod_{i=0}^{n+1}\begin{bmatrix} k & 1 \\ i & 0 \end{bmatrix} = \prod_{i=0}^{n}\begin{bmatrix} k & 1 \\ i & 0 \end{bmatrix}\begin{bmatrix} k & 1 \\ n+1 & 0 \end{bmatrix} = \begin{bmatrix} T_{k,n+1} & T_{k,n} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} k & 1 \\ n+1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} kT_{k,n+1} + (n+1)T_{k,n} & T_{k,n+1} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_{k,n+2} & T_{k,n+1} \\ 0 & 0 \end{bmatrix},$$

for every positive integer $n \geq 0$. □

For the next theorem, it is enough to observe that:

$$\prod_{i=0}^{n}\begin{bmatrix} 1 & 1 \\ pi & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_{p,1} & T_{p,0} \\ 0 & 0 \end{bmatrix}.$$

Or even that:

$$\prod_{i=0}^{1}\begin{bmatrix} 1 & 1 \\ pi & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ p & 0 \end{bmatrix} = \begin{bmatrix} 1+p & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_{p,2} & T_{p,1} \\ 0 & 0 \end{bmatrix}.$$

and:

$$\prod_{i=0}^{2}\begin{bmatrix} 1 & 1 \\ pi & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ p & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2p & 0 \end{bmatrix} = \begin{bmatrix} 1+p & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2p & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3p & 1+p \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_{p,3} & T_{p,2} \\ 0 & 0 \end{bmatrix},$$

$$\prod_{i=0}^{4}\begin{bmatrix} 1 & 1 \\ pi & 0 \end{bmatrix} = \begin{bmatrix} T_{p,4} & T_{p,3} \\ 0 & 0 \end{bmatrix}$$

Theorem 2.3. *Consider the generalized telephone numbers* $\{T_{p,n}\}_{n\in\mathbb{N}}$*. For every positive integer* $n \geq 0$*, the following relation holds* $\prod_{i=0}^{n}\begin{bmatrix} 1 & 1 \\ p & 0 \end{bmatrix} = \begin{bmatrix} T_{p,n+1} & T_{p,n} \\ 0 & 0 \end{bmatrix}$.

Proof. By mathematical induction, we will assume that: $\prod_{i=0}^{n}\begin{bmatrix} 1 & 1 \\ pi & 0 \end{bmatrix} = \begin{bmatrix} T_{p,n+1} & T_{p,n} \\ 0 & 0 \end{bmatrix}$.

For $n+1$, we have:

$$\prod_{i=0}^{n+1}\begin{bmatrix} 1 & 1 \\ pi & 0 \end{bmatrix} = \prod_{i=0}^{n+1}\begin{bmatrix} 1 & 1 \\ pi & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ p+1 & 0 \end{bmatrix} = \begin{bmatrix} T_{p,n+1} & T_{p,n} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ p+1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} T_{p,n+1} + (p+1)T_{p,n} & T_{p,n+1} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_{p,n+2} & T_{p,n+1} \\ 0 & 0 \end{bmatrix},$$

for every positive integer $n \geq 0$. □

Lemma 2.1. *Consider the numbers* $\{T_n\}_{n\in\mathbb{N}}$, $\{T_{k,n}\}_{n\in\mathbb{N}}$ *and* $\{T_{p,n}\}_{n\in\mathbb{N}}$. *For every positive integer* $n \geq 0$, *the following relations are valid:*

(a) $$\begin{bmatrix} T_1+\ldots+T_{n+1} & T_0+T_1+\ldots+T_n \\ 0 & 0 \end{bmatrix} = \prod_{k=0}^{0}\begin{bmatrix} 1 & 1 \\ k & 0 \end{bmatrix} + \prod_{k=0}^{1}\begin{bmatrix} 1 & 1 \\ k & 0 \end{bmatrix} + \prod_{k=0}^{2}\begin{bmatrix} 1 & 1 \\ k & 0 \end{bmatrix} + \ldots + \prod_{k=0}^{n}\begin{bmatrix} 1 & 1 \\ k & 0 \end{bmatrix};$$

(b) $$\begin{bmatrix} T_{k,1}+\ldots+T_{k,n+1} & T_{k,0}+T_{k,1}+\ldots+T_{k,n} \\ 0 & 0 \end{bmatrix} = \prod_{i=0}^{0}\begin{bmatrix} k & 1 \\ i & 0 \end{bmatrix} + \prod_{i=0}^{1}\begin{bmatrix} k & 1 \\ i & 0 \end{bmatrix} + \prod_{i=0}^{2}\begin{bmatrix} k & 1 \\ i & 0 \end{bmatrix} + \ldots + \prod_{i=0}^{n}\begin{bmatrix} k & 1 \\ i & 0 \end{bmatrix};$$

(c) $$\begin{bmatrix} T_{p,1}+\ldots+T_{p,n+1} & T_{p,0}+T_{p,1}+\ldots+T_{p,n} \\ 0 & 0 \end{bmatrix} = \prod_{i=0}^{0}\begin{bmatrix} 1 & 1 \\ pi & 0 \end{bmatrix} + \prod_{i=0}^{1}\begin{bmatrix} 1 & 1 \\ pi & 0 \end{bmatrix} + \prod_{i=0}^{2}\begin{bmatrix} 1 & 1 \\ pi & 0 \end{bmatrix} + \ldots + \prod_{i=0}^{n}\begin{bmatrix} 1 & 1 \\ pi & 0 \end{bmatrix}.$$

Proof. We immediately verify that:

$$\begin{bmatrix} T_1+\ldots+T_{n+1} & T_0+T_1+\ldots+T_n \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1 & T_0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} T_2 & T_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} T_{n+1} & T_n \\ 0 & 0 \end{bmatrix},$$

and, in view of Theorem 2.1, the result follows for every positive integer $n \geq 0$. Similarly, we verify items (b) and (c). □

Lemma 2.2. *Consider the numbers* $\{T_{k,n}\}_{n\in\mathbb{N}}$. *For every positive integer* $n \geq 0$, *the following relations are valid:*

(a) $$\left(\prod_{k=0}^{n}\begin{bmatrix} 1 & 1 \\ k & 0 \end{bmatrix}\right)^{n+1} = T_{n+1}^{n}\begin{bmatrix} T_{n+1} & T_n \\ 0 & 0 \end{bmatrix};$$

(b) $$\left(\prod_{i=0}^{n}\begin{bmatrix} k & 1 \\ i & 0 \end{bmatrix}\right)^{n+1} = T_{k,n+1}^{n}\begin{bmatrix} T_{k,n+1} & T_{k,n} \\ 0 & 0 \end{bmatrix};$$

(c) $$\left(\prod_{i=0}^{n}\begin{bmatrix} 1 & 1 \\ pi & 0 \end{bmatrix}\right)^{n+1} = T_{p,n+1}^{n}\begin{bmatrix} T_{p,n+1} & T_{p,n} \\ 0 & 0 \end{bmatrix}.$$

Proof. Just observe that $\prod_{k=0}^{n}\begin{bmatrix} 1 & 1 \\ k & 0 \end{bmatrix}\prod_{k=0}^{n}\begin{bmatrix} 1 & 1 \\ k & 0 \end{bmatrix} = \begin{bmatrix} T_{n+1} & T_n \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_{n+1} & T_n \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_{n+1}^2 & T_{n+1}T_n \\ 0 & 0 \end{bmatrix}$ $= T_{n+1}\begin{bmatrix} T_{n+1} & T_n \\ 0 & 0 \end{bmatrix}$ i.e., we can write that $\prod_{k=0}^{n}\begin{bmatrix} 1 & 1 \\ k & 0 \end{bmatrix}\prod_{k=0}^{n}\begin{bmatrix} 1 & 1 \\ k & 0 \end{bmatrix}^2 = T_{n+1}\begin{bmatrix} T_{n+1} & T_n \\ 0 & 0 \end{bmatrix}$.
Similarly, items (b) and (c) follow. □

Lemma 2.3. *Consider the numbers* $\{T_{k,n}\}_{n\in\mathbb{N}}$. *For every positive integer* $n \geq 0$, *the following relations are valid:*

(a) $$\begin{bmatrix} T_{n+1}+T_{m+1} & T_n+T_m \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_{n+1} & T_n \\ 0 & 0 \end{bmatrix}\left(I \begin{bmatrix} 1 & 1 \\ n+1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ n+2 & 0 \end{bmatrix}\cdots\begin{bmatrix} 1 & 1 \\ n+m & 0 \end{bmatrix}\right);$$

(b) $$\begin{bmatrix} T_{k,n+1}+T_{k,m+1} & T_{k,n}+T_{k,m} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_{k,n+1} & T_{k,n} \\ 0 & 0 \end{bmatrix}\left(I \begin{bmatrix} 1 & 1 \\ n+1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ n+2 & 0 \end{bmatrix}\cdots\begin{bmatrix} 1 & 1 \\ n+m & 0 \end{bmatrix}\right);$$

(c) $$\begin{bmatrix} T_{p,n+1}+T_{p,m+1} & T_{p,n}+T_{p,m} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_{p,n+1} & T_{p,n} \\ 0 & 0 \end{bmatrix}\left(I \begin{bmatrix} 1 & 1 \\ n+1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ n+2 & 0 \end{bmatrix}\cdots\begin{bmatrix} 1 & 1 \\ n+m & 0 \end{bmatrix}\right).$$

Proof. Just observe that:

$$\begin{bmatrix} T_{n+1}+T_{m+1} & T_n+T_m \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_{n+1} & T_n \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} T_{m+1} & T_m \\ 0 & 0 \end{bmatrix} = \prod_{k=0}^{n}\begin{bmatrix} 1 & 1 \\ k & 0 \end{bmatrix} + \prod_{k=0}^{m}\begin{bmatrix} 1 & 1 \\ k & 0 \end{bmatrix}$$
$$= \prod_{k=0}^{n}\begin{bmatrix} 1 & 1 \\ k & 0 \end{bmatrix}\left(I + \begin{bmatrix} 1 & 1 \\ n+1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ n+2 & 0 \end{bmatrix}\cdots\begin{bmatrix} 1 & 1 \\ n+m & 0 \end{bmatrix}\right).$$

Similarly, items (b) and (c) follow. □

3 Other properties with matrices

Now, we will consider the following indicated products

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} T_1 & T_1 \\ 1\cdot T_0 & 1\cdot T_0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2\cdot 1 & 2\cdot 1 \end{bmatrix} = \begin{bmatrix} T_2 & T_2 \\ 2\cdot T_1 & 2\cdot T_1 \end{bmatrix}.$$

Or, more easily, we determine that:

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 3\cdot 2 & 3\cdot 2 \end{bmatrix} = \begin{bmatrix} T_3 & T_3 \\ 3\cdot T_2 & 3\cdot T_2 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ 6 & 6 \end{bmatrix} = \begin{bmatrix} 10 & 10 \\ 4\cdot 4 & 4\cdot 4 \end{bmatrix} = \begin{bmatrix} T_4 & T_4 \\ 4\cdot T_3 & 4\cdot T_3 \end{bmatrix}.$$

Next, we will define the following product

$$\prod_{k=0}^{n}\begin{bmatrix} 1 & 1 \\ n-k & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ n & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ n-1 & 0 \end{bmatrix}\cdots\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Theorem 3.1. *For every positive integer $n \geq 1$, the following matrix properties hold:*

$$\prod_{k=0}^{n} \begin{bmatrix} 1 & 1 \\ n-k & 0 \end{bmatrix} = \begin{bmatrix} T_n & T_n \\ n\cdot T_{n-1} & n\cdot T_{n-1} \end{bmatrix}.$$

Proof. Just see that:

$$\prod_{k=-1}^{n} \begin{bmatrix} 1 & 1 \\ n-k & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ n+1 & 0 \end{bmatrix} \prod_{k=0}^{n} \begin{bmatrix} 1 & 1 \\ n-k & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ n+1 & 0 \end{bmatrix} \begin{bmatrix} T_n & T_n \\ n\cdot T_{n-1} & n\cdot T_{n-1} \end{bmatrix}$$
$$= \begin{bmatrix} T_n + n\cdot T_{n-1} & n\cdot T_{n-1} \\ (n+1)\cdot T_n & (n+1)\cdot T_n \end{bmatrix} = \begin{bmatrix} T_{n+1} & T_{n+1} \\ (n+1)\cdot T_n & n\cdot T_n \end{bmatrix}.$$ □

On the other hand, we also see that $\prod_{i=0}^{1} \begin{bmatrix} k & 1 \\ i & 0 \end{bmatrix} = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} k^2 & k \\ k & 1 \end{bmatrix}$.
Thus, we can verify that $\begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} k^2 & k \\ k & 1 \end{bmatrix} = \begin{bmatrix} k\cdot T_{k,1} & T_{k,1} \\ 1\cdot k\cdot T_{k,0} & 1\cdot T_{k,0} \end{bmatrix}$.
Or again, we easily find that:

$$\begin{bmatrix} k & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} k & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} k^2 & k \\ k & 1 \end{bmatrix} = \begin{bmatrix} k^3+k & k^2+1 \\ 2k^2 & 2k \end{bmatrix} = \begin{bmatrix} k(k^2+1) & k^2+1 \\ 2k\cdot k & 2k \end{bmatrix}$$
$$= \begin{bmatrix} k\cdot T_{k,2} & T_{k,2} \\ 2k\cdot T_{k,1} & 2\cdot T_{k,1} \end{bmatrix}$$

and that $\prod_{i=0}^{3} \begin{bmatrix} k & 1 \\ 3-i & 0 \end{bmatrix} = \begin{bmatrix} k & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} k & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} k\cdot T_{k,3} & T_{k,3} \\ 3\cdot T_{k,2} & 3\cdot T_{k,2} \end{bmatrix}$.

Theorem 3.2. *For every positive integer $n \geq 1$, the following matrix properties hold:*

$$\prod_{i=0}^{n} \begin{bmatrix} k & 1 \\ n-i & 0 \end{bmatrix} = \begin{bmatrix} k\cdot T_{k,n} & T_{k,n} \\ n\cdot kT_{k,n-1} & n\cdot T_{k,n-1} \end{bmatrix}.$$

Proof. Just see that:

$$\prod_{i=0}^{n+1} \begin{bmatrix} k & 1 \\ n-i & 0 \end{bmatrix} = \begin{bmatrix} k & 1 \\ n+1 & 0 \end{bmatrix} \prod_{i=0}^{n} \begin{bmatrix} k & 1 \\ n-i & 0 \end{bmatrix} = \begin{bmatrix} k & 1 \\ n+1 & 0 \end{bmatrix} \begin{bmatrix} k\cdot T_{k,n} & T_{k,n} \\ n\cdot kT_{k,n-1} & n\cdot T_{k,n-1} \end{bmatrix}$$
$$= \begin{bmatrix} k\cdot (k\cdot T_{k,n} + n\cdot T_{k,n-1}) & k\cdot T_{k,n} + n\cdot T_{k,n-1} \\ (n+1)k\cdot T_{k,n} & (n+1)k\cdot T_{k,n} \end{bmatrix}$$
$$= \begin{bmatrix} k\cdot T_{k,n+1} & T_{k,n+1} \\ (n+1)k\cdot T_{k,n} & (n+1)k\cdot T_{k,n} \end{bmatrix}.$$ □

Finally, let us verify that:

$$\prod_{i=0}^{1}\begin{bmatrix}1 & 1\\(n-i)p & 0\end{bmatrix}=\begin{bmatrix}1 & 1\\p & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}=\begin{bmatrix}1 & 1\\p & p\end{bmatrix}=\begin{bmatrix}T_{p,1} & T_{p,1}\\p\cdot T_{p,0} & p\cdot T_{p,0}\end{bmatrix},$$

$$\prod_{i=0}^{2}\begin{bmatrix}1 & 1\\(n-i)p & 0\end{bmatrix}=\begin{bmatrix}1+p & 1+p\\2p & 2p\end{bmatrix}\begin{bmatrix}T_{p,2} & T_{p,2}\\p\cdot T_{p,1} & p\cdot T_{p,1}\end{bmatrix},$$

$$\prod_{i=0}^{3}\begin{bmatrix}1 & 1\\(n-1)p & 0\end{bmatrix}=\begin{bmatrix}1+3p & 1+3p\\3p(1+p) & 3p(1+p)\end{bmatrix}=\begin{bmatrix}T_{p,3} & T_{p,3}\\p\cdot T_{p,2} & p\cdot T_{p,2}\end{bmatrix}.$$

Theorem 3.3. *Consider the numbers* $\{T_{p,n}\}_{n\in\mathbb{N}}$ *and for every positive integer* $n\geq 1$*, the the following property of matrices holds:*

$$\prod_{i=0}^{n}\begin{bmatrix}1 & 1\\(n-i)p & 0\end{bmatrix}=\begin{bmatrix}T_{p,n} & T_{p,n}\\n\cdot pT_{p,n-1} & n\cdot pT_{p,n-1}\end{bmatrix}.$$

Proof. Just see that:

$$\prod_{i=0}^{n+1}\begin{bmatrix}1 & 1\\(n-i)p & 0\end{bmatrix}=\begin{bmatrix}1 & 1\\n+1 & 0\end{bmatrix}\prod_{i=0}^{n+1}\begin{bmatrix}1 & 1\\(n-i)p & 0\end{bmatrix}=\begin{bmatrix}1 & 1\\n+1 & 0\end{bmatrix}\begin{bmatrix}T_{p,n} & T_{p,n}\\n\cdot pT_{p,n-1} & n\cdot pT_{p,n-1}\end{bmatrix}$$

$$=\begin{bmatrix}T_{p,n}+n\cdot pT_{p,n-1} & T_{p,n}+n\cdot pT_{p,n-1}\\(n+1)T_{p,n} & (n+1)T_{p,1}\end{bmatrix}=\begin{bmatrix}T_{p,n+1} & T_{p,n+1}\\(n+1)T_{p,n} & (n+1)T_{p,n}\end{bmatrix}.\square$$

Lemma 3.1. *Consider the numbers* $\{T_n\}_{n\in\mathbb{N}}$*,* $\{T_{k,n}\}_{n\in\mathbb{N}}$ *and* $\{T_{p,n}\}_{n\in\mathbb{N}}$*. For every positive integer* $n\geq 0$*, the following relations are valid:*

(a) $$\begin{bmatrix}T_1+T_2+\ldots+T_n & T_1+T_2+\ldots+T_n\\T_0+2T_1+\ldots+nT_{n-1} & T_0+2T_1+\ldots+nT_{n-1}\end{bmatrix}$$
$$=\prod_{k=0}^{1}\begin{bmatrix}1 & 1\\1-k & 0\end{bmatrix}+\prod_{k=0}^{2}\begin{bmatrix}1 & 1\\2-k & 0\end{bmatrix}+\ldots+\prod_{k=0}^{n}\begin{bmatrix}1 & 1\\n-k & 0\end{bmatrix};$$

(b) $$\begin{bmatrix}T_{k,1}+T_{k,2}+\ldots+T_{k,n} & T_{k,1}+T_{k,2}+\ldots+T_{k,n}\\T_{k,0}+2T_{k,1}+\ldots+nT_{k,n-1} & T_{k,0}+2T_{k,1}+\ldots+nT_{k,n-1}\end{bmatrix}$$
$$=\prod_{i=0}^{1}\begin{bmatrix}1 & 1\\(1-i)k & 0\end{bmatrix}+\prod_{i=0}^{2}\begin{bmatrix}1 & 1\\(2-i)k & 0\end{bmatrix}+\ldots+\prod_{i=0}^{n}\begin{bmatrix}1 & 1\\(n-i)k & 0\end{bmatrix};$$

(c) $$\begin{bmatrix}T_{p,1}+T_{p,2}+\ldots+T_{p,n} & T_{p,1}+T_{p,2}+\ldots+T_{p,n}\\T_{p,0}+2T_{p,1}+\ldots+nT_{p,n-1} & T_{p,0}+2T_{p,1}+\ldots+nT_{p,n-1}\end{bmatrix}$$
$$=\prod_{i=0}^{1}\begin{bmatrix}1 & 1\\(1-i)p & 0\end{bmatrix}+\prod_{i=0}^{2}\begin{bmatrix}1 & 1\\(2-i)p & 0\end{bmatrix}+\ldots+\prod_{i=0}^{n}\begin{bmatrix}1 & 1\\(n-i)p & 0\end{bmatrix}.$$

Proof. Just use Theorems 3.1, 3.2, 3.3. □

4 Conclusion

In the previous sections we demonstrated new relationships involving the finite products of matrices of the type:

$$\prod_{i=0}^{n}\begin{bmatrix} k & 1 \\ i & 0 \end{bmatrix}, \quad \prod_{i=0}^{n}\begin{bmatrix} k & 1 \\ n-i & 0 \end{bmatrix}, \quad \prod_{i=0}^{n}\begin{bmatrix} k & 1 \\ pi & 0 \end{bmatrix}.$$

We demonstrated that such products allow us to describe matrices of order 2, which generate telephone numbers and had not yet been introduced in the literature.

For future studies, application in other disciplines is suggested according to the works [4] and [6].

Acknowledgements

The part of research development in Brazil had the financial support of the National Council for Scientific and Technological Development – CNPq and the Ceará Foundation for Support to Scientific and Technological Development (Funcap).

The research development aspect in Portugal is financed by National Funds through FCT – Fundação para a Ciência e Tecnologia I.P., within the scope of the UID / CED / 00194/2020 project.

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