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Objects generated by an arbitrary natural number. Part 3: Standard modal-topological aspect

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Abstract: The set $\underline{SET}(n)$, generated by an arbitrary natural number n, was defined in [3]. There, and in [4], some arithmetic functions and arithmetic operators of a modal type are defined over the elements of $\underline{SET}(n)$. Here, over the elements of $\underline{SET}(n)$ arithmetic operators of a topological type are defined and some of their basic properties are studied. Perspectives for future research are discussed.
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1 Introduction

The present paper is a continuation of previous legs of the author's research, [3, 4]. Here, we give only the basic definitions from them and after this, in Section 2, we will introduce new arithmetic operators, this time of topological type.

As it was discussed in [3], the set $\underline{SET}(n)$ is generated by a fixed arbitrary natural number n. Similarly to [3], let everywhere below the fixed arbitrary natural number $n \geq 2$ has the canonical form

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where k, α_1, α_2, ..., $\alpha_k \geq 1$ are natural numbers and $p_1, p_2, \ldots, p_k$ are different prime numbers. In [1, 3, 4], the following notations related to n that we will use below, are introduced:

$$\begin{aligned}
\underline{set}(n) &= \{p_1, p_2, \ldots, p_k\}, \\
\underline{mult}(n) &= \prod_{i=1}^{k} p_i, \\
\omega(n) &= k, \\
\underline{SET}(n) &= \{m | m = \prod_{i=1}^{k} p_i^{\beta_i} \ \& \ \delta(n) \leq \beta_i \leq \Delta(n)\},
\end{aligned}$$

where *

$$\begin{aligned}
\delta(n) &= \min(\alpha_1, \ldots, \alpha_k), \\
\Delta(n) &= \max(\alpha_1, \ldots, \alpha_k), \\
\boxdot\, n &= (\underline{mult}(n))^{\delta(n)}, \\
\boxplus\, n &= (\underline{mult}(n))^{\Delta(n)},
\end{aligned}$$

for each $m \in \underline{SET}(n)$, i.e., for $m = \prod_{i=1}^{k} p_i^{\beta_i}$, where $\delta(m) \leq \beta_i \leq \Delta(m)$:

$$\begin{aligned}
\neg m &= \prod_{i=1}^{k} p_i^{\Delta(n)+\delta(n)-\beta_i}, \\
\Box\, m &= (\underline{mult}(n))^{\delta(m)}, \\
\Diamond m &= (\underline{mult}(n))^{\Delta(m)}.
\end{aligned}$$

While in [3, 4] it was shown that the new objects have properties specific to algebra and for modal logic†, respectively, here we will discuss their topological properties.

2 New arithmetic functions, defined in $\underline{SET}(n)$

Let us define for the above n and m:

$$\nabla(n) = \frac{\delta(n) + \Delta(n)}{2},$$

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* Other authors (see, e.g. [12]), denote the functions δ and Δ by h and H, respectively.

† As we discussed in [4], operators $\Box$ and $\Diamond$ are in some sense analogues of the modal operators "necessity" and "possibility", respectively, see, e.g., [7, 9, 10].

$$\mathcal{C}(m) = \prod_{i=1}^{k} p_i^{\max(\lceil \nabla(n) \rceil, \beta_i)},$$

$$\mathcal{I}(m) = \prod_{i=1}^{k} p_i^{\min(\lfloor \nabla(n) \rfloor, \beta_i)},$$

where $\lfloor x \rfloor$ is the floor function, or the integer part of the real number x and

$$\lceil x \rceil = \begin{cases} x, & \text{if } x \text{ is integer} \\ \lfloor x \rfloor + 1, & \text{otherwise} \end{cases}.$$

In some sense, operators $\mathcal{C}$ and $\mathcal{I}$ are analogues of the topological operators "closure" and "interior", see, e.g., [8, 11].

Following [4], let us define the operation multiplication for the natural numbers $l, m \in \underline{SET}(n)$ with canonical forms:

$$l = \prod_{i=1}^{k} p_i^{\beta_i}, \quad m = \prod_{i=1}^{k} p_i^{\gamma_i},$$

(and hence, $\delta(n) \leq \beta_i, \gamma_i \leq \Delta(n)$) by:

$$l \times m = \prod_{i=1}^{k} p_i^{\min(\beta_i + \gamma_i, \Delta(n))}.$$

The operations the Great Common Divisor and the Least Common Multiple have the standard forms:

$$(l, m) = \prod_{i=1}^{k} p_i^{\min(\beta_i, \gamma_i)}, \quad [l, m] = \prod_{i=1}^{k} p_i^{\max(\beta_i, \gamma_i)},$$

because

$$\delta(n) \leq \min(\beta_i, \gamma_i) \leq \max(\beta_i, \gamma_i) \leq \Delta(n).$$

Below, both operations will be denote by "$(.)$" and "[.]", respectively.

The following assertions are valid.

Theorem 1. *For the given* $l, m \in \underline{SET}(n)$

$$(l, m) \times [l, m] = l \times m,$$
$$\neg[\neg l, \neg m] = (l, m),$$
$$\neg(\neg l, \neg m) = [l, m].$$

Proof: Let $l, m \in \underline{SET}(n)$ be given. Then, for the first equality we obtain:

$$\begin{aligned} (l, m) \times [l, m] &= \prod_{i=1}^{k} p_i^{\min(\beta_i, \gamma_i)} \times \prod_{i=1}^{k} p_i^{\max(\beta_i, \gamma_i)} \\ &= \prod_{i=1}^{k} p_i^{\min(\min(\beta_i, \gamma_i) + \max(\beta_i, \gamma_i), \Delta(n))} \\ &= \prod_{i=1}^{k} p_i^{\min(\beta_i + \gamma_i, \Delta(n))} = l \times m. \end{aligned}$$

For the second equality we obtain:

$$\begin{aligned}
\neg[\neg l, \neg m] &= \neg\left[\neg\prod_{i=1}^{k} p_i^{\beta_i}, \neg\prod_{i=1}^{k} p_i^{\gamma_i}\right] \\
&= \neg\left[\prod_{i=1}^{k} p_i^{\Delta(n)+\delta(n)-\beta_i}, \prod_{i=1}^{k} p_i^{\Delta(n)+\delta(n)-\gamma_i}\right] \\
&= \neg\prod_{i=1}^{k} p_i^{\max(\Delta(n)+\delta(n)-\beta_i,\Delta(n)+\delta(n)-\gamma_i)} \\
&= \prod_{i=1}^{k} p_i^{\Delta(n)+\delta(n)-\max(\Delta(n)+\delta(n)-\beta_i,\Delta(n)+\delta(n)-\gamma_i)} \\
&= \prod_{i=1}^{k} p_i^{\min(\beta_i,\gamma_i)} = (l, m).
\end{aligned}$$

The third equality is proved in the same manner. □

Theorem 2. *For a given $m \in \underline{SET}(n)$:*

(a) $\mathcal{C}(\mathcal{C}(m)) = \mathcal{C}(m)$,

(b) $\mathcal{C}(\mathcal{I}(m)) = (\underline{mult}(n))^{\lceil\nabla(n)\rceil}$,

(c) $\mathcal{I}(\mathcal{C}(m)) = (\underline{mult}(n))^{\lfloor\nabla(n)\rfloor}$,

(d) $\mathcal{I}(\mathcal{I}(m)) = \mathcal{I}(m)$,

(e) $\mathcal{I}(m) \leq m \leq \mathcal{C}(m)$,

(f) $\neg\mathcal{C}(\neg m) = \mathcal{I}(m)$,

(g) $\neg\mathcal{I}(\neg m) = \mathcal{C}(m)$.

Proof: For (a) and (b) we obtain, respectively:

$$\begin{aligned}
\mathcal{C}(\mathcal{C}(m)) &= \mathcal{C}(\prod_{i=1}^{k} p_i^{\max(\lceil\nabla(n)\rceil,\beta_i)}) \\
&= \prod_{i=1}^{k} p_i^{\max(\lceil\nabla(n)\rceil,\max(\lceil\nabla(n)\rceil,\beta_i)} \\
&= \prod_{i=1}^{k} p_i^{\max(\lceil\nabla(n)\rceil,\beta_i)} = \mathcal{C}(m),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{C}(\mathcal{I}(m)) &= \mathcal{C}(\prod_{i=1}^{k} p_i^{\min(\lfloor\nabla(n)\rfloor,\beta_i)}) \\
&= \prod_{i=1}^{k} p_i^{\max(\lceil\nabla(n)\rceil,\min(\lfloor\nabla(n)\rfloor,\beta_i))} \\
&= \prod_{i=1}^{k} p_i^{\lceil\nabla(n)\rceil} \\
&= (\underline{mult}(n))^{\lceil\nabla(n)\rceil}.
\end{aligned}$$

Equalities (c) and (d) are proved in the same manner. For (e) and (f) we obtain, respectively:

$$\mathcal{I}(m) = \prod_{i=1}^{k} p_i^{\min(\lfloor \nabla(n) \rfloor, \beta_i)} \leq \prod_{i=1}^{k} p_i^{\beta_i} \leq \prod_{i=1}^{k} p_i^{\max(\lceil \nabla(n) \rceil, \beta_i)} = \mathcal{C}(m),$$

and

$$\begin{aligned}
\neg\mathcal{C}(\neg m) &= \neg\mathcal{C}(\prod_{i=1}^{k} p_i^{\Delta(n)+\delta(n)-\beta_i}) \\
&= \neg \prod_{i=1}^{k} p_i^{\max(\lceil \nabla(n) \rceil, \Delta(n)+\delta(n)-\beta_i)} \\
&= \prod_{i=1}^{k} p_i^{\Delta(n)+\delta(n)-\max(\lceil \nabla(n) \rceil, \Delta(n)+\delta(n)-\beta_i)}.
\end{aligned}$$

There are two cases. If $\lceil \nabla(n) \rceil \geq \Delta(n) + \delta(n) - \beta_i$, then $\beta_i \geq 2\nabla(n) - \lceil \nabla(n) \rceil = \lfloor \nabla(n) \rfloor$, and

$$\begin{aligned}
\neg\mathcal{C}(\neg m) &= \prod_{i=1}^{k} p_i^{2\nabla(n)-\lceil \nabla(n) \rceil} \\
&= \prod_{i=1}^{k} p_i^{\lfloor \nabla(n) \rfloor} \\
&= \prod_{i=1}^{k} p_i^{\min(\lfloor \nabla(n) \rfloor, \beta_i)} = \mathcal{I}(m).
\end{aligned}$$

If $\lceil \nabla(n) \rceil < \Delta(n) + \delta(n) - \beta_i$, then $\beta_i < 2\nabla(n) - \lceil \nabla(n) \rceil = \lfloor \nabla(n) \rfloor$, and

$$\begin{aligned}
\neg\mathcal{C}(\neg m) &= \prod_{i=1}^{k} p_i^{2\nabla(n)-(2\nabla(n)-\beta_i)} \\
&= \prod_{i=1}^{k} p_i^{\beta_i} \\
&= \prod_{i=1}^{k} p_i^{\min(\lfloor \nabla(n) \rfloor, \beta_i)} = \mathcal{I}(m).
\end{aligned}$$

Equality (g) is proved in the same manner. □

3 Two standard modal topological structures over $\underline{SET}$

In [5], on the basis of the definitions from [8, 11], the concept of a Modal Topological Structure (MTS) is introduced and some structures for the case of an intuitionistic fuzzy sets (see, [2]) are described. Here, we will use the definitions from [5], but over the special universe $\underline{SET}$.

Following and modifiying the basic definition from [5], we will give the following definition.

Definition 1. A cl-MTS is the object

$$\langle X, \mathcal{O}, \circ, \bullet, e_\circ \rangle,$$

for which:

– X is a set of natural numbers;

– $\mathcal{O} : X \to X$ is a topological operator of type "closure" (in sense of, e.g., [11]);

– $\circ : X \times X \to X$ is an operation for which exists another operation $*$ such that for every two $l, m \in X$:

$$l * m = \neg(\neg l \circ \neg m),$$
$$l \circ m = \neg(\neg l * \neg m);$$

– $\bullet : X \to X$ is a modal operator of type "possibility" (in sense of, e.g., [9]);

– $e_\circ$ is the unit element of X about operation $\circ$ and e_* – about operation $*$;

– the following nine conditions hold:

C1 $\mathcal{O}(l \circ m) = \mathcal{O}(l) \circ \mathcal{O}(m)$,

C2 $l \le \mathcal{O}(l)$,

C3 $\mathcal{O}(e_\circ) = e_\circ$,

C4 $\mathcal{O}(\mathcal{O}(l)) = \mathcal{O}(l)$,

C5 $\bullet(l * m) = \bullet l * \bullet m$,

C6 $l \le \bullet l$,

C7 $\bullet e_* = e_*$,

C8 $\bullet \bullet l = \bullet l$,

C9 $\bullet\, \mathcal{O}(l) = \mathcal{O}(\bullet l)$.

The name cl-MTS is given, because the topological operator is of closure type.

Theorem 3. $\langle \underline{SET}(n), \mathcal{C}, (.), \diamondsuit, \boxplus \rangle$ *is a* cl*-MTS.*

Proof: Let $l, m \in \underline{SET}(n)$. We check sequentially the conditions C1–C9.

$$\begin{aligned}
\text{C1:}\quad \mathcal{C}((l,m)) &= \mathcal{C}(\prod_{i=1}^{k} p_i^{\min(\beta_i,\gamma_i)}) \\
&= \prod_{i=1}^{k} p_i^{\max(\lceil\nabla(n)\rceil,\min(\beta_i,\gamma_i))} \\
&= \prod_{i=1}^{k} p_i^{\min(\max(\lceil\nabla(n)\rceil,\beta_i),\max(\max(\lceil\nabla(n)\rceil,\gamma_i))} \\
&= \left(\prod_{i=1}^{k} p_i^{\max(\lceil\nabla(n)\rceil,\beta_i)}, \prod_{i=1}^{k} p_i^{\max(\lceil\nabla(n)\rceil,\gamma_i)}\right) \\
&= (\mathcal{C}(l), \mathcal{C}(m));
\end{aligned}$$

$$\begin{aligned}
\text{C2:}\quad \mathcal{C}(l) &= \prod_{i=1}^{k} p_i^{\max(\lceil\nabla(n)\rceil,\beta_i)} \\
&\ge \prod_{i=1}^{k} p_i^{\beta_i} = l;
\end{aligned}$$

$$\begin{aligned}
\text{C3:}\quad \mathcal{C}(\boxplus(l)) &= \mathcal{C}((\underline{mult}(n))^{\Delta(l)}) \\
&= (\underline{mult}(n))^{\max(\lceil\nabla(n)\rceil,\Delta(l))} \\
&= (\underline{mult}(n))^{\Delta(l)} = \boxplus(l);
\end{aligned}$$

$$
\begin{aligned}
\text{C4:}\quad \mathcal{C}(\mathcal{C}(l)) &= \mathcal{C}\left(\prod_{i=1}^{k} p_i^{\max(\lceil\nabla(n)\rceil,\beta_i)}\right)\\
&= \prod_{i=1}^{k} p_i^{\max(\lceil\nabla(n)\rceil,(\max(\lceil\nabla(n)\rceil,\beta_i))}\\
&= \prod_{i=1}^{k} p_i^{\max(\lceil\nabla(n)\rceil,\beta_i)} = \mathcal{C}(l)
\end{aligned}
$$

$$
\begin{aligned}
\text{C5:}\quad \diamondsuit([l,m]) &= \diamondsuit\left(\prod_{i=1}^{k} p_i^{\max(\beta_i,\gamma_i)}\right)\\
&= (\underline{mult}(n))^{\max\limits_{1\le i\le k}(\beta_i,\gamma_i)}\\
&= \left[(\underline{mult}(n))^{\max\limits_{1\le i\le k}\beta_i}, (\underline{mult}(n))^{\max\limits_{1\le i\le k}\gamma_i}\right]\\
&= [\,\boxast\, l,\ \boxast\, m];
\end{aligned}
$$

$$
\begin{aligned}
\text{C6:}\quad \diamondsuit l &= \diamondsuit \prod_{i=1}^{k} p_i^{\beta_i}\\
&= (\underline{mult}(n))^{\max\limits_{1\le i\le k}\beta_i}\\
&\geq \prod_{i=1}^{k} p_i^{\beta_i} = l;
\end{aligned}
$$

$$
\begin{aligned}
\text{C7:}\quad \diamondsuit \boxcircle &= \diamondsuit(\underline{mult}(n))^{\delta(n)}\\
&= (\underline{mult}(n))^{\delta(n)} = \boxcircle\,;
\end{aligned}
$$

$$
\begin{aligned}
\text{C8:}\quad \diamondsuit\diamondsuit l &= \diamondsuit\,(\underline{mult}(n))^{\max\limits_{1\le i\le k}\beta_i}\\
&= (\underline{mult}(n))^{\Delta(l)} = \diamondsuit l;
\end{aligned}
$$

$$
\begin{aligned}
\text{C9:}\quad \diamondsuit\mathcal{C}(l) &= \diamondsuit \prod_{i=1}^{k} p_i^{\max(\lceil\nabla(n)\rceil,\beta_i)}\\
&= \prod_{i=1}^{k} p_i^{\max\limits_{1\le i\le k}(\max(\lceil\nabla(n)\rceil,\beta_i))}\\
&= \prod_{i=1}^{k} p_i^{\max(\lceil\nabla(n)\rceil,\max\limits_{1\le i\le k}\beta_i)}\\
&= \prod_{i=1}^{k} p_i^{\max(\lceil\nabla(n)\rceil,\Delta(l))}\\
&= \mathcal{C}\left(\prod_{i=1}^{k} p_i^{\Delta(l))}\right) = \mathcal{C}(\diamondsuit l).
\end{aligned}
$$

Therefore, Theorem 3 is valid. $\square$

Definition 2. A in-MTS is the object

$$\langle X, \mathcal{Q}, *, \blacksquare\,, e_* \rangle,$$

for which:

- X is the set, $*$ is the operation, and e_* is the unit element of X about operation $*$, defined in Definition 1;
- $\mathcal{Q} : X \to X$ is a topological operator of type "interior" (in sense of, e.g., [11]);

- $\blacksquare : X \to X$ is a modal operator of type "necessity" (in sense of, e.g., [9]);
- the following nine conditions hold:

I1 $\mathcal{Q}(l * m) = \mathcal{Q}(l) * \mathcal{Q}(m)$,

I2 $l \geq \mathcal{Q}(l)$,

I3 $\mathcal{Q}(e_*) = e_*$,

I4 $\mathcal{Q}(\mathcal{Q}(l)) = \mathcal{Q}(l)$,

I5 $\blacksquare (l \circ m) = \blacksquare\, l \circ \blacksquare\, m$,

I6 $l \geq \blacksquare\, l$,

I7 $\blacksquare\, e_\circ = e_\circ$,

I8 $\blacksquare \blacksquare\, l = \blacksquare\, l$,

I9 $\blacksquare\, \mathcal{Q}(l) = \mathcal{Q}(\blacksquare\, l)$.

Theorem 4. $\langle \underline{SET}(n), \mathcal{I}, [.], \square, \boxcircle \rangle$ *is an* in*-MTS.*

Proof: Let $l, m \in \underline{SET}(n)$. We check sequentially the conditions I1–I9.

$$\begin{aligned}
\text{I1:}\quad \mathcal{I}([l,m]) &= \mathcal{I}(\prod_{i=1}^{k} p_i^{\max(\beta_i,\gamma_i)}) \\
&= \prod_{i=1}^{k} p_i^{\min(\lfloor \nabla(n) \rfloor, \max(\beta_i,\gamma_i))} \\
&= \prod_{i=1}^{k} p_i^{\max(\min(\lfloor \nabla(n) \rfloor,\beta_i),\min(\lfloor \nabla(n) \rfloor,\gamma_i))} \\
&= \left[\prod_{i=1}^{k} p_i^{\min(\lfloor \nabla(n) \rfloor,\beta_i)}, \prod_{i=1}^{k} p_i^{\min(\lfloor \nabla(n) \rfloor,\gamma_i)}\right] \\
&= [\mathcal{I}(l), \mathcal{I}(m)];
\end{aligned}$$

$$\begin{aligned}
\text{I2:}\quad \mathcal{I}(l) &= \prod_{i=1}^{k} p_i^{\min(\lfloor \nabla(n) \rfloor,\beta_i)} \\
&\leq \prod_{i=1}^{k} p_i^{\beta_i} = l;
\end{aligned}$$

$$\begin{aligned}
\text{I3:}\quad \mathcal{I}(\boxcircle(l)) &= \mathcal{I}((\underline{mult}(n))^{\delta(l)}) \\
&= (\underline{mult}(n))^{\min(\lfloor \nabla(n) \rfloor,\delta(l))} \\
&= (\underline{mult}(n))^{\delta(l)} = \boxcircle(l);
\end{aligned}$$

$$\begin{aligned}
\text{I4:}\quad \mathcal{I}(\mathcal{I}(l)) &= \mathcal{I}\left(\prod_{i=1}^{k} p_i^{\min(\lfloor \nabla(n) \rfloor,\beta_i)}\right) \\
&= \prod_{i=1}^{k} p_i^{\min(\lfloor \nabla(n) \rfloor,(\min(\lfloor \nabla(n) \rfloor,\beta_i))} \\
&= \prod_{i=1}^{k} p_i^{\min(\lfloor \nabla(n) \rfloor,\beta_i)} = \mathcal{I}(l)
\end{aligned}$$

$$
\begin{aligned}
\text{I5:}\quad \Box\,((l,m)) &= \Box\left(\prod_{i=1}^{k} p_i^{\min(\beta_i,\gamma_i)}\right)\\
&= (\underline{mult}(n))^{\min\limits_{1\le i\le k}(\beta_i,\gamma_i)}\\
&= \left((\underline{mult}(n))^{\min\limits_{1\le i\le k}\beta_i}, (\underline{mult}(n))^{\min\limits_{1\le i\le k}\gamma_i}\right)\\
&= (\Box\, l, \Box\, m);
\end{aligned}
$$

$$
\begin{aligned}
\text{I6:}\quad \Box\, l &= \Box\prod_{i=1}^{k} p_i^{\beta_i}\\
&= (\underline{mult}(n))^{\min\limits_{1\le i\le k}\beta_i}\\
&\le \prod_{i=1}^{k} p_i^{\beta_i} = l;
\end{aligned}
$$

$$
\begin{aligned}
\text{I7:}\quad \Box\,\boxdot &= \Box\,(\underline{mult}(n))^{\delta(n)}\\
&= (\underline{mult}(n))^{\delta(n)} = \boxdot\,;
\end{aligned}
$$

$$
\begin{aligned}
\text{I8:}\quad \Box\,\Box\, l &= \Box\,(\underline{mult}(n))^{\min\limits_{1\le i\le k}\beta_i}\\
&= (\underline{mult}(n))^{\delta(l)} = \Box\, l;
\end{aligned}
$$

$$
\begin{aligned}
\text{I9:}\quad \Box\,\mathcal{I}(l) &= \Box\prod_{i=1}^{k} p_i^{\min(\lfloor\nabla(n)\rfloor,\beta_i)}\\
&= \prod_{i=1}^{k} p_i^{\min\limits_{1\le i\le k}(\min(\lfloor\nabla(n)\rfloor,\beta_i))}\\
&= \prod_{i=1}^{k} p_i^{\min(\lfloor\nabla(n)\rfloor,\min\limits_{1\le i\le k}\beta_i)}\\
&= \prod_{i=1}^{k} p_i^{\min(\lfloor\nabla(n)\rfloor,\delta(l))}\\
&= \mathcal{I}\left(\prod_{i=1}^{k} p_i^{\delta(l))}\right) = \mathcal{I}(\Box\, l).
\end{aligned}
$$

Therefore, Theorem 4 is valid. $\Box$

4 Conclusion

In the present research, we show that for arbitrary natural number n, over set $\underline{SET}(n)$ we can define two different modal topological structures. They do not have analogues among the existing topological structures.

In the future, over the set $\underline{SET}$ other new topological structures will be defined. They will be based over the present ones and will be their modifications. So, the present topological structures are called standard ones.

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