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Hybrid hyper-Fibonacci and hyper-Lucas numbers

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Abstract: Different number systems have been studied lately. Recently, many researchers have considered the hybrid numbers which are generalization of the complex, hyperbolic and dual number systems. In this paper, we define the hybrid hyper-Fibonacci and hyper-Lucas numbers. Furthermore, we obtain some algebraic properties of these numbers such as the recurrence relations, the generating functions, the Binet's formulas, the summation formulas, the Catalan's identity, the Cassini's identity and the d'Ocagne's identity.
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1 Introduction

The sequences of integers have been investigated by many researchers. The most widely studied sequences of numbers are the Fibonacci and the Lucas sequences. The generalizations of Fibonacci and Lucas numbers have been considered in recent years. In this paper, we consider the hyper-Fibonacci and the hyper-Lucas numbers that were introduced by Mező and Dil in [10]. The hyper-Fibonacci sequences $F_n^{(r)}$ are defined by the following recurrence relation for $n \geq 0$ and $r \geq 1$,

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)}, \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \quad F_1^{(r)} = 1\ , \tag{1}$$

where F_n is the n-th term of the Fibonacci sequence. It is clear that taking $r = 0$ in (1), the Fibonacci numbers are obtained. Similarly, the hyper-Lucas sequences $L_n^{(r)}$ are defined for $n \geq 0$ and $r \geq 1$ as,

$$L_n^{(r)} = \sum_{k=0}^{n} L_k^{(r-1)}, \;\; L_n^{(0)} = L_n, \;\; L_0^{(r)} = 2, \;\; L_1^{(r)} = 2r + 1 \,, \tag{2}$$

where L_n is the n-th term of the Lucas sequence.

The generating functions of the hyper-Fibonaci numbers and hyper-Lucas numbers are, respectively,

$$\sum_{n=0}^{\infty} F_n^{(r)} t^n = \frac{t}{(1-t-t^2)(1-t)^r}, \tag{3}$$

$$\sum_{n=0}^{\infty} L_n^{(r)} t^n = \frac{2-t}{(1-t-t^2)(1-t)^r}, \tag{4}$$

which are given in [10]. Also, $F_n^{(r)}$ and $L_n^{(r)}$ satisfy the following identity,

$$L_n^{(r)} = F_{n-1}^{(r)} + F_{n+1}^{(r)} + \binom{n+r-1}{r-1}. \tag{5}$$

Many authors have defined new generalizations of the Fibonacci and the Lucas numbers. In [10], the authors defined the hyper-Fibonacci and the hyper-Lucas numbers and gave the generating functions of these numbers. Some combinatorial properties of the hyper-Fibonacci and the hyper-Lucas numbers were given by the authors in [4]. In [6], Cao and Zhao obtained some identities for the hyper-Fibonacci and the hyper-Lucas numbers. In [9], the authors gave basic identities of the hyper-Fibonacci sequences. Komatsu and Szalay studied a new formula for the hyper-Fibonacci numbers in [13]. In [2], the authors defined the $(a, b)-$hyper-Fibonacci numbers and obtained some identities for these numbers. Some combinatorial identities of a new generalization of the hyper-Lucas numbers are obtained in [3]. In [5], the authors considered r-circulant matrices with the hyper-Fibonacci and the hyper-Lucas numbers and calculated the spectral norms of these matrices.

There are most identities of the Fibonacci, the Lucas, the hyper-Fibonacci and the hyper-Lucas numbers. Now, we give several identities and for more information, please refer to [3, 6, 9, 10, 14, 27].

$$L_{n+2} + L_n = 5F_{n+1}, \tag{6}$$

$$L_{r+s} - (-1)^s L_{r-s} = 5F_r F_s, \tag{7}$$

$$L_{r+s} + (-1)^s L_{r-s} = L_r L_s, \tag{8}$$

$$F_{r+s} + (-1)^s F_{r-s} = F_r L_s, \tag{9}$$

$$F_{r+s} - (-1)^s F_{r-s} = F_s L_r, \tag{10}$$

$$L_{2k} - 2(-1)^k = 5F_k^2, \tag{11}$$

$$\sum_{i=0}^{n} F_{ki+j} = \begin{cases} \frac{F_{nk+k+j}-(-1)^k F_{nk+j}-F_j-(-1)^j F_{k-j}}{L_k-(-1)^k-1}, & j<k \\ \frac{F_{nk+k+j}-(-1)^k F_{nk+j}-F_j+(-1)^k F_{j-k}}{L_k-(-1)^k-1}, & \text{otherwise} \end{cases}, \quad k \geq 1, \tag{12}$$

$$\sum_{i=0}^{n} L_{ki+j} = \begin{cases} \frac{L_{nk+k+j}-(-1)^k L_{nk+j}-L_j+(-1)^j L_{k-j}}{L_k-(-1)^k-1}, & j<k \\ \frac{L_{nk+k+j}-(-1)^k L_{nk+j}-L_j+(-1)^k L_{j-k}}{L_k-(-1)^k-1}, & \text{otherwise} \end{cases}, \quad k \geq 1, \tag{13}$$

$$F_n^{(r)} = F_{n-1}^{(r)} + F_n^{(r-1)}, \tag{14}$$

$$F_{n+2}^{(r)} = F_{n+1}^{(r)} + F_n^{(r)} + \binom{n+r}{r-1}, \tag{15}$$

$$F_n^{(r)} = F_{n+2r} - \sum_{k=0}^{r-1} \binom{n+r+k}{r-1-k}, \tag{16}$$

$$\sum_{s=0}^{r} F_n^{(s)} = F_{n+1}^{(r)} - F_{n-1}, \tag{17}$$

$$L_n^{(r)} = L_{n-1}^{(r)} + L_n^{(r-1)}, \tag{18}$$

$$L_n^{(r)} = L_{n+2r} + \binom{n+r-1}{r-1} - \sum_{k=0}^{r-1} \left(\binom{n+r+k-1}{r-1-k} + \binom{n+r+k+1}{r-1-k} \right), \tag{19}$$

$$\sum_{s=0}^{r} L_n^{(s)} = L_{n+1}^{(r)} - L_{n-1}. \tag{20}$$

A few values of the hyper-Fibonacci and the hyper-Lucas numbers are given in the following Table 1.

Table 1. A few values of the hyper-Fibonacci and the hyper-Lucas numbers.

n	0	1	2	3	4	5	6	7	8	9	...
$F_n^{(0)}$	0	1	1	2	3	5	8	13	21	34	...
$L_n^{(0)}$	2	1	3	4	7	11	18	29	47	76	...
$F_n^{(1)}$	0	1	2	4	7	12	20	33	54	88	...
$L_n^{(1)}$	2	3	6	10	17	28	46	75	122	198	...
$F_n^{(2)}$	0	1	3	7	14	26	46	79	133	221	...
$L_n^{(2)}$	2	5	11	21	38	66	112	187	309	507	...
$F_n^{(3)}$	0	1	4	11	25	51	97	176	309	530	...
$L_n^{(3)}$	2	7	18	39	77	143	255	442	751	1258	...

From the last century, two dimensional number systems have been taken into consideration by many researchers. Especially complex, hyperbolic and dual numbers are among the most studied topics. In [17], the author introduced the new non-commutative number system that contains of all three number system together. This number system called hybrid numbers and denoted by $\mathbb{K}$. The set of new hybrid numbers is defined as follows:

$$\mathbb{K}=\left\{Z=z_1+z_2\mathbf{i}+z_3\epsilon+z_4\mathbf{h}: z_1,z_2,z_3,z_4\in\mathbb{R}, \mathbf{i}^2=-1, \epsilon^2=0, \mathbf{h}^2=1, \mathbf{ih}=-\mathbf{hi}=\epsilon+\mathbf{i}\right\}.$$

Two hybrid numbers are equal, if all their components are equal one by one. The sum of two hybrid numbers is defined by summing their components. Addition operation in $\mathbb{K}$ is both commutative and associative. $0=0+0\mathbf{i}+0\epsilon+0\mathbf{h}\in\mathbb{K}$ is the additive identity element. The inverse element of Z is $-Z$. Therefore, $(\mathbb{K},+)$ is an Abelian group. The hybridian product is obtained by distributing the terms to right, preserving the order of multiplication of units and then writing the values of the following replacing each product of units by equalities $\mathbf{i}^2=-1$, $\epsilon^2=0$, $\mathbf{h}^2=1$, $\mathbf{ih}=-\mathbf{hi}=\epsilon+\mathbf{i}$. The multiplication table of units of hybrid numbers are in the following Table 2.

Table 2. The multiplication table of units of hybrid numbers.

$\times$	$\mathbf{1}$	$\mathbf{i}$	ϵ	$\mathbf{h}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	ϵ	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{1}-\mathbf{h}$	$\epsilon+\mathbf{i}$
ϵ	ϵ	$\mathbf{h}+\mathbf{1}$	0	$-\epsilon$
$\mathbf{h}$	$\mathbf{h}$	$-\epsilon-\mathbf{i}$	ϵ	$\mathbf{1}$

The above table shows us that the multiplication operation of the hybrid numbers is not commutative, but it has the property of associativity. $1=1+0\mathbf{i}+0\epsilon+0\mathbf{h}\in\mathbb{K}$ is the multiplicative identity element. The set of the hybrid numbers form a non-commutative ring with identity.

In recent years, many authors have investigated the special number sequences with different number systems. In [19,21,22], the authors defined hybrid numbers whose coefficients of basis are the Fibonacci, the Pell, the Pell–Lucas, the Jacobsthal and the Jacobsthal–Lucas numbers. Also, they obtained some identities of these numbers. The authors presented a new hybrid numbers with the Leonardo numbers and gave some identities of the hybrid Leonardo numbers in [1]. In [23], the authors introduced the generalized Mersenne hybrid numbers. A special kind of spacelike hybrid number which is $F(p,n)-$Fibonacci hybrid number was defined and gave some of their properties in [24]. The authors considered the hybrid numbers with the Padovan numbers and obtained some identities of these numbers in [16]. Tasci and Sevgi defined the Mersenne hybrid numbers and obtained some relations among the Mersenne, the Jacobsthal and the Jacobsthal–Lucas hybrid numbers in [26]. In [8], the author considered the hybrid numbers with generalization of the Fibonacci numbers and acquired some results. The bi-periodic Horadam hybrid number was defined in [25]. In [11], Kizilates introduced the other new generalization of the hybrid number which called the $q-$Fibonacci hybrid numbers and the $q-$Lucas hybrid numbers and obtained some important algebraic properties of these numbers. Other generalization of the hybrid numbers with special number sequences was given by Szynal in [18]. In [15,20], the authors introduced the hybrinomials via the Fibonacci, the Lucas and the Pell polynomials and obtained interesting results of these polynomials. Cerda-Morales defined and studied the third-order Jacobsthal and modified third-order Jacobsthal hybrinomials in [7]. In [12], the author defined the new generalization of hybrinomial whose coefficients of basis are the Horadam polynomials and acquired some results of the Horadam hybrinomials.

Inspired by all above papers, we introduce a new hybrid numbers with the hyper-Fibonacci and the hyper-Lucas numbers. In this paper, we obtain the reccurrence relation, the Binet's formula of these numbers. Also, we give the generating function for these numbers. In addition, the summation formula, the Cassini's identity, the Catalan's identity and the d'Ocagne's identity for the hybrid hyper-Fibonacci and the hyper-Lucas numbers are given. The hyper-Fibonacci and the hyper-Lucas numbers are generalization of the Fibonacci and the Lucas numbers. So, all results obtained in this study can be reduced to the hybrid Fibonacci and the hybrid Lucas sequences.

2 Hybrid hyper-Fibonacci numbers

In this section, we introduce the hybrid hyper-Fibonacci numbers. We obtain the generating function, the Binet's formula, the summation formulas, the Catalan's identity, the Cassini's identity, the d'Ocagne's identity and other identities.

Definition 2.1. *For $n \geq 1$, the recursive definition of the n-th hybrid hyper-Fibonacci numbers $HF_n^{(r)}$ are as follows:*

$$HF_n^{(r)} = F_n^{(r)} + \mathbf{i}F_{n+1}^{(r)} + \epsilon F_{n+2}^{(r)} + \mathbf{h}F_{n+3}^{(r)}. \tag{21}$$

From the recurrence relation (21) and (16), we have,

$$\begin{aligned} HF_n^{(r)} = F_{n+2r} - \sum_{k=0}^{r-1}\binom{n+r+k}{r-1-k} + \mathbf{i}\left(F_{n+1+2r} - \sum_{k=0}^{r-1}\binom{n+r+k+1}{r-1-k}\right) \\ +\epsilon\left(F_{n+2+2r} - \sum_{k=0}^{r-1}\binom{n+r+k+2}{r-1-k}\right) + \mathbf{h}\left(F_{n+3+2r} - \sum_{k=0}^{r-1}\binom{n+r+k+3}{r-1-k}\right). \end{aligned}$$

Hence

$$HF_n^{(r)} = HF_{n+2r} - HA_n^{(r)}, \tag{22}$$

where

$$HA_n^{(r)} = \sum_{k=0}^{r-1}\left(\binom{n+r+k}{r-1-k} + \mathbf{i}\binom{n+r+k+1}{r-1-k} + \epsilon\binom{n+r+k+2}{r-1-k} + \mathbf{h}\binom{n+r+k+3}{r-1-k}\right). \tag{23}$$

Taking $r = 0$ in (23), we find $HA_n^{(0)} = 0$. Also, considering (21) and (14), we can obtain as follows

$$\begin{aligned} HF_n^{(r)} &= F_n^{(r)} + \mathbf{i}F_{n+1}^{(r)} + \epsilon F_{n+2}^{(r)} + \mathbf{h}F_{n+3}^{(r)} \\ &= \left(F_{n-1}^{(r)} + F_n^{(r-1)}\right) + \mathbf{i}\left(F_n^{(r)} + F_{n+1}^{(r-1)}\right) + \epsilon\left(F_{n+1}^{(r)} + F_{n+2}^{(r-1)}\right) + \mathbf{h}\left(F_{n+2}^{(r)} + F_{n+3}^{(r-1)}\right) \\ &= HF_{n-1}^{(r)} + HF_n^{(r-1)}. \end{aligned}$$

The generating function for the hybrid hyper-Fibonacci numbers are obtained in the following theorem.

Theorem 2.1. *The generating function for the hybrid hyper-Fibonacci numbers $HF_n^{(r)}$ is given by*

$$g\left(t\right)=\frac{t^{3}+\mathbf{i}t^{2}+\epsilon t+\mathbf{h}}{t^{2}\left(1-t-t^{2}\right)\left(1-t\right)^{r}}-\frac{\epsilon t+\mathbf{h}\left(1+rt+t\right)}{t^{2}}. \tag{24}$$

Proof. We begin with the formal power series representation of the generating function for $\left\{HF_n^{(r)}\right\}_{n=0}^{\infty}$,

$$g\left(t\right)=\sum_{n=0}^{\infty}HF_n^{(r)}t^{n}.$$

That is

$$g\left(t\right)=\sum_{n=0}^{\infty}\left(F_n^{(r)}+\mathbf{i}F_{n+1}^{(r)}+\epsilon F_{n+2}^{(r)}+\mathbf{h}F_{n+3}^{(r)}\right)t^{n}.$$

Hence

$$g\left(t\right)=-\epsilon\frac{1}{t}-\mathbf{h}\left(\frac{1}{t^{2}}+\frac{r+1}{t}\right)+\left(1+\mathbf{i}\frac{1}{t}+\epsilon\frac{1}{t^{2}}+\mathbf{h}\frac{1}{t^{3}}\right)\sum_{n=0}^{\infty}\left(F_n^{(r)}t^{n}\right).$$

Using (3), the result is obtained. □

The next theorem gives the Binet's formula for the hybrid hyper-Fibonacci numbers.

Theorem 2.2. *For the any integer $n\geq 0$, the n-th hybrid hyper-Fibonacci numbers is*

$$HF_n^{(r)}=\frac{\underline{\alpha}\alpha^{n+2r}-\underline{\beta}\beta^{n+2r}}{\alpha-\beta}-HA_n^{(r)}, \tag{25}$$

where α and β are the roots of the characteristic equation of the Fibonacci sequence, $\underline{\alpha}=1+\mathbf{i}\alpha+\epsilon\alpha^{2}+\mathbf{h}\,\alpha^{3}$and $\underline{\beta}=1+\mathbf{i}\beta+\epsilon\beta^{2}+\mathbf{h}\beta^{3}$.

Proof. Binet's formula of the hybrid Fibonacci numbers in [22] is

$$HF_n=\frac{\underline{\alpha}\alpha^{n}-\underline{\beta}\beta^{n}}{\alpha-\beta},$$

where HF_n is the n-th hybrid Fibonacci number. Considering the identity between the hybrid Fibonacci and the hybrid hyper-Fibonacci numbers (22), the result is obtained. □

We give some results concerning sums of terms of the hybrid hyper-Fibonacci sequence by using some identities of the hyper-Fibonacci sequences.

Theorem 2.3. *For $n\geq 0$, the summation formulas of the hybrid hyper-Fibonacci numbers are*

$$\begin{aligned}
\sum_{k=0}^{n}HF_k^{(r)}&=HF_n^{(r+1)}-\epsilon-\mathbf{h}\left(2+r\right),\\
\sum_{s=0}^{r}HF_n^{(s)}&=HF_{n+1}^{(r)}-HF_{n-1},\\
\sum_{k=0}^{n}HF_{2k}^{(r)}&=HF_{2n+2r+1}-HF_{2r-1}-\sum_{k=0}^{n}HA_{2k}^{(r)},\quad r\geq 1,\\
\sum_{k=0}^{n}HF_{2k+1}^{(r)}&=HF_{2n+2r+2}-HF_{2r}-\sum_{k=0}^{n}HA_{2k+1}^{(r)},\quad r\geq 1.
\end{aligned}$$

Proof. Let us prove the first summation formula. Using the reccurrence relation of the hybrid hyper-Fibonacci numbers, we have

$$\sum_{k=0}^{n} HF_k^{(r)} = \sum_{k=0}^{n} F_k^{(r)} + \mathbf{i}\sum_{k=0}^{n} F_{k+1}^{(r)} + \epsilon\sum_{k=0}^{n} F_{k+2}^{(r)} + \mathbf{h}\sum_{k=0}^{n} F_{k+3}^{(r)}.$$

Considering (1) and (21), we obtain

$$\begin{aligned}\sum_{k=0}^{n} HF_k^{(r)} &= F_n^{(r+1)} + \mathbf{i}F_{n+1}^{(r+1)} + \epsilon\left(F_{n+2}^{(r+1)} - 1\right) + \mathbf{h}\left(F_{n+3}^{(r+1)} - 2 - r\right)\\ &= HF_n^{(r+1)} - \epsilon - \mathbf{h}\left(2 + r\right).\end{aligned}$$

Similarly we can obtain the other summation formulas for the hybrid hyper-Fibonacci numbers using (12), (16), (17), (21), (22) and (23). □

Theorem 2.4. *For positive integers $n,\ m$ and s with $m \geq s$, we have*

$$\begin{aligned}HF_{n+m}^{(r)}HF_{n+s}^{(r)} - HF_n^{(r)}HF_{n+m+s}^{(r)} &= HF_n^{(r)}HA_{n+m+s}^{(r)} - HA_{n+m}^{(r)}HF_{n+s}^{(r)}\\ &\quad -HF_{n+m+2r}HA_{n+s}^{(r)} + HA_n^{(r)}HF_{n+m+s+2r} \qquad (26)\\ &\quad +(-1)^n F_m\left(F_s\left(\mathbf{i}+3\epsilon+\mathbf{4h}\right) - L_s\left(\mathbf{i}+2\epsilon-\mathbf{h}\right)\right),\end{aligned}$$

where F_n, L_n and HF_n are the n-th Fibonacci number, the n-th Lucas number and the n-th hybrid Fibonacci number, respectively.

Proof. Using the Binet's formula (25) to left hand side (LHS), we obtain

$$\begin{aligned}LHS &= \left(\frac{\underline{\alpha}\alpha^{n+m+2r} - \underline{\beta}\beta^{n+m+2r}}{\alpha-\beta} - HA_{n+m}^{(r)}\right)\left(\frac{\underline{\alpha}\alpha^{n+s+2r} - \underline{\beta}\beta^{n+s+2r}}{\alpha-\beta} - HA_{n+s}^{(r)}\right)\\ &\quad -\left(\frac{\underline{\alpha}\alpha^{n+2r} - \underline{\beta}\beta^{n+2r}}{\alpha-\beta} - HA_n^{(r)}\right)\left(\frac{\underline{\alpha}\alpha^{n+m+s+2r} - \underline{\beta}\beta^{n+m+s+2r}}{\alpha-\beta} - HA_{n+m+s}^{(r)}\right).\end{aligned}$$

Using following equations,

$$\underline{\alpha}\underline{\beta} = \mathbf{i}\left(1+\sqrt{5}\right) + \epsilon\left(3+2\sqrt{5}\right) + \mathbf{h}\left(\mathbf{4}-\sqrt{5}\right), \qquad (27)$$

$$\underline{\beta}\underline{\alpha} = \mathbf{i}\left(1-\sqrt{5}\right) + \epsilon\left(3-2\sqrt{5}\right) + \mathbf{h}\left(\mathbf{4}+\sqrt{5}\right). \qquad (28)$$

Also, using (6), we get

$$\begin{aligned}LHS &= HF_{n+2r}HA_{n+m+s}^{(r)} + HA_n^{(r)}HF_{n+m+s+2r} - HA_n^{(r)}HA_{n+m+s}^{(r)}\\ &\quad -HF_{n+m+2r}HA_{n+s}^{(r)} - HA_{n+m}^{(r)}HF_{n+s+2r} + HA_{n+m}^{(r)}HA_{n+s}^{(r)}\\ &\quad +(-1)^{n+s+1}\left(\frac{1}{5}L_{m-s}\left(\mathbf{i}+3\epsilon+\mathbf{4h}\right) + F_{m-s}\left(\mathbf{i}+2\epsilon-\mathbf{h}\right)\right)\\ &\quad +(-1)^n\left(\frac{1}{5}L_{m+s}\left(\mathbf{i}+3\epsilon+\mathbf{4h}\right) - F_{m+s}\left(\mathbf{i}+2\epsilon-\mathbf{h}\right)\right).\end{aligned}$$

If we consider (7), (9) and (22) the result is obtained. □

Taking $m=-k$ and $s=k$ in (26), it can be obtained the Catalan's identity of the hybrid hyper-Fibonacci numbers as following corollary.

Corollary 2.1 (Catalan's identity)**.** *For positive integers n and k, with $n \geq k$, the following identity is true:*

$$\begin{aligned} HF_{n-k}^{(r)}HF_{n+k}^{(r)} - \left(HF_n^{(r)}\right)^2 = & HF_n^{(r)}HA_n^{(r)} - HA_{n-k}^{(r)}HF_{n+k}^{(r)} \\ & - HF_{n-k+2r}HA_{n+k}^{(r)} + HA_n^{(r)}HF_{n+2r} \\ & - (-1)^{n-k}\left(F_k^2(\mathbf{i}+\mathbf{3}\epsilon+\mathbf{4h}) - F_{2k}(\mathbf{i}+\mathbf{2}\epsilon-\mathbf{h})\right). \end{aligned} \tag{29}$$

Taking $k=1$ in (29), the Cassini's identity of the hybrid hyper-Fibonacci numbers is obtained.

Corollary 2.2 (Cassini's identity)**.** *For $n \geq 1$, the following equality holds:*

$$\begin{aligned} HF_{n-1}^{(r)}HF_{n+1}^{(r)} - \left(HF_n^{(r)}\right)^2 = & HF_n^{(r)}HA_n^{(r)} - HA_{n-1}^{(r)}HF_{n+1}^{(r)} \\ & - HF_{n-1+2r}HA_{n+1}^{(r)} + HA_n^{(r)}HF_{n+2r} \\ & - (-1)^{n-1}(\epsilon+\mathbf{5h}). \end{aligned} \tag{30}$$

Theorem 2.5 (d'Ocagne's identity)**.** *For positive integers n and m with $n \geq m$, the following equality is true*

$$\begin{aligned} HF_n^{(r)}HF_{m+1}^{(r)} - HF_{n+1}^{(r)}HF_m^{(r)} = & HF_{n+1}^{(r)}HA_m^{(r)} - HF_{n+2r}HA_{m+1}^{(r)} \\ & - HA_n^{(r)}HF_{m+1}^{(r)} + HA_{n+1}^{(r)}HF_{m+2r} \\ & + (-1)^m\left(F_{n-m}(\mathbf{i}+3\epsilon+\mathbf{4h}) + L_{n-m}(\mathbf{i}+2\epsilon-\mathbf{h})\right). \end{aligned} \tag{31}$$

Proof. Using the Binet's formula (25) to left hand side (LHS), we have

$$\begin{aligned} LHS = & \left(\frac{\underline{\alpha}\alpha^{n+2r} - \underline{\beta}\beta^{n+2r}}{\alpha-\beta} - HA_n^{(r)}\right)\left(\frac{\underline{\alpha}\alpha^{m+1+2r} - \underline{\beta}\beta^{m+1+2r}}{\alpha-\beta} - HA_{m+1}^{(r)}\right) \\ & - \left(\frac{\underline{\alpha}\alpha^{n+1+2r} - \underline{\beta}\beta^{n+1+2r}}{\alpha-\beta} - HA_{n+1}^{(r)}\right)\left(\frac{\underline{\alpha}\alpha^{m+2r} - \underline{\beta}\beta^{m+2r}}{\alpha-\beta} - HA_m^{(r)}\right). \end{aligned}$$

Using (27), (28) and (6), we get

$$\begin{aligned} LHS = & \left(HF_{n+1+2r} - HA_{n+1}^{(r)}\right)HA_m^{(r)} + HA_{n+1}^{(r)}HF_{m+2r} \\ & - HF_{n+2r}HA_{m+1}^{(r)} + HA_n^{(r)}\left(HA_{m+1}^{(r)} - HF_{m+1+2r}\right) \\ & + (-1)^m\left(F_{n-m}(\mathbf{i}+3\epsilon+\mathbf{4h}) + L_{n-m}(\mathbf{i}+2\epsilon-\mathbf{h})\right). \end{aligned}$$

In the last step, considering (22), the result is obtained. □

Taking $r=0$ in (29), (30) and (31), we have the Catalan' s, the Cassini's and the d'Ocagne's identity for the hybrid Fibonacci numbers, respectively. This conclusions are seen in [22].

Theorem 2.6. *For positive integers n and m with $n \geq m$, the following equalities hold:*

$$HF_{n+m}^{(r)} + (-1)^m HF_{n-m}^{(r)} = L_m HF_{n+2r} - HA_{n+m}^{(r)} - (-1)^m HA_{n-m}^{(r)}, \tag{32}$$

$$HF_{n+m}^{(r)} - (-1)^m HF_{n-m}^{(r)} = F_m HL_{n+2r} - HA_{n+m}^{(r)} + (-1)^m HA_{n-m}^{(r)}, \tag{33}$$

where F_n, L_n, HF_n and HL_n are the n-th Fibonacci number, the n-th Lucas number, the n-th hybrid Fibonacci number and the n-th hybrid Lucas number, respectively.

Proof. For the proof of (32), using (16), we have

$$LHS = \left(F_{n+m+2r} - \sum_{k=0}^{r-1}\binom{n+m+r+k}{r-1-k}\right) + \mathbf{i}\left(F_{n+m+2r+1} - \sum_{k=0}^{r-1}\binom{n+m+r+k+1}{r-1-k}\right)$$
$$+\,\epsilon\left(F_{n+m+2r+2} - \sum_{k=0}^{r-1}\binom{n+m+r+k+2}{r-1-k}\right) + \mathbf{h}\left(F_{n+m+2r+3} - \sum_{k=0}^{r-1}\binom{n+m+r+k+3}{r-1-k}\right)$$
$$+\,(-1)^m\left(\left(F_{n-m+2r} - \sum_{k=0}^{r-1}\binom{n-m+r+k}{r-1-k}\right) + \mathbf{i}\left(F_{n-m+2r+1} - \sum_{k=0}^{r-1}\binom{n-m+r+k+1}{r-1-k}\right)\right)$$
$$+\,(-1)^m\left(\epsilon\left(F_{n-m+2r+2} - \sum_{k=0}^{r-1}\binom{n-m+r+k+2}{r-1-k}\right) + \mathbf{h}\left(F_{n-m+2r+3} - \sum_{k=0}^{r-1}\binom{n-m+r+k+3}{r-1-k}\right)\right).$$

Using (9) and (23), we obtain

$$HF_{n+m}^{(r)} + (-1)^m HF_{n-m}^{(r)} = L_m HF_{n+2r} - HA_{n+m}^{(r)} - (-1)^m HA_{n-m}^{(r)}.$$

The proof of (33) is similar. □

Remark 2.1. *Taking $r = 0$ in (32) and (33), we obtain the following identities between the hybrid Fibonacci numbers:*

$$HF_{n+m} + (-1)^m HF_{n-m} = L_m HF_n,$$
$$HF_{n+m} - (-1)^m HF_{n-m} = F_m HL_n.$$

Theorem 2.7. *For positive integers n and m with $n \geq 1, m \geq 1$ and $n \geq m$, we have*

$$\begin{aligned} HF_{m+1}^{(r)}HF_{n+1}^{(r)} - HF_{m-1}^{(r)}HF_{n-1}^{(r)} = {} & HF_{m-1+2r}HA_{n-1}^{(r)} - HA_{m+1}^{(r)}HF_{n+1+2r} \\ & - HF_{m+1}^{(r)}HA_{n+1}^{(r)} + HA_{m-1}^{(r)}HF_{n-1}^{(r)} \\ & + L_{m+n+5+4r} - 2F_{m+n+4r} + 2HF_{m+n+4r}, \end{aligned} \tag{34}$$

where F_n, L_n and HF_n are the n-th Fibonacci number, the n-th Lucas number and the n-th hybrid Fibonacci number, respectively.

Proof. If we use Binet's formula of hybrid hyper-Fibonacci numbers to (LHS), we obtain

$$LHS = \left(\frac{\underline{\alpha}\alpha^{m+1+2r} - \underline{\beta}\beta^{m+1+2r}}{\alpha-\beta} - HA_{m+1}^{(r)}\right)\left(\frac{\underline{\alpha}\alpha^{n+1+2r} - \underline{\beta}\beta^{n+1+2r}}{\alpha-\beta} - HA_{n+1}^{(r)}\right)$$
$$-\left(\frac{\underline{\alpha}\alpha^{m-1+2r} - \underline{\beta}\beta^{m-1+2r}}{\alpha-\beta} - HA_{m-1}^{(r)}\right)\left(\frac{\underline{\alpha}\alpha^{n-1+2r} - \underline{\beta}\beta^{n-1+2r}}{\alpha-\beta} - HA_{n-1}^{(r)}\right).$$

Using (7) and following equations,

$$\underline{\alpha}^2 = 1 - \alpha^2 + 2\alpha^3 + \alpha^6 + 2\alpha\left(\mathbf{i} + \epsilon\alpha + \mathbf{h}\alpha^2\right), \tag{35}$$
$$\underline{\beta}^2 = 1 - \beta^2 + 2\beta^3 + \beta^6 + 2\beta\left(\mathbf{i} + \epsilon\beta + \mathbf{h}\beta^2\right), \tag{36}$$

we have

$$\begin{aligned} LHS = {} & \left(HA_{m+1}^{(r)} - HF_{m+1+2r}\right)HA_{n+1}^{(r)} + HF_{m-1+2r}HA_{n-1}^{(r)} \\ & + HA_{m-1}^{(r)}\left(HF_{n-1+2r} - HA_{n-1}^{(r)}\right) - HA_{m+1}^{(r)}HF_{n+1+2r} \\ & + F_{m+n+7+4r} - F_{m+n+3+4r} + 2\mathbf{i}F_{m+n+1+4r} + 2\epsilon F_{m+n+2+4r} + 2\mathbf{h}F_{m+n+3+4r}. \end{aligned}$$

Afterwards, considering (10) and (22), the result is obtained. □

Remark 2.2. *For $r = 0$ in above theorem, we obtain the following identities between the hybrid Fibonacci numbers:*

$$HF_{m+1}HF_{n+1} - HF_{m-1}HF_{n-1} = L_{m+n+5} + 2\mathbf{i}F_{m+n+1} + 2\epsilon F_{m+n+2} + 2\mathbf{h}F_{m+n+3}.$$

Theorem 2.8. *For positive integers m, k and s with $m \geq k$ and $m \geq s$, the following identity is true:*

$$\begin{aligned} HF_{m+k}^{(r)}HF_{m-k}^{(r)} - HF_{m+s}^{(r)}HF_{m-s}^{(r)} = {} & HF_{m+s+2r}HA_{m-s}^{(r)} - HA_{m+k}^{(r)}HF_{m-k+2r} \\ & -HF_{m+k}^{(r)}HA_{m-k}^{(r)} + HA_{m+s}^{(r)}HF_{m-s}^{(r)} \\ & + (\mathbf{i} + 3\epsilon + 4\mathbf{h}) \left((-1)^{m-s} F_s^2 - (-1)^{m-k} F_k^2 \right) \\ & + (\mathbf{i} + 2\epsilon - \mathbf{h}) \left((-1)^{m-s} F_{2s} - (-1)^{m-k} F_{2k} \right), \end{aligned} \tag{37}$$

where F_n and HF_n are the n-th Fibonacci number and the n-th hybrid Fibonacci number, respectively.

Proof. From the Binet's formula of the hybrid hyper-Fibonacci numbers to left hand side, we have

$$\begin{aligned} LHS = {} & \left(\frac{\underline{\alpha}\alpha^{m+k+2r} - \underline{\beta}\beta^{m+k+2r}}{\alpha - \beta} - HA_{m+k}^{(r)} \right) \left(\frac{\underline{\alpha}\alpha^{m-k+2r} - \underline{\beta}\beta^{m-k+2r}}{\alpha - \beta} - HA_{m-k}^{(r)} \right) \\ & - \left(\frac{\underline{\alpha}\alpha^{m+s+2r} - \underline{\beta}\beta^{m+s+2r}}{\alpha - \beta} - HA_{m+s}^{(r)} \right) \left(\frac{\underline{\alpha}\alpha^{m-s+2r} - \underline{\beta}\beta^{m-s+2r}}{\alpha - \beta} - HA_{m-s}^{(r)} \right). \end{aligned}$$

Hence

$$\begin{aligned} LHS = {} & \left(HA_{m+k}^{(r)} - HF_{m+k+2r} \right) HA_{m-k}^{(r)} - HA_{m+k}^{(r)}HF_{m-k+2r} \\ & +HA_{m+s}^{(r)} \left(HF_{m-s+2r} - HA_{m-s}^{(r)} \right) + HF_{m+s+2r}HA_{m-s}^{(r)} \\ & +\frac{1}{5} (\mathbf{i} + 3\epsilon + 4\mathbf{h}) \left((-1)^{m-s} L_{2s} - (-1)^{m-k} L_{2k} \right) \\ & + (\mathbf{i} + 2\epsilon - \mathbf{h}) \left((-1)^{m-s} F_{2s} - (-1)^{m-k} F_{2k} \right). \end{aligned}$$

From (22) and (11), the result is obtained. □

Remark 2.3. *We can give special cases of above theorem for $r = 0$,*

$$\begin{aligned} HF_{m+k}HF_{m-k} - HF_{m+s}HF_{m-s} = {} & (\mathbf{i} + 3\epsilon + 4\mathbf{h}) \left((-1)^{m-s} F_s^2 - (-1)^{m-k} F_k^2 \right) \\ & + (\mathbf{i} + 2\epsilon - \mathbf{h}) \left((-1)^{m-s} F_{2s} - (-1)^{m-k} F_{2k} \right). \end{aligned}$$

3 Hybrid hyper-Lucas numbers

In this part of the study, we define the hybrid hyper-Lucas numbers. Then, we give the generating function for the hybrid hyper-Lucas numbers. Finally, we obtain some important identities for these numbers.

Definition 3.1. *For $n \geq 1$, the n-th hybrid hyper-Lucas numbers $HL_n^{(r)}$ are defined by*

$$HL_n^{(r)} = L_n^{(r)} + \mathbf{i}L_{n+1}^{(r)} + \epsilon L_{n+2}^{(r)} + \mathbf{h}L_{n+3}^{(r)} \tag{38}$$

From (5) and (38), we have

$$\begin{aligned} HL_n^{(r)} &= F_{n-1}^{(r)} + F_{n+1}^{(r)} + \binom{n+r-1}{r-1} + \mathbf{i}\left(F_n^{(r)} + F_{n+2}^{(r)} + \binom{n+r}{r-1}\right) \\ &\quad + \epsilon\left(F_{n+1}^{(r)} + F_{n+3}^{(r)} + \binom{n+r+1}{r-1}\right) + \mathbf{h}\left(F_{n+2}^{(r)} + F_{n+4}^{(r)} + \binom{n+r+2}{r-1}\right). \end{aligned}$$

Using (5), (21) and (22), we get

$$HL_n^{(r)} = HL_{n+2r} - HK_n^{(r)} \tag{39}$$

where

$$HK_n^{(r)} = HA_{n-1}^{(r)} + HA_{n+1}^{(r)} - \left(\binom{n+r}{r-1} + \mathbf{i}\binom{n+r+1}{r-1} + \epsilon\binom{n+r+2}{r-1} + \mathbf{h}\binom{n+r+3}{r-1}\right). \tag{40}$$

Taking $r = 0$ in (40), we find $HK_n^{(0)} = 0$. Considering (18) and (38), the following expression is obtained

$$\begin{aligned} HL_n^{(r)} &= L_n^{(r)} + \mathbf{i}L_{n+1}^{(r)} + \epsilon L_{n+2}^{(r)} + \mathbf{h}L_{n+3}^{(r)} \\ &= \left(L_{n-1}^{(r)} + L_n^{(r-1)}\right) + \mathbf{i}\left(L_n^{(r)} + L_{n+1}^{(r-1)}\right) + \epsilon\left(L_{n+1}^{(r)} + L_{n+2}^{(r-1)}\right) + \mathbf{h}\left(L_{n+2}^{(r)} + L_{n+3}^{(r-1)}\right) \\ &= HL_{n-1}^{(r)} + HL_n^{(r-1)}. \end{aligned}$$

In the next theorem, we obtain the generating function for the hybrid hyper-Lucas numbers.

Theorem 3.1. *The generating function for the hybrid hyper-Lucas numbers $HL_n^{(r)}$ is*

$$\begin{aligned} g(t) = &\frac{(2-t)\left(t^3 + \mathbf{i}t^2 + \epsilon\mathbf{t} + \mathbf{h}\right)}{t^3\left(1-t-t^2\right)(1-t)^r} - \mathbf{i}\frac{2}{t} - \epsilon\left(\frac{2+t(2r+1)}{t^2}\right) \\ &- \mathbf{h}\left(\frac{2+t(2r+1)+t^2\left(r^2+2r+3\right)}{t^3}\right). \end{aligned}$$

Proof. We begin with the formal power series representation of the generating function for $\left\{HL_n^{(r)}\right\}_{n=0}^{\infty}$ by $g(t)$:

$$g(t) = \sum_{n=0}^{\infty}\left(L_n^{(r)} + \mathbf{i}L_{n+1}^{(r)} + \epsilon L_{n+2}^{(r)} + \mathbf{h}L_{n+3}^{(r)}\right)t^n.$$

Hence

$$\begin{aligned} g(t) = &-\mathbf{i}\frac{2}{t} - \epsilon\left(\frac{2+t(2r+1)}{t^2}\right) - \mathbf{h}\left(\frac{2+t(2r+1)+t^2\left(r^2+2r+3\right)}{t^3}\right) \\ &+ \left(1 + \mathbf{i}\frac{1}{t} + \epsilon\frac{1}{t^2} + \mathbf{h}\frac{1}{t^3}\right)\sum_{n=0}^{\infty}\left(L_n^{(r)}t^n\right). \end{aligned}$$

Using (4), the result is obtained. ☐

In the following theorem, we obtain Binet's formula for the hybrid hyper-Lucas numbers.

Theorem 3.2. *For the any integer $n \geq 0$, the n-th hybrid hyper-Lucas numbers is*

$$HL_n^{(r)} = \underline{\alpha}\alpha^{n+2r} + \underline{\beta}\beta^{n+2r} - HK_n^{(r)}, \tag{41}$$

where α and β are the roots of the characteristic equation Lucas sequence, $\underline{\alpha} = 1 + \mathbf{i}\alpha + \epsilon\alpha^2 + \mathbf{h}\alpha^3$ and $\underline{\beta} = 1 + \mathbf{i}\beta + \epsilon\beta^2 + \mathbf{h}\beta^3$.

Proof. We know that the Binet's formula of hybrid Lucas numbers is

$$HL_n = \underline{\alpha}\alpha^n + \underline{\beta}\beta^n,$$

where HL_n is the n-th hybrid Lucas number in [22]. Considering the identity between the hybrid Lucas and the hybrid hyper-Lucas numbers (39), the result is obtained. □

Now, we give the summation formula of the hybrid hyper-Lucas numbers by using some identities of the hyper-Lucas numbers.

Theorem 3.3. *For $n \geq 0$, the summation formulas of the hybrid hyper-Lucas numbers are*

$$\sum_{k=0}^{n} HL_k^{(r)} = HL_n^{(r+1)} - 2\mathbf{i} - \epsilon(3 + 2r) - \mathbf{h}\left(r^2 + 4r + 6\right),$$

$$\sum_{s=0}^{r} HL_n^{(s)} = HL_{n+1}^{(r)} - HL_{n-1},$$

$$\sum_{k=0}^{n} HL_{2k}^{(r)} = HL_{2n+2r+1} - HL_{2r-1} - \sum_{k=0}^{n} HK_{2k}^{(r)}, \quad r \geq 1,$$

$$\sum_{k=0}^{n} HL_{2k+1}^{(r)} = HL_{2n+2r+2} - HL_{2r} - \sum_{k=0}^{n} HK_{2k+1,}^{(r)} \quad r \geq 1.$$

Proof. The proof of the second summation formula is as follows:

$$\sum_{s=0}^{r} HL_n^{(s)} = \sum_{s=0}^{r} L_n^{(s)} + \mathbf{i}\sum_{s=0}^{r} L_{n+1}^{(s)} + \epsilon\sum_{s=0}^{r} L_{n+2}^{(s)} + \mathbf{h}\sum_{s=0}^{r} L_{n+3}^{(s)}.$$

From (20), we have

$$LHS = L_{n+1}^{(r)} - L_{n-1} + \mathbf{i}\left(L_{n+2}^{(r)} - L_n\right) + \epsilon\left(L_{n+3}^{(r)} - L_{n+1}\right) + \mathbf{h}\left(L_{n+4}^{(r)} - L_{n+2}\right).$$

Using the recurrence relation of hybrid hyper-Lucas numbers, the result is obtained.

Similarly, we can acquire other summation formulas for hybrid hyper-Lucas numbers using (2), (13), (19), (38) and (40) . □

Theorem 3.4. *For positive integers n, m and s with $m \geq s$, we have*

$$\begin{aligned} HL_{n+m}^{(r)}HL_{n+s}^{(r)} - HL_n^{(r)}HL_{n+m+s}^{(r)} = {} & HL_n^{(r)}HK_{n+m+s}^{(r)} - HK_{n+m}^{(r)}HL_{n+s+2r} \\ & -HL_{n+m}^{(r)}HK_{n+s}^{(r)} + HK_n^{(r)}HL_{m+n+s+2r} \\ & +5(-1)^n F_m\left(L_s(\mathbf{i} + 2\epsilon - \mathbf{h}) - F_s(\mathbf{i} + 3\epsilon + 4\mathbf{h})\right), \end{aligned} \tag{42}$$

where F_n, L_n and HL_n are the n-th Fibonacci number, the nth Lucas number and the n-th hybrid Lucas number, respectively.

Proof. Using the Binet's formula of the hybrid hyper-Lucas numbers to left hand side (LHS), we obtain

$$\begin{aligned} LHS = &\left(\underline{\alpha}\alpha^{n+m+2r} + \underline{\beta}\beta^{n+m+2r} - HK^{(r)}_{n+m}\right)\left(\underline{\alpha}\alpha^{n+s+2r} + \underline{\beta}\beta^{n+s+2r} - HK^{(r)}_{n+s}\right) \\ &- \left(\underline{\alpha}\alpha^{n+2r} + \underline{\beta}\beta^{n+2r} - HK^{(r)}_{n}\right)\left(\underline{\alpha}\alpha^{n+m+s+2r} + \underline{\beta}\beta^{n+m+s+2r} - HK^{(r)}_{n+m+s}\right). \end{aligned}$$

Using (27), (28) and (39), we obtain

$$\begin{aligned} LHS = &HL_{n+2r}HK^{(r)}_{n+m+s} + HK^{(r)}_{n}HL_{n+m+s+2r} - HK^{(r)}_{n}HK^{(r)}_{n+m+s} \\ &-HL_{n+m+2r}HK^{(r)}_{n+s} - HK^{(r)}_{n+m}HL_{n+s+2r} + HK^{(r)}_{n+m}HK^{(r)}_{n+s} \\ &+(-1)^{n+s}\left(L_{m-s}\left(\mathbf{i}+3\epsilon+\mathbf{4h}\right) + 5F_{m-s}\left(\mathbf{i}+2\epsilon-\mathbf{h}\right)\right) \\ &-(-1)^{n}\left(L_{m+s}\left(\mathbf{i}+3\epsilon+\mathbf{4h}\right) - 5F_{m+s}\left(\mathbf{i}+2\epsilon-\mathbf{h}\right)\right). \end{aligned}$$

In the last step, considering (7), (9) and (39), the result is obtained. □

Taking $m = -k$ and $s = k$ in (42), the Catalan's identity for the hybrid hyper-Lucas numbers is obtained.

Corollary 3.1 (Catalan's identity)**.** *For positive integers n and k with $n \geq k$, the following identity is true:*

$$\begin{aligned} HL^{(r)}_{n-k}HL^{(r)}_{n+k} - \left(HL^{(r)}_{n}\right)^2 = &HL^{(r)}_{n}HK^{(r)}_{n} - HK^{(r)}_{n-k}HL_{n+k+2r} \\ &-HL^{(r)}_{n-k}HK^{(r)}_{n+k} + HK^{(r)}_{n}HL_{n+2r} \\ &-5\begin{pmatrix} (-1)^{n-k}F_{2k}(\mathbf{i}+\mathbf{2}\epsilon-\mathbf{h}) \\ -(-1)^{n+k}F^2_{k}(\mathbf{i}+\mathbf{3}\epsilon+\mathbf{4h}) \end{pmatrix}. \end{aligned} \tag{43}$$

Taking $k = 1$ in (43), the Cassini's identity of the hybrid hyper-Lucas numbers is obatined.

Corollary 3.2 (Cassini's identity)**.** *For $n \geq 1$, the following equality holds:*

$$\begin{aligned} HL^{(r)}_{n-1}HL^{(r)}_{n+1} - \left(HL^{(r)}_{n}\right)^2 = &HL^{(r)}_{n}HK^{(r)}_{n} - HK^{(r)}_{n-1}HL_{n+1+2r} \\ &-HL^{(r)}_{n-1}HK^{(r)}_{n+1} + HK^{(r)}_{n}HL_{n+2r} \\ &-5\left(-1\right)^{n}\left(\epsilon+\mathbf{5h}\right). \end{aligned} \tag{44}$$

Theorem 3.5 (d'Ocagne's identity)**.** *For positive integers n and m, with $n \geq m$, the following equality holds:*

$$\begin{aligned} HL^{(r)}_{n}HL^{(r)}_{m+1} - HL^{(r)}_{n+1}HL^{(r)}_{m} = &HL^{(r)}_{n+1}HK^{(r)}_{m} - HL^{(r)}_{n}HK^{(r)}_{m+1} \\ &-HK^{(r)}_{n}HL_{m+2r+1} + HK^{(r)}_{n+1}HL_{m+2r} \\ &-5(-1)^{m}\left(L_{n-m}\left(\mathbf{i}+2\epsilon-\mathbf{h}\right) + F_{n-m}\left(\mathbf{i}+3\epsilon+\mathbf{4h}\right)\right). \end{aligned} \tag{45}$$

Proof. Using the Binet's formula (41), we get

$$\begin{aligned} &HL^{(r)}_{n}HL^{(r)}_{m+1} - HL^{(r)}_{n+1}HL^{(r)}_{m} \\ &\quad= \left(\underline{\alpha}\alpha^{n+2r} + \underline{\beta}\beta^{n+2r} - HK^{(r)}_{n}\right)\left(\underline{\alpha}\alpha^{m+1+2r} + \underline{\beta}\beta^{m+1+2r} - HK^{(r)}_{m+1}\right) \\ &\quad\quad- \left(\underline{\alpha}\alpha^{n+1+2r} + \underline{\beta}\beta^{n+1+2r} - HK^{(r)}_{n+1}\right)\left(\underline{\alpha}\alpha^{m+2r} + \underline{\beta}\beta^{m+2r} - HK^{(r)}_{m}\right). \end{aligned}$$

Using (27) and (28), we have

$$\begin{aligned}HL_n^{(r)}HL_{m+1}^{(r)} - HL_{n+1}^{(r)}HL_m^{(r)} = &\left(HL_{n+2r+1} - HK_{n+1}^{(r)}\right)HK_m^{(r)} - HK_n^{(r)}HL_{m+2r+1}\\ &+HK_{n+1}^{(r)}HL_{m+2r} + \left(HK_n^{(r)} - HL_{n+2r}\right)HK_{m+1}^{(r)}\\ &-5(-1)^m\left(F_{n-m}\left(\mathbf{i}+3\epsilon+\mathbf{4h}\right) + L_{n-m}\left(\mathbf{i}+2\epsilon-\mathbf{h}\right)\right).\end{aligned}$$

From (39), the result is obtained. □

Theorem 3.6. *For positive integers n and m, with $n \geq m$. Then, the following equalities hold:*

$$HL_{n+m}^{(r)} + (-1)^m HL_{n-m}^{(r)} = L_m HL_{n+2r} - HK_{n+m}^{(r)} - (-1)^m HK_{n-m}^{(r)}, \tag{46}$$

$$HL_{n+m}^{(r)} - (-1)^m HL_{n-m}^{(r)} = 5F_m HF_{n+2r} - HK_{n+m}^{(r)} + (-1)^m HK_{n-m}^{(r)}, \tag{47}$$

where F_n, L_n, HF_n and HL_n are the n-th Fibonacci number, the n-th Lucas number, the n-th hybrid Fibonacci number and the n-th hybrid Lucas number, respectively.

Proof. For the proof of (46) and (47), using the recurrence relation of hybrid hyper-Lucas numbers and (7), (8) respectively, the results are obtained. □

Remark 3.1. *Taking $r = 0$ in* (46) *and* (47)*, we obtain*

$$HL_{n+m} + (-1)^m HL_{n-m} = L_m HL_n,$$

$$HL_{n+m} - (-1)^m HL_{n-m} = 5F_m HF_n.$$

Theorem 3.7. *For positive integers n and m with $n \geq 1, m \geq 1$ and $n \geq m$, we have*

$$\begin{aligned}HL_{m+1}^{(r)}HL_{n+1}^{(r)} - HL_{m-1}^{(r)}HL_{n-1}^{(r)} = &\, HL_{m-1+2r}HK_{n-1}^{(r)} - HK_{m+1}^{(r)}HL_{n+1}^{(r)}\\ &-HL_{m+1+2r}HK_{n+1}^{(r)} + HK_{m-1}^{(r)}HL_{n-1}^{(r)}\\ &+5L_{m+n+5+4r} - 10F_{m+n+4r} + 10HF_{m+n+4r},\end{aligned} \tag{48}$$

where F_n, L_n, HF_n and HL_n are the n-th Fibonacci number, the nth Lucas number, the n-th hybrid Fibonacci number and the n-th hybrid Lucas number, respectively.

Proof. Using Binet's formula to (LHS), we have

$$\begin{aligned}LHS = &\left(\underline{\alpha}\alpha^{m+1+2r} + \underline{\beta}\beta^{m+1+2r} - HK_{m+1}^{(r)}\right)\left(\underline{\alpha}\alpha^{n+1+2r} + \underline{\beta}\beta^{n+1+2r} - HK_{n+1}^{(r)}\right)\\ &-\left(\underline{\alpha}\alpha^{m-1+2r} + \underline{\beta}\beta^{m-1+2r} - HK_{m-1}^{(r)}\right)\left(\underline{\alpha}\alpha^{n-1+2r} + \underline{\beta}\beta^{n-1+2r} - HK_{n-1}^{(r)}\right).\end{aligned}$$

Then, using (7), (35) and (36), we get

$$\begin{aligned}LHS = &\, HK_{m-1}^{(r)}\left(HL_{n-1+2r} - HK_{n-1}^{(r)}\right) + HL_{m-1+2r}HK_{n-1}^{(r)}\\ &-HK_{m+1}^{(r)}\left(HL_{n+1+2r} - HK_{n+1}^{(r)}\right) - HL_{m+1+2r}HK_{n+1}^{(r)}\\ &+5F_{m+n+7+4r} - 5F_{m+n+3+4r} + 10\mathbf{i}F_{m+n+1+4r} + 10\epsilon F_{m+n+2+4r} + 10\mathbf{h}F_{m+n+3+4r}.\end{aligned}$$

Afterwards, considering (10) and (39), the result is obtained. □

Remark 3.2. *If we take $r = 0$ in* (48), *we obtain the following identity for hybrid Lucas numbers*

$$HL_{m+1}HL_{n+1} - HL_{m-1}HL_{n-1} = 5L_{m+n+5} + 10\mathbf{i}F_{m+n+1} + 10\epsilon F_{m+n+2} + 10\mathbf{h}F_{m+n+3}.$$

Theorem 3.8. *For the positive integers m, k and s with $m \geq k$ and $m \geq s$, the following identity holds:*

$$\begin{aligned} HL_{m+k}^{(r)}HL_{m-k}^{(r)} - HL_{m+s}^{(r)}HL_{m-s}^{(r)} = {} & HK_{m+s}^{(r)}HL_{m-s+2r} - HL_{m+k}^{(r)}HK_{m-k}^{(r)} \\ & - HK_{m+k}^{(r)}HL_{m-k+2r} + HL_{m+s}^{(r)}HK_{m-s}^{(r)} \\ & + (\mathbf{i} + 3\epsilon + 4\mathbf{h})\left((-1)^{m-k}L_{2k} - (-1)^{m-s}L_{2s}\right) \\ & + 5(\mathbf{i} + 2\epsilon - \mathbf{h})\left((-1)^{m-k}F_{2k} - (-1)^{m-s}F_{2s}\right). \end{aligned} \tag{49}$$

where F_n, L_n and HL_n are the n-th Fibonacci number,the n-th Lucas number and the n-th hybrid Lucas number, respectively.

Proof. From the Binet's formula, we have

$$\begin{aligned} LHS = {} & \left(\underline{\alpha}\alpha^{m+k+2r} + \underline{\beta}\beta^{m+k+2r} - HK_{m+k}^{(r)}\right)\left(\underline{\alpha}\alpha^{m-k+2r} + \underline{\beta}\beta^{m-k+2r} - HK_{m-k}^{(r)}\right) \\ & - \left(\underline{\alpha}\alpha^{m+s+2r} + \underline{\beta}\beta^{m+s+2r} - HK_{m+s}^{(r)}\right)\left(\underline{\alpha}\alpha^{m-s+2r} + \underline{\beta}\beta^{m-s+2r} - HK_{m-s}^{(r)}\right). \end{aligned}$$

Hence

$$\begin{aligned} LHS = {} & \left(HL_{m+s+2r} - HK_{m+s}^{(r)}\right)HK_{m-s}^{(r)} + HK_{m+s}^{(r)}HL_{m-s+2r} \\ & - \left(HL_{m+k+2r} - HK_{m+k}^{(r)}\right)HK_{m-k}^{(r)} - HK_{m+k}^{(r)}HL_{m-k+2r} \\ & + (\mathbf{i} + 3\epsilon + 4\mathbf{h})\left((-1)^{m-k}L_{2k} - (-1)^{m-s}L_{2s}\right) \\ & + 5(\mathbf{i} + 2\epsilon - \mathbf{h})\left((-1)^{m-k}F_{2k} - (-1)^{m-s}F_{2s}\right). \end{aligned}$$

From (39), the result is obtained. □

Remark 3.3. *For $r = 0$ in* (49)*, the following equality for the hybrid-Lucas numbers is true:*

$$\begin{aligned} HL_{m+k}HL_{m-k} - HL_{m+s}HL_{m-s} = {} & (\mathbf{i} + 3\epsilon + 4\mathbf{h})\left((-1)^{m-k}L_{2k} - (-1)^{m-s}L_{2s}\right) \\ & + 5(\mathbf{i} + 2\epsilon - \mathbf{h})\left((-1)^{m-k}F_{2k} - (-1)^{m-s}F_{2s}\right). \end{aligned}$$

4 Conclusion

In this research, we have defined a new type of hybrid numbers whose coefficients of basis are the hyper-Fibonacci and the hyper-Lucas numbers. We have derived the Catalan's, the Cassini's and the d'Ocagne's identities for these numbers. For future works, more identities can be obtained. In addition, the applications of the hybrid hyper-Fibonacci and the hybrid hyper-Lucas numbers can be considered.

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