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Corrigendum to: "Some modular considerations regarding odd perfect numbers – Part II" [Notes on Number Theory and Discrete Mathematics, 2020, Vol. 26, No. 3, 8–24]

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Abstract: In [2], the authors proposed a theorem which they recently found out to contradict Chen and Luo's results [1]. In the present paper, we provide the correct form of this theorem.
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1 Introduction

Let $\sigma(x) = \sigma_1(x)$ denote the *classical sum of divisors* of the positive integer x.

An odd number n satisfying $\sigma(n) = 2n$ is called an *odd perfect number*. Euler showed that a hypothetical odd perfect number n, if one exists, must necessarily have the form

$$n = p^k m^2,$$

where p is the special prime satisfying $p \equiv k \equiv 1 \pmod 4$ and $\gcd(p, m) = 1$.

In a part of the abstract of the paper [2], it is claimed that

1. $\sigma(m^2) \equiv 3 \pmod 8$ holds only if $p - k \equiv 4 \pmod{16}$.

On page 15 of the paper [2], it is stated (without proof) that if $n = p^k m^2$ is an odd perfect number with special prime p satisfying $\sigma(m^2) \equiv 3 \pmod 8$, then exactly one of the following conditions hold:

1. $p \equiv 1 \pmod{16}, k \equiv 13 \pmod{16}$,
2. $p \equiv 5 \pmod{16}, k \equiv 1 \pmod{16}$,
3. $p \equiv 9 \pmod{16}, k \equiv 5 \pmod{16}$,
4. $p \equiv 13 \pmod{16}, k \equiv 9 \pmod{16}$.

2 The correct form of the Abstract and the Theorem

We now give the corrected form of the part of the abstract that is in error and the corresponding theorem:

2.1 Corrected Abstract

If $n = p^k m^2$ is an odd perfect number with special prime p, then

1. $\sigma(m^2) \equiv 3 \pmod 8$ holds only if $p - k \equiv 12 \pmod{16}$.

2.2 Corrected Theorem

Theorem 2.1. *Suppose that $n = p^k m^2$ is an odd perfect number with special prime p satisfying $\sigma(m^2) \equiv 3 \pmod 8$. This implies that exactly one of the following conditions hold:*

1. $p \equiv 1 \pmod{16}$, $k \equiv 5 \pmod{16}$,
2. $p \equiv 5 \pmod{16}$, $k \equiv 9 \pmod{16}$,
3. $p \equiv 9 \pmod{16}$, $k \equiv 13 \pmod{16}$,
4. $p \equiv 13 \pmod{16}$, $k \equiv 1 \pmod{16}$.

3 A Proof of Theorem 2.1

(This section refers to the results (i.e. lemmas) as stated in the paper [2].)

Let $n = p^k m^2$ be an odd perfect number with special prime p, satisfying $\sigma(m^2) \equiv 3 \pmod 8$. By Lemma 3.1, $p \equiv k + 4 \pmod 8$ holds.

We now consider each of the resulting possible congruences for p and k modulo 16:

1. $p \equiv 1 \pmod{16}$, $k \equiv 5 \pmod{16}$,
2. $p \equiv 1 \pmod{16}$, $k \equiv 13 \pmod{16}$,
3. $p \equiv 5 \pmod{16}$, $k \equiv 1 \pmod{16}$,
4. $p \equiv 5 \pmod{16}$, $k \equiv 9 \pmod{16}$,
5. $p \equiv 9 \pmod{16}$, $k \equiv 5 \pmod{16}$,
6. $p \equiv 9 \pmod{16}$, $k \equiv 13 \pmod{16}$,
7. $p \equiv 13 \pmod{16}$, $k \equiv 1 \pmod{16}$,
8. $p \equiv 13 \pmod{16}$, $k \equiv 9 \pmod{16}$.

Recall that we have the equation

$$2D(m^2)s(m^2) = (\gcd(m^2, \sigma(m^2)))^2 D(p^k)s(p^k) \tag{$*$}$$

where $D(x) = 2x - \sigma(x)$ is the *deficiency* of x and $s(x) = \sigma(x) - x$ is the *aliquot sum* of x.

First, suppose that $p \equiv 1 \pmod{16}$, $k \equiv 5 \pmod{16}$. By Lemma 3.3, $D(p^k) \equiv 12 \pmod{16}$. By Lemma 3.5, $D(m^2) \equiv 7 \pmod{8}$. By Lemma 3.4, $s(p^k) \equiv 5 \pmod{16}$. By Lemma 3.6, $s(m^2) \equiv 2 \pmod{8}$.

Thus, symbolically we obtain from Equation ($*$) that

$$2(8a+7)(8b+2) = (8c+1)(16d+12)(16e+5)$$

which is solvable over the positive integers, per WolframAlpha.

Next, suppose that $p \equiv 1 \pmod{16}$, $k \equiv 13 \pmod{16}$. By Lemma 3.3, $D(p^k) \equiv 4 \pmod{16}$. By Lemma 3.5, $D(m^2) \equiv 7 \pmod{8}$. By Lemma 3.4, $s(p^k) \equiv 13 \pmod{16}$. By Lemma 3.6, $s(m^2) \equiv 2 \pmod{8}$.

Thus, symbolically we obtain from Equation ($*$) that

$$2(8a+7)(8b+2) = (8c+1)(16d+4)(16e+13)$$

which is NOT solvable over the positive integers, per WolframAlpha.

Next, suppose that $p \equiv 5 \pmod{16}$, $k \equiv 1 \pmod{16}$. By Lemma 3.3, $D(p^k) \equiv 4 \pmod{16}$. By Lemma 3.5, $D(m^2) \equiv 7 \pmod{8}$. By Lemma 3.4, $s(p^k) \equiv 1 \pmod{16}$. By Lemma 3.6, $s(m^2) \equiv 2 \pmod{8}$.

Thus, symbolically we obtain from Equation ($*$) that

$$2(8a+7)(8b+2) = (8c+1)(16d+4)(16e+1)$$

which is NOT solvable over the positive integers, per WolframAlpha.

Next, suppose that $p \equiv 5 \pmod{16}$, $k \equiv 9 \pmod{16}$. By Lemma 3.3, $D(p^k) \equiv 12 \pmod{16}$. By Lemma 3.5, $D(m^2) \equiv 7 \pmod{8}$. By Lemma 3.4, $s(p^k) \equiv 9 \pmod{16}$. By Lemma 3.6, $s(m^2) \equiv 2 \pmod{8}$.

Thus, symbolically we obtain from Equation ($*$) that

$$2(8a+7)(8b+2) = (8c+1)(16d+12)(16e+9)$$

which is solvable over the positive integers, per WolframAlpha.

Next, suppose that $p \equiv 9 \pmod{16}$, $k \equiv 5 \pmod{16}$. By Lemma 3.3, $D(p^k) \equiv 4 \pmod{16}$. By Lemma 3.5, $D(m^2) \equiv 7 \pmod{8}$. By Lemma 3.4, $s(p^k) \equiv 5 \pmod{16}$. By Lemma 3.6, $s(m^2) \equiv 2 \pmod{8}$.

Thus, symbolically we obtain from Equation ($*$) that

$$2(8a+7)(8b+2) = (8c+1)(16d+4)(16e+5)$$

which is NOT solvable over the positive integers, per WolframAlpha.

Next, suppose that $p \equiv 9 \pmod{16}$, $k \equiv 13 \pmod{16}$.By Lemma 3.3, $D(p^k) \equiv 12 \pmod{16}$. By Lemma 3.5, $D(m^2) \equiv 7 \pmod{8}$. By Lemma 3.4, $s(p^k) \equiv 13 \pmod{16}$. By Lemma 3.6, $s(m^2) \equiv 2 \pmod{8}$.

Thus, symbolically we obtain from Equation ($*$) that

$$2(8a+7)(8b+2) = (8c+1)(16d+12)(16e+13)$$

which is solvable over the positive integers, per WolframAlpha.

Next, suppose that $p \equiv 13 \pmod{16}$, $k \equiv 1 \pmod{16}$. By Lemma 3.3, $D(p^k) \equiv 12 \pmod{16}$. By Lemma 3.5, $D(m^2) \equiv 7 \pmod{8}$. By Lemma 3.4, $s(p^k) \equiv 1 \pmod{16}$. By Lemma 3.6, $s(m^2) \equiv 2 \pmod{8}$.

Thus, symbolically we obtain from Equation ($*$) that

$$2(8a+7)(8b+2) = (8c+1)(16d+12)(16e+1)$$

which is solvable over the positive integers, per WolframAlpha.

Lastly, suppose that $p \equiv 13 \pmod{16}$, $k \equiv 9 \pmod{16}$. By Lemma 3.3, $D(p^k) \equiv 4 \pmod{16}$. By Lemma 3.5, $D(m^2) \equiv 7 \pmod{8}$. By Lemma 3.4, $s(p^k) \equiv 9 \pmod{16}$. By Lemma 3.6, $s(m^2) \equiv 2 \pmod{8}$.

Thus, symbolically we obtain from Equation ($*$) that

$$2(8a+7)(8b+2) = (8c+1)(16d+4)(16e+9)$$

which is NOT solvable over the positive integers, per WolframAlpha.

4 Conclusion

To summarize: If $\sigma(m^2) \equiv 3 \pmod{8}$, then exactly one of the following conditions hold:

1. $p \equiv 1 \pmod{16}$, $k \equiv 5 \pmod{16}$,
2. $p \equiv 5 \pmod{16}$, $k \equiv 9 \pmod{16}$,
3. $p \equiv 9 \pmod{16}$, $k \equiv 13 \pmod{16}$,
4. $p \equiv 13 \pmod{16}$, $k \equiv 1 \pmod{16}$.

In other words, the congruence $\sigma(m^2) \equiv 3 \pmod{8}$ holds only if $p \equiv k+12 \pmod{16}$.

Our findings now match Chen and Luo's results [1]. The rest of the paper [2] is unaffected.

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References

[1] Chen, S.-C., & Luo, H. (2013). Odd multiperfect numbers. *Bulletin of the Australian Mathematical Society*, 88(1), 56–63.

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