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A note on edge irregularity strength of firefly graph

Umme Salma[1], **H. M. Nagesh**[2] **and D. Prahlad**[3]

[1] Department of Science & Humanities,
PES University, Bangalore, India
e-mail: ummesalma@pesu.pes.edu

[2] Department of Science & Humanities,
PES University, Bangalore, India
e-mail: nageshhm@pes.edu

[3] Department of Electronics and Communication Engineering,
PES University, Bangalore, India
e-mail: prahlad@pes.edu

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Abstract: Let G be a simple graph. A vertex labeling $\psi : V(G) \to \{1, 2, \ldots, \alpha\}$ is called α-labeling. For an edge $uv \in G$, the weight of uv, written $z_\psi(uv)$, is the sum of the labels of u and v, i.e., $z_\psi(uv) = \psi(u) + \psi(v)$. A vertex α-labeling is said to be an edge irregular α-labeling of G if for every two distinct edges a and b, $z_\psi(a) \neq z_\psi(b)$. The minimum α for which the graph G contains an edge irregular α-labeling is known as the edge irregularity strength of G and is denoted by $\mathrm{es}(G)$. In this paper, we find the exact value of edge irregularity strength of different cases of firefly graph $F_{s,t,n-2s-2t-1}$ for any $s \geq 1, t \geq 1, n - 2s - 2t - 1 \geq 1$.
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1 Introduction

Let $V(G)$ and $E(G)$, respectively, be the vertex set and edge set of a simple and connected graph G. An assignment of integers (positive) to a set of vertices or edges, or both, subject to certain constraints is called *graph labeling*.

The authors in [4] defined irregular labeling for a graph G as an assignment of labels from the set of natural numbers to the edges of G such that the sums of the labels assigned to the edges of each vertex are distinct. The minimum value of the largest label of an edge over all existing irregular labelings is known as the *irregularity strength* of G and denoted by $s(G)$.

Determination of $s(G)$ seems to be hard even for a simple structure of a graph G [3,4]. As an example, Figure 1 shows that the irregularity strength of the Petersen graph is 5.

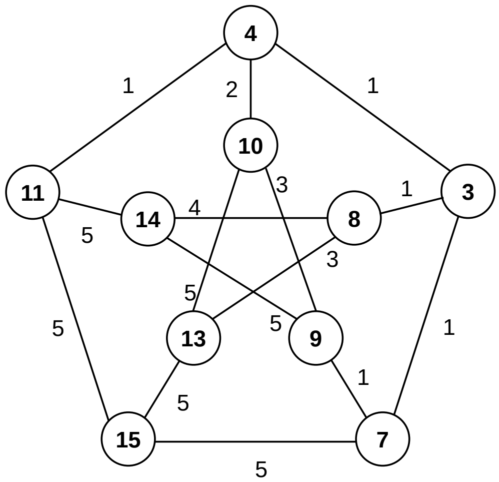

Figure 1. Irregularity strength of the Petersen graph

Motivated by the work of Chartrand et al. [4], authors in [1] introduced the concept of edge irregularity strength of graphs as follows.

Let G be a simple graph. A vertex labeling $\psi : V(G) \to \{1, 2, \ldots, \alpha\}$ is called α-labeling. For an edge $uv \in G$, the weight of uv, written $z_\psi(uv)$, is the sum of the labels of u and v, i.e., $z_\psi(uv) = \psi(u) + \psi(v)$. A vertex α-labeling is said to be an edge irregular α-labeling of G if for every two distinct edges a and b, $z_\psi(a) \neq z_\psi(b)$. The minimum α for which the graph G contains an edge irregular α-labeling is known as the edge irregularity strength of G and is denoted by $\mathrm{es}(G)$.

Clearly, $s(G)$ is an edge labeling of a graph G such that the distinct vertices have distinct weights; and $\mathrm{es}(G)$ is a vertex labeling of a graph G such for every two different edges their weights are distinct.

2 Preliminary results

For a given graph G, the authors in [1] estimated the bounds for $\mathrm{es}(G)$ and also found exact value of $\mathrm{es}(G)$ for several families of graphs such as paths P_n of order $n \geq 2$; cycle C_n of order $n \geq 3$; star graph $K_{1,n}$ of order $n+1, n \geq 1$; double star $S_{m,n}$ with $3 \leq m \leq n$; and Cartesian product of two paths $P_n \times P_m$ of order $m, n \geq 2$. The authors in [2] determined the edge irregularity strength of Toeplitz graphs. The exact value of edge irregularity strength of corona product of graphs with paths is determined in [9]. The authors in [10] determined the exact value of edge irregularity strength of disjoint union of graphs. The authors in [8] determined the edge irregularity strength of sunlet graph. The edge irregularity strength of ladder related graphs are computed in [7]. Recently, in [6] the edge irregularity strength of line graph and line cut-vertex graph of comb graph is determined.

Motivated by the studies as mentioned above, we find the exact value of edge irregularity strength of different cases of firefly graph $F_{s,t,n-2s-2t-1}$ for any $s \geq 1, t \geq 1, n-2s-2t-1 \geq 1$.

The authors in [5] introduced the concept of firefly graph as follows.

Definition 2.1. *A* firefly graph*, written* $F_{s,t,n-2s-2t-1}$ $(s \geq 0, t \geq 0, n-2s-2t-1 \geq 0)$*, is a graph of order* n *that consists of* s *triangles,* t *pendent paths of length* 2*, and* $n-2s-2t-1$ *pendent edges sharing a vertex in common.*

Figure 2 shows an example of a firefly graph.

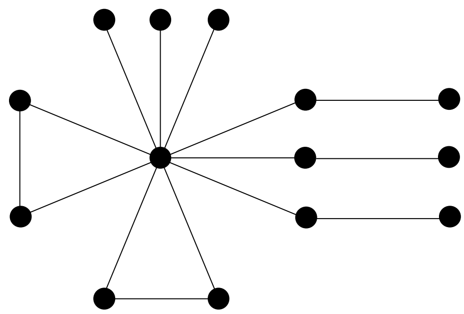

Figure 2. The firefly graph $F(2, 3, 3)$

Clearly, $|V(F_{s,t,n-2s-2t-1})| = n$; $|E(F_{s,t,n-2s-2t-1})| = n+s-1$; and $\Delta = n-t-1$.

The following theorem in [1] establishes the lower bound for the edge irregularity strength of a graph G.

Theorem 2.2. *Let* $G = (V, E)$ *be a simple graph with maximum degree* $\Delta(G)$*. Then*

$$\mathrm{es}(G) \geq \max\{\lceil \tfrac{|E(G)|+1}{2} \rceil, \Delta(G)\}.$$

3 Edge irregularity strength of firefly graph

With respect to Figure 2, $\mathrm{es}(F_{1,0,0}) = 3$ since $F_{1,0,0} \cong K_3$. Similarly, $\mathrm{es}(F_{0,1,0}) = 2$ since $F_{0,1,0} \cong P_3$; and $\mathrm{es}(F_{0,0,1}) = 1$ since $F_{0,0,1} \cong K_2$. Based on this observation, we determine $\mathrm{es}(F_{s,t,n-2s-2t-1})$ for different cases of $F_{s,t,n-2s-2t-1}$ for any $s \geq 1, t \geq 1, n-2s-2t-1 \geq 1$.

In the next theorem, we find the exact value of edge irregularity strength of firefly graph $F_{s,t,n-2s-2t-1}$ for any $s \geq 1, t \geq 1, n-2s-2t-1 \geq 1$ and $s = t = n-2s-2t-1$.

Theorem 3.1. *Let* $G = F_{s,t,n-2s-2t-1}$ $(s \geq 1, t \geq 1, n-2s-2t-1 \geq 1)$ *be the firefly graph. Then* $\mathrm{es}(G) = n-t-1$ *for* $s = t = 2s-2t-1$.

Proof. Let $G = F_{s,t,n-2s-2t-1}$ $(s \geq 1, t \geq 1, n - 2s - 2t - 1 \geq 1)$ be the firefly graph. For $s = t = n - 2s - 2t - 1$, let us consider the vertex set $V(G)$ and edge set $E(G)$ of G as follows:

$$V(G) = \{x, x_i : 1 \leq i \leq n - t - 1\} \cup \{y_j : 1 \leq j \leq t\};$$
$$E(G) = \{xx_i : 1 \leq i \leq n - t - 1\} \cup \{x_l x_{l+1} : l = 2i - 1, 1 \leq i \leq s\} \cup$$
$$\{x_i y_j : n - 2t \leq i \leq n - t - 1, 1 \leq j \leq t\}.$$

Since $|V(G)| = n$, $|E(G)| = n + s - 1$, and the maximum degree $\Delta = n - t - 1$, according to the Theorem 2.2, $\mathrm{es}(G) \geq \max\{\lceil \frac{n+s}{2} \rceil, n - t - 1\}$. Since $n - t - 1 > \lceil \frac{n+s}{2} \rceil$ for $s = t = n - 2s - 2t - 1$, $\mathrm{es}(G) \geq n - t - 1$.

To prove the equality, it suffices to prove the existence of an edge irregular $(n - t - 1)-$labeling. Define a labeling ψ on vertex set of G as follows:

Let $\psi : V(G) \to \{1, 2, \ldots, n - t - 1\}$ such that

$$\psi(x) = 1;$$
$$\psi(x_i) = (n - 2s - t - 1) + i \text{ for } 1 \leq i \leq 2s;$$
$$\psi(x_i) = i - 2s \text{ for } 2s + 1 \leq i \leq n - t - 1;$$
$$\psi(y_j) = (n - 2t - 1) + j \text{ for } 1 \leq j \leq t.$$

The edge weights are as follows:

$$z_\psi(xx_i) = (n - 2s - t) + i \text{ for } 1 \leq i \leq 2s;$$
$$z_\psi(xx_i) = i - 2s + 1 \text{ for } 2s + 1 \leq i \leq n - t - 1;$$
$$z_\psi(x_l x_{l+1}) = 2(n - 2s - t) + 2l - 1 \text{ for } l = 2i - 1, 1 \leq i \leq s;$$
$$z_\psi(x_i y_j) = (n - 2s - 2t - 1) + i + j \text{ for } n - 2t \leq i \leq n - t - 1, 1 \leq j \leq t.$$

On the basis of above calculations we see that the edge weights are distinct for all pairs of distinct edges. Therefore, $\mathrm{es}(G) = n - t - 1$. This completes the proof. □

As an example, for the graph G of Figure 3, $s = 2, t = 2, n - 2s - 2t - 1 = 2$. Hence, $\mathrm{es}(G) = 8$.

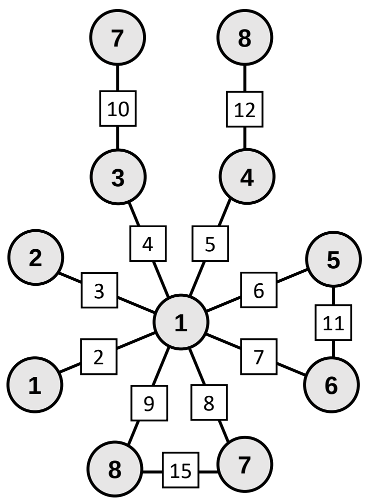

Figure 3. Edge irregularity strength of $F_{2,2,2}$

In the next theorem, we determine the exact value of edge irregularity strength of firefly graph for any $s \geq 1, t \geq 1, n-2s-2t-1 \geq 1$ and $n-2s-2t-1 > s+t, t \geq s$.

Theorem 3.2. *Let* $G = F_{s,t,n-2s-2t-1}$ $(s \geq 1, t \geq 1, n-2s-2t-1 \geq 1)$ *be the firefly graph. Then* $\mathrm{es}(G) = n-t-1$ *for* $n-2s-2t-1 > s+t, t \geq s$.

Proof. Let $G = F_{s,t,n-2s-2t-1}$ $(s \geq 1, t \geq 1, n-2s-2t-1 \geq 1)$ be the firefly graph, where $n-2s-2t-1 > s+t, t \geq s$. Let us consider the vertex set $V(G)$ and edge set $E(G)$ of G as follows:

$$V(G) = \{x, x_i : 1 \leq i \leq n-t-1\} \cup \{y_j : 1 \leq j \leq t\};$$

$$E(G) = \{xx_i : 1 \leq i \leq n-t-1\} \cup \{x_l x_{l+1} : l = n-2s-t+2(i-1), 1 \leq i \leq s\} \cup \{x_i y_j : n-2s-2t \leq i \leq n-2s-t-1, 1 \leq j \leq t\}.$$

According to the Theorem 2.2, $\mathrm{es}(G) \geq n-t-1$. To prove the equality, it suffices to prove the existence of an edge irregular $(n-t-1)-$labeling.

Let $\psi : V(G) \to \{1, 2, \ldots, n-t-1\}$ be the labeling on vertex set of G such that:

$$\psi(x) = 1;$$

$$\psi(x_i) = i \text{ for } 1 \leq i \leq n-t-1;$$

$$\psi(y_j) = 2s+t+1 \text{ for } 1 \leq j \leq t.$$

The edge weights are as follows:

$$z_\psi(xx_i) = i+1 \text{ for } 1 \leq i \leq n-t-1;$$

$$z_\psi(x_l x_{l+1}) = 2l+1 \text{ for } l = n-2s-t-2+2i \ , 1 \leq i \leq s;$$

$$z_\psi(x_i y_j) = n-t+j \text{ for } n-2t-2s \leq i \leq n-t-2s-1, 1 \leq j \leq t.$$

On the basis of above calculations we see that the edge weights are distinct for all pairs of distinct edges. Therefore, $\mathrm{es}(G) = n-t-1$ for $n-2s-2t-1 > s+t, t \geq s$. This completes the proof. $\square$

As an example, for the graph G of Figure 4, $s = 2, t = 3, n-2s-2t-1 = 6$. Hence, $\mathrm{es}(G) = 13$. Similarly, for the graph G of Figure 5, $s = 1, t = 1, n-2s-2t-1 = 3$. Hence, $\mathrm{es}(G) = 6$.

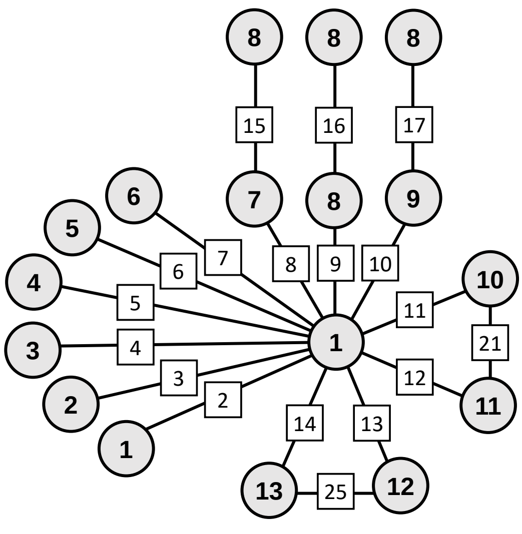

Figure 4. Edge irregularity strength of $F(2,3,6)$

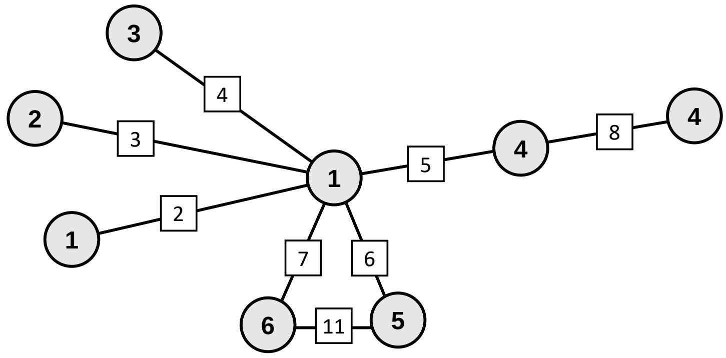

Figure 5. Edge irregularity strength of $F(1, 1, 3)$

In the next theorem, we determine the exact value of edge irregularity strength of firefly graph for any $s \geq 1, t \geq 1, n-2s-2t-1 \geq 1$ and $t \geq s+(n-2s-2t-1), n-2s-2t-1 \geq 1, s = 1$.

Theorem 3.3. *Let* $G = F_{s,t,n-2s-2t-1}$ $(s \geq 1, t \geq 1, n-2s-2t-1 \geq 1)$ *be the firefly graph. Then* $\mathrm{es}(G) = n-t-1$ *for* $t \geq s+(n-2s-2t-1)$ *and* $n-2s-2t-1 \geq 1, s = 1$.

Proof. The proof is similar to the proof of Theorem 3.2 and so the proof is omitted here. $\square$

As an example, for the graph G of Figure 6, $s = 1, t = 3, n-2s-2t-1 = 1$. Hence, $\mathrm{es}(G) = 6$. Similarly, for the graph G of Figure 7, $s = 1, t = 5, n-2s-2t-1 = 4$. Hence, $\mathrm{es}(G) = 11$.

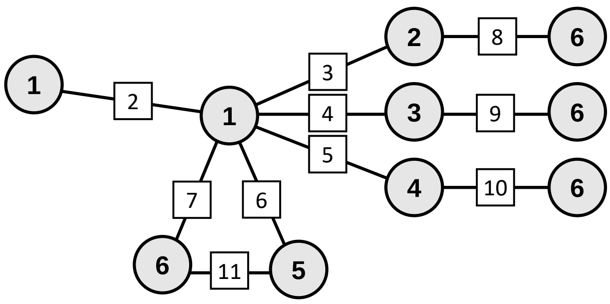

Figure 6. Edge irregularity strength of $F(1, 3, 1)$

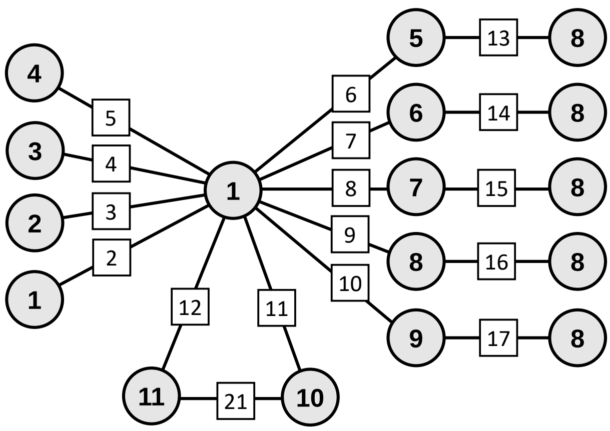

Figure 7. Edge irregularity strength of $F(1, 5, 4)$

4 Conclusion

In this note, the exact value of edge irregularity strength of different cases of firefly graph $F_{s,t,n-2s-2t-1}$ for any $s \geq 1, t \geq 1, n - 2s - 2t - 1 \geq 1$ is computed. However, determining the exact value of edge irregularity strength of firefly graph for many other cases, such as $s \geq t+(n-2s-2t-1)$ and $t \geq n-2s-2t-1$; $s \geq t+(n-2s-2t-1)$ and $n-2s-2t-1 > t$, etc., still remain open.

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