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On certain equations and inequalities involving the arithmetical functions $\varphi(n)$ and $d(n)$ – II

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Abstract: In papers [3] and [5] we have studied certain equations and inequalities involving the arithmetical functions $\varphi(n)$ and $d(n)$. In this paper we will consider some other equations. Some open problems will be stated, too.
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1 Introduction

Let $\varphi(n)$ and $d(n)$ denote the Euler totient function and the number of divisors functions, respectively. It is well-known that $\varphi(r) = d(1) = 1$, and for $n = p_1^{a_1} \dots p_r^{a_r} > 1$ (prime factorization), we have

$$\varphi(n) = p_1^{a_1-1} \cdots p_r^{a_r-1} \cdot (p_1 - 1) \cdots (p_r - 1) \text{ and } d(n) = (a_1 + 1) \cdots (a_r + 1) \tag{1}$$

with $p_1(i = \overline{1, r})$ distinct primes, and $a_i(i = \overline{1, r})$ positive integers. In paper [3] we have studied the solutions of the equation $\varphi(n) + d(n) = n$, and proved certain related inequalities. In paper [5] we have solved the equation $\varphi(n) + d(n) = \frac{n}{2}$, and studied the related inequalities. Another equation solved in [5] was $\varphi(n) + d^2(n) = 2n$.

The aim of this paper is to consider more equations for the arithmetical function $\varphi(n)$ and $d(n)$. Some open problems and conjectures will be stated, too.

2 Main results

First we study two equations, which can be solved by the methods of [3]:

Theorem 1. *The equation*

$$2\varphi(n) + d(n) = 2n \tag{2}$$

has the only solutions as $n =$ primes. The equation

$$\varphi(n) + 2d(n) = 2n \tag{3}$$

has the only solutions as: $n = 3, 4$.

Proof. Let $n = p =$ prime. Then, as $\varphi(p) = p - 1$ and $d(p) = 2$, as $2 \cdot (p - 1) + 2 = 2p$, clearly $n = p$ is a solution. Let now n be a composite number. Then, by the known inequalities (see [3]) $\varphi(n) \leq n - \sqrt{n}$ and $d(n) < 2\sqrt{n}$ we get $2\varphi(n) + d(n) < 2n - 2\sqrt{n} + 2\sqrt{n} = 2n$, so there are no composite solutions for (2).

By the same argument, if $n = p$ is a solution of (3), then $p - 1 + 4 = 2p$, so $p = 3$, and this is the only prime solution of (3). Now, if n is composite, then as $\varphi(n) + 2d(n) < n - \sqrt{n} + 4\sqrt{n}$ $= n + 3\sqrt{n}$, if $n + 3\sqrt{n} \leq 2n$ (i.e., $n \geq 3\sqrt{n}$, or $\sqrt{n} \geq 3$ or $n \geq 9$), there are no solutions. Now an easy verification for the composite numbers $n \in \{4, 6, 8\}$ we get that only $n = 4$ is a solution. This finishes the proof of Theorem 1. □

Theorem 2. *The solution of the equation*

$$2\varphi(n) + d(n) = n \tag{4}$$

are $n = 18$ and $n = 8 \cdot p$, where $p \geq 3$ is a prime.

Proof. An easy verification shows that $n = p$ (prime) and $n = 2^k$ $(k \geq 1)$ are not solutions. Indeed, $2 \cdot (p - 1) + 2 = 2p > p$ and $2 \cdot 2^{k-1} + k + 1 > 2^k$. Let now $n = 2^k \cdot p$, when we get $2^k \cdot (p - 1) + 2 \cdot (k + 1) = 2^k \cdot p$, or $2^{k-1} = k + 1$. This has the only solution $k = 3$, as $k \neq 1, k \neq 2$, and for $k \geq 4$ one has $2^{k-1} > k + 1$. Therefore, $n = 2^3 \cdot p = 8p$ is always a solution for $p \geq 3$ a prime. Let now $n = 2^k \cdot N$ where N is odd and composite. The equation (3) becomes

$$2^k \cdot \varphi(N) + (k + 1)d(N) = 2^k \cdot N. \tag{5}$$

Now, as $k + 1 \leq 2^k$ and $\varphi(N) + d(N) \leq N$ (with equality only for $N = 9$, see [3]), we get that the left side of (5) is $\leq$ right side, with equality only for $k = 1$ and $N = 9$. Thus the solution $n = 2^1 \cdot 9 = 18$ is obtained. Finally, if n is odd remark that $d(n)$ should be odd, so $n = m^2$. Then we get that m should divide $d(m^2)$, so $m = 3$ (see [3]), and we do not obtain solutions. □

Theorem 3. *The equation*

$$\varphi(n) + 2d(n) = n \tag{6}$$

has the only solutions as $n = 14, 18, 20, 24$.

Proof. As in the proof of Theorem 2, it is immediate that $n = p$ (prime) and $n = 2^k$ are not solutions. On the other hand, it is also immediate that the only solution of the form $n = p \cdot 2^k$ is $n = 3 \cdot 2^3 = 24$. Indeed, the equation in this case becomes $2^{k-1} \cdot (p+1) = 4 \cdot (k+1)$, and as $2^{k-1} \geq k+1$ for $k \geq 3$ and $p+1 \geq 4$, the result follows for $k \geq 3$. For $k = 1$ we get the solution $p = 7$, while for $k = 2$ we get $p = 5$. Therefore, all solutions of the form $n = 2^k \cdot p$ are $n = 14, 20, 24$. Let now $n = 2^k \cdot N$, with $N \geq 3$ odd and composite. The equation (6) becomes

$$2^{k+1} \cdot \varphi(N) + (k+1)d(N) = 2^k \cdot N. \tag{7}$$

Remark that for $k \geq 3$ one has $k+1 \leq 2^{k-1}$, so the left side of (7) is $\leq 2^{k-1} \cdot [\varphi(N) + d(N)] \leq 2^{k-1} \cdot N$, by the known inequality $\varphi(N) + d(N) \leq N$. Thus, if (7) is true, then $2^k \cdot N \leq 2^{k-1} \cdot N$, which is impossible.

For $k = 1$ the equation becomes

$$\varphi(N) + 4d(N) = 2N \tag{8}$$

By the inequality $\varphi(N) + d(N) \leq N$ (with equality only for $N = 9$), remark that if the following inequality would be true:

$$3d(N) \leq N, \tag{9}$$

then the equation could have a single solution, namely $N = 9$.

As $d(N) < 2\sqrt{N}$, and $2\sqrt{N} \leq \frac{N}{3}$ if $N \geq 36$, a simple verification for $N \in \{9, 15, 21, 25, 27, 33, 35\}$ shows that (9) holds true, with equality only for $N = 9$. Thus we have obtained in this case the solution $n = 2 \cdot 9 = 18$.

Let now $k = 2$, when we get $n = 4 \cdot N$, so the equation becomes $2\varphi(N) + 6d(N) = 4N$, or

$$\varphi(N) + 3d(N) = 2N. \tag{10}$$

As $\varphi(N) + d(N) \leq N$, and $2d(N) < N$ by (9) (as $\frac{N}{3} < \frac{N}{2}$), clearly (10) is impossible. Thus, in this case no solution is obtained, and as clearly there are no odd solutions, the proof of Theorem 3 is complete. □

Theorem 4. *The equation*

$$(\varphi(n))^{\varphi(n)} + (d(n))^{d(n)} = n^n \tag{11}$$

has no solutions.

The only solution to the equation

$$(\varphi(n))^{\varphi(n)} \cdot (d(n))^{d(n)} = n^n \tag{12}$$

is $n = 2$

Proof. Equation (11) is a particular case of a general equation $a^a + b^b = c^c$. □

Lemma 1. *The equation*

$$a^a + b^b = c^c \tag{13}$$

cannot be solved in positive integers a, b, c.

Proof. Suppose that $b \geq a$. Then $a^a + b^b \leq 2b^b$. On the other hand, clearly from (13) it follows that $c > b$, so $c \geq b+1$ as c, b are positive integers. This implies

$$c^c \geq (b+1)^{b+1} = (b+1)\cdot(b+1)^b > (b+1)\cdot b^b \geq 3b^b,$$

if $b \geq 2$. This is a contradiction, as the left side of (13) is $\leq 2b^b$. If $b = 1$, then $a = 1$ and (13) becomes $2 = c^c$, which has no solutions, as for $c = 1, c^c = 1$ and for $c \geq 2$, one has $c^c \geq 4$ □

Remark 1. *The similar equation*

$$a^a \cdot b^b = c^c \tag{14}$$

is not known to be solved in the general cases (see, e.g., [1], Section D13). Thus equation (12) *must be treated separately.*

Using the representation (1), *as equation* (12) *implies*

$$(\varphi(n))^{\varphi(n)} \mid n^n, \tag{15}$$

we get that

$$p_1^{na_1-\varphi(n).(a_1-1)} \cdots p_r^{n.a_r-\varphi(n).(a_r-1)} = K\cdot(p_1-1)^{\varphi(n)} \cdots (p_r-1)^{\varphi(n)} \tag{16}$$

where $K \geq 1$ is an integer. Let $p_1 < p_2 < \cdots < p_r$, $r \geq 2$. As $p_r - 1 > p_{r-1}$, excepting $p_1 = 2$, $p_2 = 3$, when $p_2 - 1 = p_1$; clearly $p_r - 1$ cannot divide the left side of (16)*, thus we get that we can have only $p_r = 3$, $p_{r-1} = 2$; i.e., n has the form $n = 2^a.3^b$.*

In this case, equation (12) *becomes*

$$(a+1)^{d(n)}\cdot(b+1)^{d(n)} = 2^{a(n-\varphi(n))}\cdot 3^{b(n-\varphi(n))}. \tag{17}$$

As, $n - \varphi(n) = 2^a\cdot 3^b - 2^a\cdot 3^{b-1} = 2^{a+1}\cdot 3^{b-1} = 2\varphi(n)$, (17) *can be written as*

$$(a+1)^{d(n)}\cdot(b+1)^{d(n)} = 2^{2a\varphi(n)}\cdot 3^{2b\varphi(n)}. \tag{18}$$

Now, $2^a \geq a+1$ and $3^b > b+1$, it will be sufficient to consider the inequality

$$2\varphi(n) \geq d(n). \tag{19}$$

Lemma 2. *Inequality* (19) *holds true for any $n \geq 1$, with equality only for $n = 2$ and $n = 6$.*

Proof. Remark that $2p_1^{a_1-1}\cdot(p_1-1) \geq a_1+1$ for $p_1 \geq 2$ and $p_2^{a_2-1}\cdot(p_2-1) \geq a_2+1$ for $p_2 \geq 3$, with equality only for $p_1 = 2$, $a_1 = 1$ and $p_2 = 3$, $a_2 = 1$. On the other hand, $p_3^{a_3-1}\cdot(p_3-1) > a_3+1$, for $p_3 \geq 5$ and $a_3 \geq 1$, etc, so

$$2\varphi(n) = 2p_1^{a_1-1}\cdot(p_1-1)\cdot p_2^{a_2-1}\cdot(p_2-1)\cdots p_r^{a_r-1}\cdot(p_r-1) \geq (a_1+1)\dots(a_r+1),$$

with equality only for $r = 1$ and $r = 2$; when we get the solutions $n = 2$ and $n = 6$.

By Lemma 2 it follows that (18) is impossible so equation (12) cannot be solved when $r \geq 2$.

As for $n \geq 3$, $\varphi(n)$ is even, n should be even, and the only possibility is $n = 2$. □

Theorem 5. *The equation*

$$\varphi(n) + (d(n))^2 = n \tag{20}$$

has the solutions $n = 68, 128, 384, 864$.

Proof. First we prove that equation (20) cannot have odd solutions. Let $n = p_1^{a_1} \cdots p_r^{a_r}$ be the prime factorization of n, with $3 \le p_1 < \cdots < p_r$. Then (20) can be written as

$$(a_1+1)^2 \cdots (a_r+1)^2 = p_1^{a_1-1} \cdots p_r^{a_r-1} \cdot [p_1 \cdots p_r - (p_1-1)\cdots(p_r-1)]. \tag{21}$$

Clearly, each part of the product in the right side of (21) is odd, so the right side of (21) is an odd number. Then, if $a_r = 1$, then (21) is impossible, as the left side is even. Suppose that $a_r > 1$. Then, $(p_r, (p_1-1)\cdots(p_r-1)) = 1$, the right side of (21) can be written as $p_r^{a_r-1} \cdot X$, where $(p_r, X) = 1$. By (21), this should be a perfect square, so we should have $p_r^{a_r-1} = A^2$, $X = B^2$, where $(A, B) = 1$. But p_r being prime, $p_r^{a_r-1} = A^2$ implies that $a_r - 1$ is even, so $a_r + 1$ is even, too. Thus the left side of (21) is again even number, a contradiction.

Let now n be an even solution to (20). As n is even, by the known inequality $\varphi(n) \le \frac{n}{2}$ we get from (20) that $\frac{n}{2} \le d^2(n)$, so $d(n) \ge \sqrt{\frac{n}{2}}$. But from the known inequality $d(n) < 4 \cdot \sqrt[3]{n}$, if $4 \cdot \sqrt[3]{n} < \sqrt{\frac{n}{2}}$, this will be impossible, as then we would have $d(n) < \sqrt{\frac{n}{2}}$. As $4 \cdot \sqrt[3]{n} < \sqrt{\frac{n}{2}}$ is equivalent with $n > 8 \cdot 4^6 = 32768 = n_0$, this means that for such values $n > n_0$, there are no even solutions. It is not difficult to obtain by a computer that for n even and $n \le 32766$ the only solutions to (20) are $n = 68, 128, 384$ and 864. This finishes the proof of Theorem 5. □

In paper [5] we have studied the equation $\varphi(n) + d(n) = \frac{n}{2}$, which had a single solution. Now, we will consider the equation $\varphi(n) + d(n) = \frac{n}{4}$.

Theorem 6. *The equation*

$$\varphi(n) + d(n) = \frac{n}{4} \tag{22}$$

has all solutions of the form $n = 4m$*, where* m *is an even number. One has* $\omega(m) \ge 4$.

Proof. Clearly, n is a multiple of 4, let $n = 4m$. Then equation (22) becomes

$$\varphi(4m) + d(4m) = m. \tag{23}$$

Let us now suppose that m is odd. Then, as $(4,m)=1$, (23) can be written as $2\varphi(m) + 3d(m) = m$. This implies that $d(m)$ should be odd, so m should be a perfect square: $m = M^2$. As $\varphi(M^2) = M\varphi(M)$, this implies that

$$M \mid 3d(M^2). \tag{24}$$

First suppose that $(M, 3) = 1$. Then $M \mid d(M^2)$, and it is shown in [3] that $M = 3$, which is impossible. Let $M = 3 \cdot K$ and let $K = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ be the prime factorization of K. As (24) can be written now as

$$K \mid d(9K^2) \tag{25}$$

we get $p_1^{a_1} \cdots p_r^{a_r} \mid 3 \cdot (2a_1+1) \dots (2a_r+1)$. Remark that for any $a_1, a_2 \ge 1$ one has $5^{a_1} \cdot 7^{a_2} \ge 3 \cdot (2a_1+1)(2a_2+1)$, and as $p_r^{a_r} > 2a_r + 1$ for any $p_r \ge 11$, clearly the above divisibility cannot be true if $p_1 \geq 5$, $p_r > p_{r1} > \cdots > p_2$, since $5^{a_1} \geq 3 \cdot (2a_1 + 1)$ for $a_1 \geq 2$; we should have $a_1 = 1$ so $K = p$ (prime), and $p \mid d(3^2 \cdot p^2) = 9$ is possible only when $p = 3$. Thus, $K = 3$ and $M = 9$ and we do not get a solution.

If $p_1 = 3$, then $K = 3^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r}$ and (25) becomes

$$3^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r} \mid (2a_1 + 3)(2a_2 + 1) \cdots (2a_1 + 1). \tag{26}$$

Now, if $a_1 \geq 2$, by $3^{a_1} > 2a_1 + 3$ and $p_2^{a_2} \geq 5^{a_2} > 2a_2 + 1, \ldots$, clearly (26) will be impossible. If $a_1 = 1$, (26) becomes

$$3 \cdot p_2^{a_2} \cdots p_r^{a_r} = 5(2a_2 + 1) \cdots (2a_r + 1)$$

and as $3 \cdot 5^{a_2} \geq 5(2a_2 + 1)$, $p_3^{a_3} \geq 7^{a_3} > 2a_3 + 1, \ldots$, we can have only $a_2 = 1$ and $r = 2$, so we get again $K = 3$ and $K = 9$. These do not provide again solutions to (23). Thus, finally m cannot be odd.

Let us now assume $\omega(m) = 1$, i.e., $m = 2^k$ (as m is even). Then, as $2^{k+1} + k + 3 > 2^k$, clearly (23) is impossible.

Let $\omega(m) = 2$, i.e., $m = 2^k \cdot p^a$ where $p \geq 3$ is a prime. Then $4m = 2^{k+2} \cdot p^a$ and after elementary transformations (23) becomes $2^k \cdot p^{a-1} \cdot (2 - p) = (k + 3)(a + 1)$, which is impossible, as $2 - p < 0$.

Let $\omega(m) = 3$, i.e., $m = 2^k \cdot p^a \cdot q^b$ where $p \geq 3$, $q \geq 5$ are primes. As $4m = 2^{k+2} \cdot p^a \cdot q^b$, after elementary transformations, the equation (23) can be written as

$$2^k \cdot p^{a-1} \cdot q^{k-1} \cdot [2(p + q + 1) - p \cdot q] = (k + 3)(a + 1)(b + 1). \tag{27}$$

Remark that $pq - 2(p + q - 1) = pq - 2p - 2q + 2 = (p - 2)(q - 2) - 2 \geq 3 - 2 = 1$, as $p - 2 \geq 1$, $q - 2 \geq 3$. Thus the left side of (27) is $\leq -1 < 0$, a contradiction. □

The following theorem offers particular solutions to equation (23).

Theorem 7. *All solutions to* (23) *of the type* $m = 2^a \cdot 3^b \cdot 5^c \cdot p$ *, where* $p \geq 7$ *is a prime, are* $m = 2^2 \cdot 3^2 \cdot 5 \cdot 11$, $m = 2^2 \cdot 3 \cdot 5^2 \cdot 13$ *and* $m = 2^3 \cdot 3^2 \cdot 5 \cdot 13$.

A solution with $\omega(m) = 5$ *is* $m = 2^4 \cdot 3 \cdot 5 \cdot 17 \cdot 251$.

Proof. Equation (23) for $m = 2^a \cdot 3^b \cdot 5^c \cdot p$ can be written as

$$2^a \cdot 3^{b-1} \cdot 5^{c-1} \cdot (16 - p) = 2(a + 3)(b + 1)(c + 1). \tag{28}$$

This implies $p \leq 13$. First we prove that $p = 7$ is impossible. Indeed, in this case (28) becomes

$$2^{a-1} \cdot 3^{b+1} \cdot 5^{c-1} = (a + 3)(b + 1)(c + 1). \tag{29}$$

Remark that $5^{c-1} > c + 1$ if $c \geq 2$, $3^{b+1} > b + 1$ for $b \geq 1$, $2^{a-1} \geq a + 3$ for $a \geq 4$. If $c = 1$, then $2^{a-1} \cdot 3^{b+1} = 2(a + 3) \cdot (b + 1)$ can be rewritten as $2^{a-2} \cdot 3^{b+1} = (a + 3)(b + 1)$. For $a \geq 4$ this cannot provide solutions, and also for $a = 1$, $a = 2$, $a = 3$ we cannot find solutions. Thus for $b \geq 1$, $c \geq 1$ and $a \geq 4$ there are no solutions. For $a = 1$, the left side is odd, the right side even, for $a = 2$ we get $2 \cdot 3^{b+1} \cdot 5^{c-1} = 5(b + 1) \cdot (c + 1)$ and as $2 \cdot 5^{c-2} \geq c + 1$ for any $c \geq 3$, we should consider $c = 2$, when we get $2 \cdot 3^{b+1} = (b+1) \cdot 3$ and this cannot have solutions as $3^b > b + 1$. Finally, for $a = 3$ we get the equation $2^2 \cdot 3^{b+1} \cdot 5^{c-1} = 2 \cdot 3 \cdot (b+1) \cdot (c+1)$ or $2 \cdot 3^b \cdot 5^{c-1} = (b+1) \cdot (c+1)$. As $3^b > b+1, 2 \cdot 5^{c-1} \geq c+1$, this equation cannot have solutions.

For $p = 11$ the equation (28) becomes

$$2^{a-1} \cdot 3^{b-1} \cdot 5c = (a+3)(b+1)(c+2) \tag{30}$$

and as above, we can show, by using elementary inequalities, that $a = 2, b = 2, c = 1$.

For $p = 13$, equation (28) becomes

$$2^{a-1} \cdot 3^b \cdot 5^{c-1} = (a+3)(b+1)(c+1), \tag{31}$$

and by the above elementary considerations (which we omit here) it can be shown that $a = 2$, $b = 1$, $c = 2$ or $a = 3$, $b = 2$, $c = 1$.

Let now m have the form

$$m = 2^k \cdot p_1 \cdots p_r \tag{32}$$

where $p_1, \ldots, p_r$ are distinct odd primes.

In this case, equation (23) becomes

$$2^{k-r} \cdot [p_1 \cdots p_r - 2(p_1 - 1) \cdots (p_r - 1)] = k + 1. \tag{33}$$

Therefore, for $r \geq k + 1$, (23) has no solutions of the form (32). When $k = r$, equation (33) becomes

$$p_1 \cdots p_r - 2(p_1 - 1) \cdots (p_r - 1) = r + 1. \tag{34}$$

When $r = 4, p_1 = 3, p_2 = 5$ we get from (34) the equation $15p_3p_4 - 16(p_3 - 1)(p_4 - 1) = 5$, which can be written equivalently as $(p_3 - 16)(p_4 - 16) = 235$.

As $235 = 5 \cdot 47$ for $p_3 - 16 = 1$, $p_4 - 16 = 235$, we get the solutions $p_3 = 17$, $p_4 = 251$, which is a prime. Therefore, a solution of the form (32) is

$$m = 2^4 \cdot 3 \cdot 5 \cdot 17 \cdot 251. \qquad \square$$

Conjecture 1. *The equation* (23) *has infinitely many solutions.*

Conjecture 2. *The equation* (34) *has infinitely many solutions in primes* $p_1, \ldots, p_r$.

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