Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2023, Volume 29, Number 1, 137–146
DOI: 10.7546/nntdm.2023.29.1.137-146

Congruences for harmonic sums

Yining Yang[1] and Peng Yang[2]

[1] School of Science, University of Science and Technology Liaoning
Anshan, Liaoning 114051, P. R. China
e-mail: 202070100025@stu.ustl.edu.cn

[2] School of Science, University of Science and Technology Liaoning
Anshan, Liaoning 114051, P. R. China
e-mail: alberteyang-1@163.com

Received: 23 November 2022 **Revised:** 9 March 2023
Accepted: 20 March 2023 **Online First:** 21 March 2023

Abstract: Zhao found a curious congruence modulo p on harmonic sums. Xia and Cai generalized his congruence to a supercongruence modulo p^2. In this paper, we improve the harmonic sums

$$H_p(n) = \sum_{\substack{l_1+l_2+\cdots+l_n=p \\ l_1,l_2,\ldots,l_n>0}} \frac{1}{l_1 l_2 \cdots l_n}$$

to supercongruences modulo p^3 and p^4 for odd and even where prime $p > 8$ and $3 \le n \le p-6$.
Keywords: Harmonic sums, Bernoulli numbers, Congruence.
2020 Mathematics Subject Classification: 11A04, 11A41.

1 Introduction

In 2006, Zhao [7] found the congruence

$$H_p(3) = \sum_{\substack{i+j+k=p \\ i,j,k>0}} \frac{1}{ijk} \equiv 6\beta_{p-3} \pmod p$$

for any prime $p > 3$, where $\beta_n = \frac{B_n}{n}$ is called the divided Bernoulli number. Then Zhou and Cai [8] generalized Zhao's congruence to any positive integer $n \le p-2$,

$$H_p(n) \equiv \begin{cases} n!\beta_{p-n} \pmod{p}, & 2 \nmid n, \\ -\frac{np}{2}n!\beta_{p-n-1} \pmod{p^2}, & 2 \mid n, \end{cases}$$

where prime $p > 3$. Later in 2010, Xia and Cai [6] improved the congruence to modulo p^2 where $3 \le n \le p-3$ is an odd integer and prime $p > 3$,

$$H_p(n) \equiv n!\,(2\beta_{p-n} - \beta_{2p-n-1}) + \frac{n!p}{2} \sum_{\substack{2 \le i \le n-3 \\ i \text{ even}}} \beta_{p-i-1}\beta_{p-n+i} \pmod{p^2}.$$

After that, Shen [4] raised the power of l_n to 2 and 3 for any integer n and prime $p > n+2$

$$\sum_{\substack{l_1+l_2+\cdots+l_n=p \\ l_1,\ldots,l_n>0}} \frac{1}{l_1 l_2 \cdots l_n^2} \equiv \begin{cases} -\frac{(n+1)!(n-2)}{4n}\beta_{p-n-1} \pmod{p}, & 2 \mid n, \\ (n-1)! \sum\limits_{0 \le a \le p-n} a\binom{n+a}{n-1}\beta_a\beta_{p-n-a} \pmod{p}, & 2 \nmid n. \end{cases}$$

$$\begin{aligned} \sum_{\substack{l_1+l_2+\cdots+l_n=p \\ l_1,\ldots,l_n>0}} \frac{1}{l_1 l_2 \cdots l_n^3} \equiv \sum_{a=0}^{p-2} \sum_{b=0}^{p-n-a} (-1)^{b+n+1} \frac{(n-1)!ab}{a+1} \binom{a+b}{b} \\ \times \binom{a+b+n}{n-1} \beta_a\beta_b\beta_{p-n-a-b} \pmod{p}. \end{aligned}$$

The main purpose of this paper is to improve congruence $H_p(n)$ to the modulus p^4.

2 Some lemmas

Lemma 2.1 (Newton's formula [6]). *Let n and k be integers, $1 \le k \le n$. For any variables $x_1, x_2, \ldots, x_n$, denote*

$$\lambda_k = x_1^k + x_2^k + \cdots + x_n^k,$$
$$\sigma_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Then

$$\lambda_k = \lambda_{k-1}\sigma_1 - \lambda_{k-2}\sigma_2 + \cdots + (-1)^k\lambda_1\sigma_{k-1} + (-1)^{k+1}k\sigma_k.$$

From now on, we define S_k and A_k to be $x_i = \frac{1}{i}$ and $n = p-1$ in λ_k and σ_k, i.e.,

$$S_k = \sum_{x=1}^{p-1} \frac{1}{x^k},$$
$$A_k = \sum_{1 \le u_1 < u_2 < \cdots < u_k \le p-1} \frac{1}{u_1 u_2 \cdots u_k}.$$

Lemma 2.2 (Zhou and Cai [8]). *For every positive integer $p-2 \geq n \geq 2$ and odd prime p, we have*

$$H_p(n) = \frac{n!}{p} A_{n-1}.$$

Proof. See [8, pp. 1331–1332]. $\square$

Lemma 2.3. *Let j be a positive integer. Then*

$$A_n \equiv \begin{cases} a_{2j}p + a'_{2j}p^2 \pmod{p^3}, & n = 2j, \\ b_{2j-1}p^2 \pmod{p^3}, & n = 2j-1, \end{cases}$$

where

$$a_{2j} = -\left(\beta_{2p-2j-2} - 2\beta_{p-2j-1}\right), \; a'_{2j} = \frac{1}{2} \sum_{\substack{2 \leq i \leq 2j-2 \\ i \text{ is even}}} \beta_{p-i-1}\beta_{p-2j-1+i};$$

$$b_{2j-1} = -\frac{j}{2j+1}\beta_{p-2j-1}.$$

Proof. For n odd, see [3] or Lemma 3 in [8]; for n even, see Theorem 1 in [6]. $\square$

Lemma 2.4 (Sun [5]). *Let p be a prime greater than 7 and $1 \leq k \leq p-6$. Then*

$$S_k \equiv \begin{cases} c_k p \pmod{p^3}, & 2 \mid k, \\ d_k p^2 \pmod{p^3}, & 2 \nmid k, \end{cases}$$

and we also have

$$S_k \equiv \begin{cases} e_k p + e'_k p^3 \pmod{p^4}, & 2 \mid k, \\ f_k p^2 \pmod{p^4}, & 2 \nmid k, \end{cases}$$

where

$$c_k = k\left(\beta_{2p-2-k} - 2\beta_{p-1-k}\right);$$

$$d_k = \binom{k+1}{2}\beta_{p-2-k};$$

$$e_k = -k\left(\beta_{3p-3-k} - 3\beta_{2p-2-k} + 3\beta_{p-1-k}\right), e'_k = -\binom{k+2}{3}\beta_{p-3-k};$$

$$f_k = -\binom{k+1}{2}\left(\beta_{2p-3-k} - 2\beta_{p-2-k}\right).$$

Lemma 2.5. *Let $p \geq 7$ be a prime and $k \in \{1, 2, \ldots, p-6\}$. Then*

$$S_k \equiv \begin{cases} g_k p + g'_k p^3 \pmod{p^5}, & 2 \mid k, \\ h_k p^2 + h'_k p^4 \pmod{p^5}, & 2 \nmid k, \end{cases}$$

where

$$g_k = k\left(4\beta_{p-1-k} - 6\beta_{2p-2-k} + 4\beta_{3p-3-k} - \beta_{4p-4-k}\right),$$

$$g'_k = \binom{k+2}{3}\left(\beta_{2p-4-k} - 2\beta_{p-3-k}\right);$$

$$h_k = \binom{k+1}{2}\left(3\beta_{p-2-k} - 3\beta_{2p-3-k} + \beta_{3p-4-k}\right),$$

$$h'_k = \binom{k+3}{4}\beta_{p-4-k}.$$

Proof. Using the same argument as in the proof of Theorem 5.1 in [5], we can easily carry out the proof of this Lemma. For $m \in \mathbb{Z}^+$, it is clear that

$$\begin{aligned}
1^m + 2^m + \cdots + (p-1)^m &= \frac{1}{m+1}\sum_{r=1}^{m+1}\binom{m+1}{r}B_{m+1-r}p^r \\
&\equiv pB_m + \frac{p^2}{2}mB_{m-1} + \frac{p^3}{6}\binom{m}{2}B_{m-2} \\
&\quad + \frac{p^4}{24}\binom{m}{3}B_{m-3} + \frac{p^5}{120}\binom{m}{4}B_{m-4} \pmod{p^5}.
\end{aligned}$$

Let $k \in \{1, 2, \ldots, p-6\}$. From (5.1) in [5] and Euler's theorem, setting $m = \varphi\left(p^5\right) - k$, we have

$$\begin{aligned}
S_k &= \sum_{x=1}^{p-1}\frac{1}{x^k} \\
&\equiv \sum_{x=1}^{p-1} x^{\varphi\left(p^5\right)-k} \qquad (1)\\
&\equiv \begin{cases} pB_{\varphi(p^5)-k} + \frac{k(k+1)}{6}p^3B_{\varphi(p^5)-k-2} + \frac{k(k+1)(k+2)(k+3)}{120}p^5B_{\varphi(p^5)-k-4} \pmod{p^5}, & \text{if } 2 \mid k \\ -\frac{k}{2}p^2B_{\varphi(p^5)-k-1} - \frac{k(k+1)(k+2)}{24}p^4B_{\varphi(p^5)-k-3} \pmod{p^5}, & \text{if } 2 \nmid k \end{cases}
\end{aligned}$$

To prove our congruence, we evaluate $pB_{\varphi(p^5)-k},\ p^2B_{\varphi(p^5)-k-1},\ p^3B_{\varphi(p^5)-k-2},\ p^4B_{\varphi(p^5)-k-3},$ $p^5B_{\varphi(p^5)-k-4} \pmod{p^5}$ separately. It is easy to conclude from Corollary 4.1 in [5] that

$$\beta_{k(p-1)+b} \equiv \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\left(1-p^{r(p-1)+b-1}\right)\beta_{r(p-1)+b} \pmod{p^n},$$

hence

$$\begin{aligned}
\beta_{\varphi(p^5)-k} &= \beta_{(p^4-1)(p-1)+p-1-k} \\
&\equiv -\binom{p^4-2}{3}(1-p^{p-2-k})\beta_{p-1-k} + \binom{p^4-3}{2}(p^4-1)(1-p^{2p-3-k})\beta_{2p-2-k} \\
&\quad - \binom{p^4-1}{2}(p^4-4)(1-p^{3p-4-k})\beta_{3p-3-k} + \binom{p^4-1}{3}(1-p^{4p-5-k})\beta_{4p-4-k} \\
&\equiv 4(1-p^{p-2-k})\beta_{p-1-k} - 6(1-p^{2p-3-k})\beta_{2p-2-k} + 4(1-p^{3p-4-k})\beta_{3p-3-k} \\
&\quad - (1-p^{4p-5-k})\beta_{4p-4-k} \\
&\equiv 4\beta_{p-1-k} - 6\beta_{2p-2-k} + 4\beta_{3p-3-k} - \beta_{4p-4-k} \pmod{p^4}.
\end{aligned}$$

Therefore,

$$pB_{\varphi(p^5)-k} \equiv -k\left(4\beta_{p-1-k} - 6\beta_{2p-2-k} + 4\beta_{3p-3-k} - \beta_{4p-4-k}\right)p \pmod{p^5}. \qquad (2)$$

Similary, we have

$$p^3B_{\varphi(p^5)-k-2} \equiv -\left(k+2\right)\left(2\beta_{p-3-k} - \beta_{2p-4-k}\right)p^3 \pmod{p^5}, \qquad (3)$$

$$p^2B_{\varphi(p^5)-k-1} \equiv -\left(k+1\right)\left(3\beta_{p-2-k} - 3\beta_{2p-3-k} + \beta_{3p-4-k}\right)p^2 \pmod{p^5}. \qquad (4)$$

From Kummer's congruences we conclude that

$$\beta_{\varphi(p^5)-k-3} = \beta_{(p^4-1)(p-1)+p-4-k} \equiv \beta_{p-4-k} \pmod{p}.$$

Hence,

$$p^4 B_{\varphi(p^5)-k-3} \equiv -(k+3)\beta_{p-4-k} p^4 \pmod{p^5}. \tag{5}$$

Combining congruences (2), (4), (3), (5) with (1), leads to our congruence. □

Lemma 2.6 (Xia and Cai [6])**.** *Let m, n be integers and $1 \leq m \leq n$. For any variables $x_1, x_2, \ldots, x_n$, we have*

$$\sigma_m = (-1)^m \sum_{\substack{\alpha_1+2\alpha_2+\cdots+n\alpha_n=m \\ \alpha_1,\ldots,\alpha_n>0}} T(\alpha_1, \ldots, \alpha_n)\, \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \cdots \lambda_n^{\alpha_n},$$

where for nonnegative integers $\alpha_1, \alpha_2, \ldots, \alpha_n$,

$$T(\alpha_1, \ldots, \alpha_n) = (-1)^{\alpha_1+\alpha_2+\cdots+\alpha_n} \frac{n}{2^{\alpha_2} \cdots n^{\alpha_n} \alpha_1! \cdots \alpha_n!}.$$

3 Main results

Theorem 3.1. *Let $p > 9$ be a prime and $n \in \{3, 4, \ldots, p-6\}$. Then*

$$H_p(n) \equiv \begin{cases} -\frac{n!}{n-1}\left(h_{n-1}p + Fp^2 + Gp^3\right) \pmod{p^4}, & 2 \mid n, \\ -\frac{n!}{n-1}\left(g_{n-1} - Cp + Dp^2 - Ep^3\right) \pmod{p^4}, & 2 \nmid n, \end{cases}$$

where

$$\begin{aligned}
C &= \sum_{j=1}^{\frac{n-3}{2}} \frac{1}{2j} e_{2j} e_{n-1-2j}, \\
D &= g'_{n-1} - \frac{1}{2j} \sum_{j=1}^{\frac{n-3}{2}} \left(e_{n-1-2j} \sum_{i=1}^{j-1} a_{2i} c_{2j-2i} \right), \\
E &= \frac{1}{j} \sum_{j=1}^{\frac{n-3}{2}} e_{2j} e'_{n-1-2j} - \frac{1}{2j} \sum_{j=1}^{\frac{n-3}{2}} \left(e_{n-1-2j} \sum_{i=1}^{j-1} a'_{2i} c_{2j-2i} \right) + \sum_{j=1}^{\frac{n-1}{2}} b_{2j-1} d_{n-2j}; \\
F &= \sum_{j=1}^{\frac{n-2}{2}} \left(a_{2j} f_{n-1-2j} + \frac{1}{2j-1} b_{2j-1} c_{n-2j} \right), \\
G &= h'_{n-1} + \sum_{j=1}^{\frac{n-2}{2}} \left(a'_{2j} f_{n-1-2j} + \frac{1}{2j-1} c_{n-2j} \sum_{i=1}^{j-1} \left(a_{2i} d_{2j-1-2i} - b_{2i-1} c_{2j-2i} \right) \right).
\end{aligned}$$

Proof. First we need to evaluate the $A_{2j},\ A_{2j-1} \pmod{p^4}$. By Lemma 2.1,

$$S_{n-1} = A_1 S_{n-2} + \cdots + (-1)^k A_{n-2} S_1 + (-1)^n (n-1) A_{n-1}. \tag{6}$$

If $n = 2j+1$,

$$2jA_{2j} = -S_{2j} + A_{2j-1}S_1 - \sum_{i=1}^{j-1}\left(A_{2i}S_{2j-2i} - A_{2i-1}S_{2j+1-2i}\right). \tag{7}$$

If $i \geq j$, combining Lemma 2.3 and Lemma 2.4, we get

$$\begin{aligned} A_{2i}S_{2j-2i} &\equiv a_{2i}c_{2j-2i}p^2 + a'_{2i}c_{2j-2i}p^3 \pmod{p^4}, \\ A_{2i-1}S_{2j+1-2i} &\equiv 0 \pmod{p^4}. \end{aligned}$$

So by (7) and Lemma 2.4,

$$A_{2j} \equiv -\frac{1}{2j}\left[e_{2j}p + e'_{2j}p^3 + \sum_{i=1}^{j-1}(a_{2i}c_{2j-2i}p^2 - a'_{2i}c_{2j-2i}p^3)\right] \pmod{p^4}. \tag{8}$$

If $n = 2j$, we have

$$(2j-1)A_{2j-1} = -S_{2j-1} - \sum_{i=1}^{j-1}A_{2i}S_{2j-1-2i} + \sum_{i=1}^{j-1}A_{2i-1}S_{2j-2i}. \tag{9}$$

Now proceed as above. If $i \geq j$, by Lemma 2.3 and Lemma 2.4 we have

$$\begin{aligned} A_{2i}S_{2j-1-2i} &\equiv a_{2i}d_{2j-1-2i}p^3 \pmod{p^4}, \\ A_{2i-1}S_{2j-2i} &\equiv b_{2i-1}c_{2j-2i}p^3 \pmod{p^4}. \end{aligned}$$

Then by (9) and Lemma 2.4,

$$A_{2j-1} \equiv -\frac{1}{2j-1}\left[b_{2j-1}p^2 + \sum_{i=1}^{j-1}\left(a_{2i}d_{2j-1-2i} - b_{2i-1}c_{2j-2i}\right)p^3\right] \pmod{p^4}. \tag{10}$$

The rest of proof can be completed by the same method to the one used above. We now apply this argument again modulo p^5. If n is odd, combining Lemma 2.3 and Lemma 2.4, for $j \in \{1, 2, \ldots, \frac{n-1}{2}\}$, we get

$$\begin{aligned} A_{2j}S_{n-1-2j} &\equiv -\frac{1}{2j}\left[e_{2j}p + e'_{2j}p^3 + \sum_{i=1}^{j-1}(a_{2i}c_{2j-2i}p^2 - a'_{2i}c_{2j-2i}p^3)\right] \\ &\qquad \times \left(e_{n-1-2j}p + e'_{n-1-2j}p^3\right) \\ &\equiv -\frac{1}{2j}e_{2j}e_{n-1-2j}p^2 - \frac{1}{2j}e_{n-1-2j}\sum_{i=1}^{j-1}a_{2i}c_{2j-2i}p^3 \\ &\qquad - \frac{1}{2j}\left(2e_{2j}e'_{n-1-2j} - e_{n-1-2j}\sum_{i=1}^{j-1}a'_{2i}c_{2j-2i}\right)p^4 \pmod{p^5}, \\ A_{2j-1}S_{n-2j} &\equiv b_{2j-1}d_{n-2j}p^4 \pmod{p^5}. \end{aligned}$$

Now by (6) and Lemma 2.5,

$$\begin{aligned}(n-1)A_{n-1} &= -S_{n-1} + A_{n-2}S_1 - \sum_{j=1}^{\frac{n-3}{2}} \left(A_{2j}S_{n-1-2j} - A_{2j-1}S_{n-2j}\right)\\ &\equiv -g_{n-1}p - Cp^2 - Dp^3 + Ep^4 \pmod{p^5},\end{aligned}$$

where

$$\begin{aligned}C &= -\sum_{j=1}^{\frac{n-3}{2}} \frac{1}{2j} e_{2j}e_{n-1-2j},\\ D &= g'_{n-1} - \sum_{j=1}^{\frac{n-3}{2}} \frac{1}{2j} e_{n-1-2j} \sum_{i=1}^{j-1} a_{2i}c_{2j-2i},\\ E &= \sum_{j=1}^{\frac{n-1}{2}} b_{2j-1}d_{n-2j} + \sum_{j=1}^{\frac{n-3}{2}} \frac{1}{2j}\left(2e_{2j}e'_{n-1-2j} - e_{n-1-2j}\sum_{i=1}^{j-1} a'_{2i}c_{2j-2i}\right).\end{aligned}$$

From $(n-1,p)=1$ and Lemma 2.2,

$$H_p(n) \equiv -\frac{n!}{n-1}\left(g_{n-1} + Cp + Dp^2 - Ep^3\right) \pmod{p^4}.$$

If n is even and $j \le \frac{n-2}{2}$, combining Lemma 2.3 and Lemma 2.4,

$$\begin{aligned}A_{2j}S_{n-1-2j} &\equiv \left(a_{2j}p + a'_{2j}p^2\right) \times f_{n-1-2j}p^2\\ &\equiv a_{2j}f_{n-1-2j}p^3 + a'_{2j}f_{n-1-2j}p^4 \pmod{p^5},\\ A_{2j-1}S_{n-2j} &\equiv -\frac{1}{2j-1}c_{n-2j}\sum_{i=1}^{j-1}\left(a_{2i}d_{2j-1-2i} - b_{2i-1}c_{2j-2i}\right)p^4\\ &\quad - \frac{1}{2j-1}b_{2j-1}c_{n-2j}p^3 \pmod{p^5}.\end{aligned}$$

Then by (6) and Lemma 2.5,

$$\begin{aligned}(n-1)A_{n-1} &= -S_{n-1} - \sum_{j=1}^{\frac{n-2}{2}} \left(A_{2j}S_{n-1-2j} - A_{2j-1}S_{n-2j}\right)\\ &\equiv -h_{n-1}p^2 - Fp^3 - Gp^4 \pmod{p^5},\end{aligned}$$

where

$$\begin{aligned}F &= \sum_{j=1}^{\frac{n-2}{2}} \left(a_{2j}f_{n-1-2j} + \frac{1}{2j-1}b_{2j-1}c_{n-2j}\right),\\ G &= h'_{n-1} + \sum_{j=1}^{\frac{n-2}{2}}\left(a'_{2j}f_{n-1-2j} + \frac{1}{2j-1}c_{n-2j}\sum_{i=1}^{j-1}\left(a_{2i}d_{2j-1-2i} - b_{2i-1}c_{2j-2i}\right)\right).\end{aligned}$$

From $(n-1,p)=1$ and Lemma 2.2,

$$H_p(n) \equiv -\frac{n!}{n-1}\left(h_{n-1}p + Fp^2 + Gp^3\right) \pmod{p^4}. \qquad \square$$

All of the above numbers $a, b, c, \ldots$, whose definition can be found in Section 2, are rational linear combinations of the divided Bernoulli numbers β_n.

By a suitable modification to the proof of Theorem 3.1, we can get the result of congruence $H_p(n)$ modulo p^3, p^2 and p. The congruences of $H_p(n)$ modulo p^2 and p have been proved by Cai and Xia in different methods from this paper.

Corollary 3.1. *Let $p > 7$ be a prime and $n \in \{3, 4, \ldots, p-4\}$. Then*

$$H_p(n) \equiv \begin{cases} -\frac{n!}{n-1}\left(f_{n-1}p + Ip^2\right) \pmod{p^3}, & 2 \mid n, \\ -\frac{n!}{n-1}\left(-e_{n-1} + \sum\limits_{j=1}^{\frac{n-3}{2}} a_{2j}c_{n-1-2j}p + Kp^2\right) \pmod{p^3}, & 2 \nmid n, \end{cases}$$

where

$$I = \sum_{j=1}^{\frac{n-2}{2}} (j-i)\,\beta_{p-1-2i}\left[(2j-2i-1)\,\beta_{p-1-2(j-i)} - \frac{2i\beta_{p-1-2(j+i)}}{2i+1}\right],$$

$$K = e'_{n-1} + \sum_{j=1}^{\frac{n-2}{2}} (n-1-2j)\,\beta_{p-n+2j}a'_{2j}.$$

Proof. Combining (8), (10) and Lemma 2.2, we can prove the corollary. □

Corollary 3.2 (Xia and Cai [6])**.** *Let p be an odd prime such that $p > 3$. For every positive integer $n \le p-2$, we have*

$$H_p(n) \equiv \begin{cases} n!\beta_{p-n} \pmod{p}, & 2 \nmid n, \\ -\frac{np}{2}n!\beta_{p-n-1} \pmod{p^2}, & 2 \mid n, \end{cases}$$

and

$$H_p(n) \equiv n!\left(2\beta_{p-n} - \beta_{2p-n-1}\right) + \frac{n!p}{2}\sum_{\substack{2 \le i \le n-3 \\ i \text{ even}}} \beta_{p-i-1}\beta_{p-n+i} \pmod{p^2},$$

where $3 \le n \le p-3$ is an odd integer and $p > 3$ is prime.

Proof. Using Kummer's congruence

$$\beta_{k(p-1)+b} \equiv \beta_b \pmod{p}$$

where $k = 0, 1, 2, \ldots$, Theorem 3.1 and Corollary 3.1 imply the corollary. □

By the relationship between the $H_p(n)$ and $H_{\frac{p+1}{2}}(n)$, combined with Corollary 3.1 we can obtain the following congruences.

Corollary 3.3. *Let $p > 7$ be a prime and $3 \le n \le \frac{p+1}{2}$. Then*

$$H_{\frac{p+1}{2}}(n) \equiv \begin{cases} -\frac{2n!}{(n-1)(p+1)}\left(f_{n-1}p^2 + Ip^3\right) \pmod{p^3}, & 2 \mid n, \\ -\frac{2n!}{(n-1)(p+1)}\left(-e_{n-1}p + \sum\limits_{j=1}^{\frac{n-3}{2}} a_{2j}c_{n-1-2j}p^2 + Kp^3\right) \pmod{p^3}, & 2 \nmid n. \end{cases}$$

Proof. In order to prove the theorem, we need to know the relationship between $H_p(n)$ and $H_{\frac{p+1}{2}}(n)$. Let

$$A_n\left(\frac{p+1}{2}\right) = \sum_{1\leq u_1<u_2<\cdots<u_n\leq\frac{p+1}{2}} \frac{1}{u_1u_2\cdots u_n}.$$

By Lemma 2.2 and Lemma 2.6, we obtain

$$\begin{aligned} H_p(n) &= \frac{n!}{p}A_{n-1} \\ &= (-1)^{n-1}\frac{n!}{p}\sum_{\substack{\alpha_1+2\alpha_2+\cdots+(n-1)\alpha_{n-1}=n-1\\ \alpha_1,\ldots,\alpha_{n-1}>0}} T\left(\alpha_1,\ldots,\alpha_{n-1}\right) S_1^{\alpha_1}S_2^{\alpha_2}\cdots S_{n-1}^{\alpha_{n-1}}, \qquad (11) \\ H_{\frac{p+1}{2}}(n) &= \frac{2n!}{p+1}A_{n-1}\left(\frac{p+1}{2}\right) \\ &= (-1)^{n-1}\frac{2n!}{p+1}\sum_{\substack{\alpha_1+2\alpha_2+\cdots+(n-1)\alpha_{n-1}=n-1\\ \alpha_1,\ldots,\alpha_{n-1}>0}} T\left(\alpha_1,\ldots,\alpha_{n-1}\right) S_1^{\alpha_1}S_2^{\alpha_2}\cdots S_{n-1}^{\alpha_{n-1}}. \qquad (12) \end{aligned}$$

Combining (11) and (12) can get

$$H_{\frac{p+1}{2}}(n) = \frac{2n!}{n!}\frac{p}{p+1}H_p(n).$$

Then by Corollary 3.1 and above equation, it is easy to prove this corollary. □

4 Conclusion

The main results of this paper are a theorem and three corollaries. Theorem 3.1 establishes a supercongruence of harmonic sums $H_p(n)$ on modulo p^3 and p^4. The corollaries are generalizations of Xia and Cai's congruences in [6].

Acknowledgements

This work is supported by Foundation of Liaoning Educational Committee, Project 2019LNJC08.

References

[1] Carlitz, L. (1953). A theorem of Glaisher. *Canadian Journal of Mathematics*, 5(1), 35–39.

[2] Ireland, K., & Rosen, M. (1990). *A Classical Introduction to Modern Number Theory*. Springer, New York.

[3] Meštrović, R. (2013). Some Wolstenholme type congruences. *Mathematics for Applications*, 2, 35–42.

[4] Shen, Z. (2011). Some congruences involving multiple harmonic sums. *Journal of Zhejiang International Studies University*, 5, 95–99.

[5] Sun, Z. (2000). Congruences concerning Bernoulli numbers and Bernoulli polynomials. *Discrete Applied Mathematics*, 105(1), 193–223.

[6] Xia, B., & Cai, T. (2010). Bernoulli numbers and congruences for harmonic sums. *International Journal of Number Theory*, 6(4), 849–855.

[7] Zhao, J. (2006). Bernoulli numbers, Wolstenholme's theorem, and p^5 variations of Lucas' theorem. *Journal of Number Theory*, 123(1), 18–26.

[8] Zhou, X., & Cai, T. (2007). A generalization of a curious congruence on harmonic sums. *Proceedings of the American Mathematical Society*, 135(5), 1329–1333.