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Quotients of arithmetical functions under the Dirichlet convolution

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Abstract: We study existence of a solution of the arithmetical equation $f*h = g$ in f, where $f*h$ is the Dirichlet convolution of arithmetical functions f and h, and derive an explicit expression for the solution. As applications we obtain expressions for the Möbius function μ and the so-called totients. As applications we also present our results on the arithmetical equation $f * h = g$ in the language of Cauchy convolution and further deconvolution in discrete linear systems.
Keywords: Arithmetical equation, Dirichlet convolution, Möbius function, Totient function, Cauchy convolution, Discrete linear system.
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1 Introduction

By an arithmetical function we mean a complex-valued function on the set of positive integers. The Dirichlet convolution of two arithmetical functions f and h is defined by

$$(f * h)(n) = \sum_{d|n} f(d)h(n/d).$$

The Dirichlet convolution is associative and commutative, and the function e_0, defined by $e_0(1) = 1$ and $e_0(n) = 0$ for $n \neq 1$, serves as an identity with respect to the Dirichlet convolution.

An arithmetical function f possesses the Dirichlet inverse if and only if $f(1) \neq 0$. See, e.g., [15,20].

Quotients of arithmetical functions under the Dirichlet convolution are defined as solutions of the equation $f * h = g$ in f. It is clear that if h has the Dirichlet inverse, (that is, if $h(1) \neq 0$), then the quotient of g and h can be written as $f = g * h^{-1}$. If $h(1) = 0$, then it is more laborious to find the quotient. However, a recursive and a matrix expression for the quotient in this case was given already in 1937 by Amante [1]. These expressions were studied further by Pellegrino and Succi (see e.g. [16, 22]). Quinton and Robert [17] provide a computational approach to this problem.

Quotients with respect to certain generalized convolutions are studied, e.g., in [8,11]. Solutions to arithmetical function polynomial equations are studied in [4, 6, 21].

In this paper we present:

(i) a necessary and sufficient condition for existence of a quotient under the Dirichlet convolution, and

(ii) an explicit expression for the quotient under the Dirichlet convolution.

As applications we obtain explicit expressions for the Dirichlet inverse of an arithmetical function, the Möbius function μ and the so-called totient functions, such as the Euler totient function ϕ and the Dedekind totient function ψ. The expressions connect these functions μ, ϕ and ψ with the divisor function τ_k.

Applications to discrete linear systems arise as follows.A discrete-time signal is an arithmetical function (or a sequence of numbers). The Cauchy convolution of arithmetical functions is an analogue of the Dirichlet convolution, and the results (i) and (ii) for quotients under the Dirichlet convolution yield directly analogous results for quotients under the Cauchy convolution. The problem of finding quotients under the Cauchy convolution is, in fact, the problem of deconvolution in discrete linear systems, that is, the problem of deconvolving the input out of the output and the system impulse response.

2 On the existence of quotient

For an arithmetical function f with $f \not\equiv 0$, let $\chi(f)$ denote the smallest n for which $f(n) \neq 0$. We confine ourselves to arithmetical functions f such that $f \not\equiv 0$ and $f(n) = 0$ unless $\chi(f) \mid n$, that is, $f(n) = 0$ if $n \neq m\chi(f)$ for each positive integer m. Let A' denote the class of these kind of arithmetical functions. For example, every arithmetical function f with $f(1) \neq 0$ belongs to A'. The class of the so-called semi-multiplicative functions (see [12, 20]) is also a subclass of A'. Therefore the most important arithmetical functions belong to A'.

Theorem 2.1. *Let $g, h \in A'$. Then the equation $f * h = g$ has a solution in f if and only if $\chi(h) \mid \chi(g)$. In this case the solution is unique and is in A'.*

Proof. If $f * h = g$ has a solution in f, then $\chi(f)\chi(h) = \chi(g)$ and hence $\chi(h) \mid \chi(g)$.

Conversely, assume that $\chi(h) \mid \chi(g)$. Denote $a = \chi(g)/\chi(h)$, and define f by

$$f(n) = 0 \quad \text{if } a \nmid n \tag{1}$$

and

$$\begin{cases} f(a) = g(\chi(g))/h(\chi(h)), \\ f(na) = h(\chi(h))^{-1}\Big[g(n\chi(g)) - \sum_{\substack{d|n \\ d<n}} f(da)h(n\chi(h)/d)\Big], \quad n \geq 2. \end{cases} \tag{2}$$

By (1) and the properties

$$\begin{cases} h(n) = 0, & \text{if } \chi(h) \nmid n, \\ g(n) = 0, & \text{if } \chi(g) \nmid n, \end{cases} \tag{3}$$

and $a\chi(h) = \chi(g)$, it follows that

$$(f * h)(n) = g(n) \quad \text{ if } \chi(g) \nmid n. \tag{4}$$

From (2) it follows that

$$\sum_{d|n} f(da)h(n\chi(h)/d) = g(n\chi(g)), \quad n \geq 1. \tag{5}$$

By (1) and (3), the above identity (5) can be written as

$$(f * h)(na\chi(h)) = g(n\chi(g)), \quad n \geq 1$$

or

$$(f * h)(n\chi(g)) = g(n\chi(g)), \quad n \geq 1. \tag{6}$$

Now, combining (4) and (6) proves that f is a solution of $f * h = g$. This proves the converse part.

Next, we shall prove the uniqueness. Assume that f_1 and f_2 are solutions of $f * h = g$. Then

$$(f_1 * h)(\chi(h)) = (f_2 * h)(\chi(h))$$

or

$$f_1(1)h(\chi(h)) = f_2(1)h(\chi(h)).$$

Since $h(\chi(h)) \neq 0$, we obtain $f_1(1) = f_2(1)$. Assume inductively that $f_1(m) = f_2(m)$ for $m < n$. Since f_1 and f_2 are solutions of $f * h = g$, we have

$$(f_1 * h)(n\chi(h)) = (f_2 * h)(n\chi(h)),$$

or

$$f_1(n)h(\chi(h)) + \sum_{\substack{d|n \\ d<n}} f_1(d)h(n\chi(h)/d) = f_2(n)h(\chi(h)) + \sum_{\substack{d|n \\ d<n}} f_2(d)h(n\chi(h)/d).$$

Thus, by the inductive assumption, $f_1(n) = f_2(n)$. Therefore $f_1 = f_2$, and the solution f is unique.

Finally, it follows from (1) and (2) that $\chi(f) = a$ and further that $f \in A'$. This completes the proof of Theorem 2.1. □

Remark 2.1. Theorem 2.1 corrects the inaccuracy that exists in Theorem 3 of [7].

3 An explicit expression for quotient

From the proof of Theorem 2.1 we can directly find two expressions for the solution f of the equation $f * h = g$, that is, for the quotient of g and h. In fact, (2) gives a recursive expression for the quotient. Further, we can look upon (5) as a system of linear equations and use Cramer's rule to obtain a determinant expression for the quotient, cf. [1, 16, 22]. We do not present the details here.

The aim of this section is to derive an explicit expression for the quotient from the recursive expression (2). This explicit expression is given in Theorem 3.1.

Theorem 3.1. *Let $a = \chi(f), b = \chi(h), c = \chi(g)$. If $g, h \in A'$ with $b \mid c$, then the values of the solution f of the equation $f * h = g$ are given by*

$$f(na) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{h(b)^k} \sum_{\substack{d_1 d_2 \cdots d_k = n \\ d_2, d_3, \dots, d_k > 1}} g(d_1 c) h(d_2 b) \cdots h(d_k b), \quad n \geq 1. \tag{7}$$

Remark 3.1. In (7), the summation over k is finite. It suffices that k runs through the integers from 1 to $\Omega(n) + 1$, where $\Omega(n)$ is the total number of prime factors of n, each being counted according to its multiplicity, with $\Omega(1) = 0$.

Proof. We proceed by induction on n. For $n = 1$, the right side of (7) is $g(c)/h(b)$ and hence (7) holds.

Assume that (7) holds for $n < m$. Then, by (2),

$$f(ma) = h(b)^{-1}[g(mc) - \sum_{\substack{de=m \\ d<m}} f(da)h(eb)].$$

By the inductive assumption,

$$f(ma) = h(b)^{-1}[g(mc) - \sum_{\substack{de=m \\ d<m}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{h(b)^k} \sum_{\substack{d_1 d_2 \cdots d_k = d \\ d_2, d_3, \dots, d_k > 1}} g(d_1 c) h(d_2 b) \cdots h(d_k b) h(eb)].$$

Manipulating the above sum gives

$$\begin{aligned} f(ma) &= \frac{g(mc)}{h(b)} + \sum_{k=1}^{\infty} \frac{(-1)^{k+2}}{h(b)^{k+1}} \sum_{\substack{d_1 d_2 \cdots d_{k+1} = m \\ d_2, d_3, \dots, d_{k+1} > 1}} g(d_1 c) h(d_2 b) \cdots h(d_{k+1} b) \\ &= \frac{g(mc)}{h(b)} + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{h(b)^k} \sum_{\substack{d_1 d_2 \cdots d_k = m \\ d_2, d_3, \dots, d_k > 1}} g(d_1 c) h(d_2 b) \cdots h(d_k b) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{h(b)^k} \sum_{\substack{d_1 d_2 \cdots d_k = m \\ d_2, d_3, \dots, d_k > 1}} g(d_1 c) h(d_2 b) \cdots h(d_k b). \end{aligned}$$

This completes the induction. □

Example 3.1. If $h(1) \neq 0$, then we have

$$f(nc) = (g * h^{-1})(nc) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{h(1)^k} \sum_{\substack{d_1 d_2 \cdots d_k = n \\ d_2, d_3, \dots, d_k > 1}} g(d_1 c) h(d_2) \cdots h(d_k), \quad n \geq 1. \tag{8}$$

4 Applications to arithmetical functions

In this section we apply Theorem 3.1 to obtain explicit expressions for the Dirichlet inverse of an arithmetical function, the Möbius function μ and the so-called totient functions, such as the Euler totient function ϕ and the Dedekind totient function ψ. For material on these functions, see [15, 18–20].

4.1 The Dirichlet inverse of an arithmetical function

The Dirichlet inverse h^{-1} of an arithmetical function h is defined by $h * h^{-1} = h^{-1} * h = e_0$. The Dirichlet inverse of h exists if and only if $h(1) \neq 0$. The well-known recursive formula for h^{-1} follows directly from the definition and is given as

$$\begin{cases} h^{-1}(1) = \frac{1}{h(1)}, \\ h^{-1}(n) = \frac{-1}{h(1)} \sum_{\substack{d \mid n \\ d<n}} h^{-1}(d) h(n/d), \quad n \geq 2, \end{cases} \tag{9}$$

(see [15, 20]). By Theorem 3.1 this can be written in an explicit form as

$$\begin{cases} h^{-1}(1) = \frac{1}{h(1)}, \\ h^{-1}(n) = \sum_{k=1}^{\Omega(n)} \frac{(-1)^k}{h(1)^{k+1}} \sum_{\substack{d_1 d_2 \cdots d_k = n \\ d_1, d_2, \ldots, d_k > 1}} h(d_1) h(d_2) \cdots h(d_k), \quad n \geq 2, \end{cases} \tag{10}$$

(see [2, 10]).

4.2 The Möbius function

The Möbius function μ is the Dirichlet inverse of the constant function $\equiv 1$. The classical expression for the Möbius function is

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^r, & \text{if } n = p_1 p_2 \cdots p_r,\ p_i \neq p_j\ (i \neq j), \\ 0, & \text{if there exists a prime } p \text{ such that } p^2 \mid n. \end{cases} \tag{11}$$

(see [7, 9]). Application of (10) with $h \equiv 1$ gives

$$\mu(n) = \sum_{k=0}^{\Omega(n)} (-1)^k \Delta_k(n), \quad n \geq 1, \tag{12}$$

where $\Delta_k(n)$ is the function defined as follows. If $k \geq 1$, then $\Delta_k(n)$ is the number of k-tuples $(d_1, d_2, \ldots, d_k)$ such that $d_1 d_2 \cdots d_k = n$, $d_1, d_2, \ldots, d_k > 1$. In addition, $\Delta_0(n) = e_0(n)$ $(n \geq 1)$, that is, $\Delta_0(1) = 1$ and $\Delta_0(n) = 0$ $(n \geq 2)$.

Several properties of the function $\Delta_k(n)$ are presented in [10]. For example, $\Delta_k(n)$ is written in terms of the well-known divisor function $\tau_k(n)$ (for τ_k, see [3, 20]). For $k \geq 1$, $\tau_k(n)$ is defined as the number of k-tuples $(d_1, d_2, \ldots, d_k)$ such that $d_1 d_2 \cdots d_k = n$. In other words, for $k \geq 1$, $\tau_k = u * u * \cdots * u$ (u, k times), where $u \equiv 1$. In addition, it is convenient to define $\tau_0(n) = e_0(n)$ $(n \geq 1)$. From [10] we know the following result.

Theorem 4.1. *For $k \geq 0$,*

$$\Delta_k(n) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \tau_{k-i}(n), \quad n \geq 1. \tag{13}$$

It should be noted that some authors [20] use the notation $d_k(n)$ for $\tau_k(n)$. Also note that $\tau_2(n)$ is the classical divisor function, usually denoted by $\tau(n)$ or $d(n)$.

4.3 Totient functions

An arithmetical function f is said to be multiplicative if $f(1) = 1$ and

$$f(mn) = f(m)f(n) \tag{14}$$

whenever $(m, n) = 1$. A multiplicative function f is said to be completely multiplicative if (14) holds for all m and n. A multiplicative function f is said to be a totient function [9, 14, 15] if $f = g * h^{-1}$, where g and h are completely multiplicative functions. A totient function is thus the quotient of two completely multiplicative functions.

Theorem 4.2. *If f is a totient function of the form $f = g * h^{-1}$, then*

$$f(n) = \sum_{k=0}^{\Omega(n)} (-1)^k \left(g * (h\Delta_k)\right)(n), \quad n \geq 1. \tag{15}$$

Proof. Since $a = b = c = 1$ and h is completely multiplicative in (7), we obtain

$$\begin{aligned} f(n) &= g(n) + \sum_{k=2}^{\infty} (-1)^{k+1} \sum_{d_1 | n} g(d_1) h(n/d_1) \sum_{\substack{d_2 d_3 \cdots d_k = n/d_1 \\ d_2, d_3, \ldots, d_k > 1}} 1 \\ &= g(n) + \sum_{k=2}^{\infty} (-1)^{k+1} \sum_{d_1 | n} g(d_1) h(n/d_1) \Delta_{k-1}(n/d_1). \end{aligned}$$

This easily leads to Theorem 4.2. □

The Euler totient function $\phi(n)$ is defined as the number of integers x (mod n) such that $(x, n) = 1$. It is well known [15, 20] that $\phi = N * u^{-1} = N * \mu$, where $N(n) = n$ for all n. The Dedekind totient function ψ is defined by $\psi(n) = n \prod_{p|n}(1 + p^{-1})$. It is well known [15] that $\psi = N * \lambda^{-1} = N * \mu^2$, where λ is Liouville's function defined by $\lambda(n) = (-1)^{\Omega(n)}$. We thus obtain the following Corollary of Theorem 4.2.

Corollary 4.1. *We have*

$$\phi(n) = \sum_{k=0}^{\Omega(n)} (-1)^k (N * \Delta_k)(n), \quad n \geq 1, \tag{16}$$

$$\psi(n) = \sum_{k=0}^{\Omega(n)} (-1)^k \left(N * (\lambda\Delta_k)\right)(n), \quad n \geq 1. \tag{17}$$

The class of rational arithmetical functions is an extension of the class of totients. A multiplicative function f is said to be a rational arithmetical function of order (r, s) [14] if $f = g * h^{-1}$, where $g = g_1 * \cdots * g_r$ and $h = h_1 * \cdots * h_s$, the functions g_i and h_i being completely multiplicative. A rational arithmetical function is a quotient of this kind of functions g and h. Totients are rational arithmetical function of order $(1, 1)$, and the Möbius function is a rational arithmetical function of order $(0, 1)$.

5 Sequences and Cauchy convolution

If f and h are arithmetical functions, then their Dirichlet convolution at a prime power p^n is

$$\sum_{k=0}^{n} f(p^k)h(p^{n-k}),$$

which is the Cauchy convolution of the sequences $\{f(p^n)\}_{n=0}^{\infty}$ and $\{h(p^n)\}_{n=0}^{\infty}$. This connection between the Dirichlet convolution and the Cauchy convolution suggests that Theorems 2.1 and 3.1 can be written in terms of sequences and Cauchy convolution.

Let $\circ$ denote the Cauchy convolution of two sequences [15, 20], that is, if $\{x(n)\}_{n=0}^{\infty}$ and $\{h(n)\}_{n=0}^{\infty}$ are sequences, then $\{(x \circ h)(n)\}_{n=0}^{\infty}$ is the sequence given by

$$(x \circ h)(n) = \sum_{k=0}^{n} x(k)h(n-k).$$

We now present the Cauchy analogues of Theorems 2.1 and 3.1.

Theorem 5.1. *The equation $x \circ h = y$ has a solution in x if and only if*

$$\chi(h) \le \chi(y).$$

In this case the solution is unique.

Theorem 5.2. *Let $a = \chi(x), b = \chi(h), c = \chi(y)$. If $b \le c$, then the values of the solution x of the equation $x \circ h = y$ are given by*

$$x(n+a) = \sum_{i=1}^{n+1} \frac{(-1)^{i+1}}{h(b)^i} \sum_{\substack{k_1+k_2+\cdots+k_i=n \\ k_2,k_3,\ldots,k_i>0}} y(k_1+c)h(k_2+b)\cdots h(k_i+b), \; n \ge 0. \tag{18}$$

Theorems 5.1 and 5.2 follow from Theorems 2.1 and 3.1.

Remark 5.1. Note that the solution $\{x(n)\}$ satisfies $\chi(x) = \chi(y) - \chi(h)$. In particular, if $\chi(h) = 0$, then $\chi(x) = \chi(y)$ and (18) can be written as

$$x(n) = \sum_{i=1}^{n+1} \frac{(-1)^{i+1}}{h(0)^i} \sum_{\substack{k_1+k_2+\cdots+k_i=n \\ k_2,k_3,\ldots,k_i>0}} y(k_1)h(k_2)\cdots h(k_i), \quad n \ge \chi(y). \tag{19}$$

6 Application to discrete linear systems

Consider a linear time-invariant (LTI) system. Then the input-output relationship is

$$y(n) = (x \circ h)(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k),$$

where $\{x(n)\}$ is the input signal, $\{y(n)\}$ is the output signal and $\{h(n)\}$ is the impulse response of the system. The LTI system is completely characterized by $\{h(n)\}$. We here confine ourselves to causal systems, that is, we assume that $h(n) = 0$ for $n < 0$. We also assume that the input signals are causal, that is, $x(n) = 0$ for $n < 0$. The output thus has the form

$$y(n) = \sum_{k=0}^{n} h(k)x(n-k), \ n \geq 0$$

(see [13]).

In Section 5 we, in fact, consider the following problem. We are given the output $\{y(n)\}$ and the impulse response $\{h(n)\}$ of the system. We wish to determine the input $\{x(n)\}$ from the equation $y = x \circ h$, that is, we wish to deconvolve the input $\{x(n)\}$ out of $y = x \circ h$ (see [5]).

Theorem 5.1 shows that a causal input $\{x(n)\}$ satisfying $y = x \circ h$ exists if and only if $\chi(h) \leq \chi(y)$. Theorem 5.2 gives an explicit expression for the input. As far as we know, this kind of expression has not previously been presented in the literature. Classical expressions for the input are the recursive expression and the matrix expression (see [5]).

Example 6.1. Let $\{h(n)\}$ be the exponential impulse response, that is, $h(n) = \alpha^n$, $n \geq 0$. Then $\chi(h) = 0$ and $\chi(x) = \chi(y)$. By (19), for $n \geq \chi(x)$

$$x(n) = y(n) + \sum_{i=2}^{n+1} (-1)^{i+1} \sum_{k_1=0}^{n} y(k_1)\alpha^{n-k_1} \sum_{\substack{k_2+k_3+\cdots+k_i=n-k_1 \\ k_2,k_3,\ldots,k_i>0}} 1.$$

The inner sum is equal to $\binom{n-k_1-1}{i-2}$. Let $k = n - k_1$. Then we obtain

$$x(n) = y(n) + \sum_{k=0}^{n} y(n-k)\alpha^k \sum_{i=0}^{k-1} (-1)^{i+1} \binom{k-1}{i}.$$

The inner sum is $= -1$ for $k = 1$, and $= 0$ otherwise. Thus

$$x(n) = y(n) - \alpha y(n-1). \tag{20}$$

Note that (20) could also be derived with the aid of z-transform easily.

7 Application to probability theory

Consider the equation $Y = X + H$ in X, where X and H are independent discrete random variables. Let $P(X = n) = x(n)$ and $P(H = n) = h(n)$ for all $n \geq 0$. Then

$$y(n) = \sum_{k=0}^{n} x(k)h(n-k) = (x \circ h)(n).$$

We can thus apply Section 5 to solve the following problem. Let the probability functions $h(n)$ and $y(n)$ be given. What is the probability function $x(n)$? Theorem 5.1 shows that $x(n)$ with $\chi(x) \geq 0$ exists if and only if $\chi(y) \leq \chi(h)$. Theorem 5.2 gives an explicit expression for the solution $x(n)$.

Example 7.1. Let H be the geometric distribution, that is, $h(n) = q^{n-1}p$, $n \geq 1$. Thus $\chi(h) = 1$ and $\chi(x) = \chi(y) - 1$, where $\chi(y) \equiv c \geq 1$. By (18), for $n \geq 0$

$$x(n+c-1) = p^{-1}y(n+c) + p^{-1}\sum_{i=2}^{n+1}(-1)^{i+1}\sum_{k_1=0}^{n} y(k_1+c)q^{n-k_1} \sum_{\substack{k_2+k_3+\cdots+k_i=n-k_1 \\ k_2,k_3,\ldots,k_i>0}} 1.$$

Proceeding in a similar way to Example 6.1 we obtain

$$x(n+c-1) = p^{-1}y(n+c) + p^{-1}q\,y(n-1+c).$$

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