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A generalized computation procedure for Ramanujan-type identities and cubic Shevelev sum

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Abstract: A generalized Computation procedure for construction of the Ramanujan-type from a given general cubic equation and a cosine Ramanujan-type identity is developed from detailed analyses of the properties of Ramanujan-type cubic equations. Examples are provided together with cubic Shevelev sums.

Keywords: Ramanujan cubic polynomials, Ramanujan cubic polynomials of the second kind, Cubic Shevelev sum.

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1 Introduction

Babylonian mathematicians were familiar with cubic equations and the Iranian scholar Omar Khayyam (1048–1122) solved many cubic equations with the aid of algorithms with a conic section base [8]. However, in the West, Geronimo Cardano (1501–1576) seems to have been the first to have published an analysis of the cubic equation

$$x^3 + Px + Q = 0.$$

as in equation (4) below [1], though Cardano admitted that he had obtained a hint for this from Niccolo Tartaglia (c.1500–1557) [3]. Cardano quickly came across Ramanujan-type cubic equations, but he was unable to handle the square root of a negative number, a “casus irreducubilis” as he described it. In this paper, we are not so restricted, and we link these developments of Cardano-type cubic equations in the sixteenth century with those of Ramanujan in the twentieth century and Shevelev in this century.

Theorem 1.1 ([9]). *Let $\alpha,\ \beta$, and γ denote the distinct roots of a cubic equation*

$$x^3 - ax^2 + bx - 1 = 0. \tag{1}$$

If $\alpha,\ \beta$, and γ are real, then

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{a + 6 + 3t} \qquad \text{and} \tag{2}$$

$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = \sqrt[3]{b + 6 + 3t}, \tag{3}$$

where t is the only real root of the associated Ramanujan equation

$$t^3 - 3(a + b + 3)t - (ab + 6(a + b) + 9) = 0 \tag{4}$$

with $t \neq \alpha,\ \beta,\ \gamma$.

Liao, Saul, and Shiue [7] (see also Chen [2]):

Theorem 1.2 ([7]). *Given a polynomial equation of degree three $x^3 - 3rsx + rs(r + s) = 0$ with real coefficients, then the three solutions to this equation are*

$$x = -\sqrt[3]{rs}\left(\sqrt[3]{r} + \sqrt[3]{s}\right),\ -\sqrt[3]{rs}\left(\omega\sqrt[3]{r} + \omega^2\sqrt[3]{s}\right),\ -\sqrt[3]{rs}\left(\omega^2\sqrt[3]{r} + \omega\sqrt[3]{s}\right),$$

where $\omega = \frac{-1+\sqrt{3}i}{2}$.

When $r \neq s$, and r and s are complex conjugates, then the three distinct solutions are

$$-2\sqrt{rs}\cos\left(\frac{\theta}{3}\right),\ -2\sqrt{rs}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right),\ -2\sqrt{rs}\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right),$$

where $\theta = \mathrm{Arg}(r)$.

2 Main results

We will use the next two theorems from Shiue et al. [12] to prove our main theorems.

Theorem 2.1 ([12]). *Let* $f(x) = x^3 - ax^2 + bx - 1 = 0$, $a, b \in \mathbb{R}$, a, b *not both* 0. *Let* $f(x)$ *have three distinct real roots* α, β, *and* γ. *Let* $t^3 - 3(a + b + 3)t - (ab + 6(a + b) + 9) = 0$ *be the associated Ramanujan equation. Then*

$$t = \sqrt[3]{\frac{ab + 6(a + b) + 9 + \Delta}{2}} + \sqrt[3]{\frac{ab + 6(a + b) + 9 - \Delta}{2}}, \tag{5}$$

where the discriminant of $f(x)$ *is* $\Delta^2 = (ab)^2 - 4(a^3 + b^3) + 18ab - 27$. *Moreover,*

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{a + 6 + 3t},$$
$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = \sqrt[3]{b + 6 + 3t}.$$

Theorem 2.2 ([12]). *Let* $t^3 - 3(a + b + 3)t - (ab + 6(a + b) + 9) = 0$ *be a cubic equation, where* $a, b \in \mathbb{R}$, a, b *not both* 0. *Then only one root is real.*

In this paper, we will denote the discriminant of $f(x)$ as $D(f)$ other than Δ^2. Using Theorems 2.1 and 2.2, we give the result of obtaining Ramanujan-type identities given a general cubic equation $f(x) = x^3 + Ax^2 + Bx + C = 0$, $A, B, C \in \mathbb{R}$ and $C \neq 0$. Note that if the discriminant

$$D(f) = (AB)^2 - 4(A^3C + B^3) + 18ABC - 27C^2$$

of the cubic equation $f(x)$ is greater than zero, then $f(x) = 0$ has three distinct real roots [5].

Theorem 2.3. *Let* $f(x) = x^3 + Ax^2 + Bx + C = 0$, $A, B, C \in \mathbb{R}$, A, B *not both* 0, $C \neq 0$. *Let* $f(x)$ *have three distinct real roots* α, β, *and* γ. *Then*

(a) the associated Ramanujan equation is

$$t^3 - 3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right)t - \left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = 0; \tag{6}$$

(b) the only real root to the associated Ramanujan equation (6) *is*

$$t = \sqrt[3]{\frac{\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9 + \frac{\sqrt{D(f)}}{C}}{2}} + \sqrt[3]{\frac{\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9 - \frac{\sqrt{D(f)}}{C}}{2}}, \tag{7}$$

where

$$D(f) = (AB)^2 - 4(A^3C + B^3) + 18ABC - 27C^2; \tag{8}$$

(c) the Ramanujan-type identities are

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = -\sqrt[3]{A + 6\sqrt[3]{C} + 3t\sqrt[3]{C}}, \tag{9}$$

$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = -\frac{1}{\sqrt[3]{C}}\sqrt[3]{B + 6\sqrt[3]{C^2} + 3t\sqrt[3]{C^2}}. \tag{10}$$

Proof. Let $f(x) = x^3 + Ax^2 + Bx + C = 0$, $C \neq 0$. Dividing by $-C$ yields

$$\left(-\frac{x}{\sqrt[3]{C}}\right)^3 + \left(-\frac{A}{\sqrt[3]{C}}\right)\left(-\frac{x}{\sqrt[3]{C}}\right)^2 + \left(\frac{B}{\sqrt[3]{C^2}}\right)\left(-\frac{x}{\sqrt[3]{C}}\right) - 1 = 0.$$

Let $y = -\frac{x}{\sqrt[3]{C}}$, $a = \frac{A}{\sqrt[3]{C}}$, and $b = \frac{B}{C^{\frac{2}{3}}}$. Then

$$g(y) = y^3 - ay^2 + by - 1 = 0. \tag{11}$$

By Ramanujan [9], the associated Ramanujan equation of (11) is

$$t^3 - 3(a + b + 3)t - (ab + 6(a + b) + 9) = 0.$$

Thus the associated Ramanujan equation of $x^3 + Ax^2 + Bx + C = 0$ is

$$t^3 - 3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right)t - \left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = 0, \tag{12}$$

which is the result for (a).

For (b), by Theorem 2.1,

$$\begin{aligned}
D(g) &= (ab)^2 - 4(a^3 + b^3) + 18ab - 27 \\
&= \left(\frac{AB}{C}\right)^2 - 4\left(\frac{A^3}{C} + \frac{B^3}{C^2}\right) + \frac{18AB}{C} - 27 \\
&= \frac{1}{C^2}\left((AB)^2 - 4(A^3C + B^3) + 18ABC - 27C^2\right) \\
&= \frac{D(f)}{C^2} > 0,
\end{aligned}$$

since $f(x) = 0$ has three distinct real roots.

By Theorem 2.2, the associated Ramanujan equation $t^3-3(a+b+3)t-(ab+6(a+b)+9) = 0$ has only one real root t. Hence,

$$\begin{aligned}
t &= \sqrt[3]{\frac{ab + 6(a + b) + 9 + \sqrt{D(g)}}{2}} + \sqrt[3]{\frac{ab + 6(a + b) + 9 - \sqrt{D(g)}}{2}} \\
&= \sqrt[3]{\frac{\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9 + \frac{\sqrt{D(f)}}{C}}{2}} + \sqrt[3]{\frac{\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9 - \frac{\sqrt{D(f)}}{C}}{2}}
\end{aligned}$$

also has only one real root t. This concludes the proof for (b).

Lastly, for (c): note that the roots of (11) are $\alpha_y = -\frac{\alpha}{\sqrt[3]{C}}$, $\beta_y = -\frac{\beta}{\sqrt[3]{C}}$, and $\gamma_y = -\frac{\gamma}{\sqrt[3]{C}}$. Then, by Theorem 1.1, we have

$$\sqrt[3]{\alpha_y} + \sqrt[3]{\beta_y} + \sqrt[3]{\gamma_y} = \sqrt[3]{a + 6 + 3t}, \tag{13}$$

$$\frac{1}{\sqrt[3]{\alpha_y}} + \frac{1}{\sqrt[3]{\beta_y}} + \frac{1}{\sqrt[3]{\gamma_y}} = \sqrt[3]{b + 6 + 3t}, \tag{14}$$

where

$$t = \sqrt[3]{\frac{ab + 6(a + b) + 9 + \sqrt{D(g)}}{2}} + \sqrt[3]{\frac{ab + 6(a + b) + 9 - \sqrt{D(g)}}{2}}$$

and

$$D(g) = (ab)^2 - 4(a^3+b^3) + 18ab - 27 = \frac{D(f)}{C^2}.$$

Substituting $\alpha_y = -\frac{\alpha}{\sqrt[3]{C}}$, $\beta_y = -\frac{\beta}{\sqrt[3]{C}}$, $\gamma_y = -\frac{\gamma}{\sqrt[3]{C}}$, $a = \frac{A}{\sqrt[3]{C}}$, and $b = \frac{B}{\sqrt[3]{C^2}}$, we have

$$\sqrt[3]{-\frac{\alpha}{\sqrt[3]{C}}} + \sqrt[3]{-\frac{\beta}{\sqrt[3]{C}}} + \sqrt[3]{-\frac{\gamma}{\sqrt[3]{C}}} = \sqrt[3]{\frac{A}{\sqrt[3]{C}} + 6 + 3t}, \tag{15}$$

$$\frac{1}{\sqrt[3]{-\frac{\alpha}{\sqrt[3]{C}}}} + \frac{1}{\sqrt[3]{-\frac{\beta}{\sqrt[3]{C}}}} + \frac{1}{\sqrt[3]{-\frac{\gamma}{\sqrt[3]{C}}}} = \sqrt[3]{\frac{B}{\sqrt[3]{C^2}} + 6 + 3t}, \tag{16}$$

where t and $D(f)$ are described in (7) and (8) respectively. Finally, simplification gives

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = -\sqrt[3]{A + 6\sqrt[3]{C} + 3t\sqrt[3]{C}},$$
$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = -\frac{1}{\sqrt[3]{C}}\sqrt[3]{B + 6\sqrt[3]{C^2} + 3t\sqrt[3]{C^2}}.$$

□

Corollary 2.4. *Let $f(x) = x^3 - 3rsx + rs(r+s) = 0$ with real coefficients. Then*

(a) the associated Ramanujan equation is

$$t^3 + 9\left(\sqrt[3]{\frac{rs}{(r+s)^2}} - 1\right)t + 9\left(2\sqrt[3]{\frac{rs}{(r+s)^2}} - 1\right) = 0;$$

(b) the only real root to the associated Ramanujan equation is

$$t = \sqrt[3]{\frac{9\left(1 - 2\sqrt[3]{\frac{rs}{(r+s)^2}}\right) + \frac{\sqrt{D(f)}}{C}}{2}} + \sqrt[3]{\frac{9\left(1 - 2\sqrt[3]{\frac{rs}{(r+s)^2}}\right) - \frac{\sqrt{D(f)}}{C}}{2}}, \tag{17}$$

where

$$\frac{D(f)}{C^2} = -\frac{27(r-s)^2}{(r+s)^2}; \tag{18}$$

(c) the Ramanujan-type identities are

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = -\sqrt[3]{6\sqrt[3]{rs(r+s)} + 3t\sqrt[3]{rs(r+s)}}, \tag{19}$$

$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = \sqrt[3]{\frac{3}{r+s} - \frac{6}{\sqrt[3]{rs(r+s)}} - \frac{3t}{\sqrt[3]{rs(r+s)}}}. \tag{20}$$

Proof. Let $A = 0$, $B = -3rs$, and $C = rs(r+s)$ in the results of Theorem 2.3.

For part (a), we have

$$t^3 - 3\left(\frac{-3rs}{\sqrt[3]{rs(r+s)}} + 3\right)t - \left(6\left(\frac{-3rs}{\sqrt[3]{rs(r+s)}}\right) + 9\right) = 0.$$

Simplifying gives

$$t^3+9\left(\sqrt[3]{\frac{rs}{(r+s)^2}}-1\right)t+9\left(2\sqrt[3]{\frac{rs}{(r+s)^2}}-1\right)=0.$$

For part (b),

$$t=\sqrt[3]{\frac{9\left(1-2\sqrt[3]{\frac{rs}{(r+s)^2}}\right)+\frac{\sqrt{D(f)}}{C}}{2}}+\sqrt[3]{\frac{9\left(1-2\sqrt[3]{\frac{rs}{(r+s)^2}}\right)-\frac{\sqrt{D(f)}}{C}}{2}},$$

where

$$\begin{aligned}\frac{D(f)}{C^2}&=\frac{1}{C^2}\left((AB)^2-4(A^3C+B^3)+18ABC-27C^2\right)\\&=\frac{1}{(rs)^2(r+s)^2}\left(108(rs)^3-27(rs)^2(r+s)^2\right)=\frac{1}{(rs)^2(r+s)^2}\left(-27(rs)^2(r-s)^2\right)\\&=-\frac{27(r-s)^2}{(r+s)^2}.\end{aligned}$$

For part (c),

$$\begin{aligned}\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}&=-\sqrt[3]{A+6\sqrt[3]{C}+3t\sqrt[3]{C}}\\&=-\sqrt[3]{6\sqrt[3]{rs(r+s)}+3t\sqrt[3]{rs(r+s)}}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\sqrt[3]{\alpha}}+\frac{1}{\sqrt[3]{\beta}}+\frac{1}{\sqrt[3]{\gamma}}&=-\frac{1}{\sqrt[3]{C}}\sqrt[3]{B+6\sqrt[3]{C^2}+3t\sqrt[3]{C^2}}\\&=-\frac{1}{\sqrt[3]{rs(r+s)}}\sqrt[3]{-3rs+6\sqrt[3]{(rs)^2(r+s)^2}+3t\sqrt[3]{(rs)^2(r+s)^2}}\\&=\sqrt[3]{\frac{3}{r+s}-\frac{6}{\sqrt[3]{rs(r+s)}}-\frac{3t}{\sqrt[3]{rs(r+s)}}}.\end{aligned}$$

□

Next, we shall use Theorems 1.2 and 2.3 to obtain Theorem 2.5 to construct cosine Ramanujan-type identities. An example of the construction is given in Section 6.4.

Theorem 2.5. *Let $f(x)=x^3-3rsx+rs(r+s)=0$, where $r,s\in\mathbb{C}$ are complex conjugates with $r=\xi+\eta i$ and $\xi,\eta\neq 0$. Define $\theta=\operatorname{Arg}(r)$. Then the cosine Ramanujan-type identities can be obtained via the following:*

$$\sqrt[3]{\cos\left(\frac{\theta}{3}\right)}+\sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}+\sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}=\sqrt[9]{\frac{2\xi}{|r|}}\sqrt[3]{3+\frac{3t}{2}},\tag{21}$$

$$\frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3}\right)}}+\frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}}+\frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}}=\sqrt[3]{\frac{3}{\xi}}\sqrt[3]{-|r|+2\sqrt[3]{4\xi^2|r|}+t\sqrt[3]{4\xi^2|r|}},\tag{22}$$

where

$$t=\sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2|r|}{\xi^2}}\right)+3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}}+\sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2|r|}{\xi^2}}\right)-3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}}.\tag{23}$$

Proof. By Theorem 1.2, the three distinct roots of $f(x) = 0$ are

$$\alpha = -2\sqrt{rs}\cos\left(\frac{\theta}{3}\right), \beta = -2\sqrt{rs}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right), \gamma = -2\sqrt{rs}\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right).$$

Note that $rs = |r|^2$ and $r + s = 2\xi$. Using (12), the associated Ramanujan equation is

$$\begin{aligned} 0 &= t^3 - 3\left(\frac{-3rs}{\sqrt[3]{(rs)^2(r+s)^2}} + 3\right)t - 6\left(\frac{-3rs}{\sqrt[3]{(rs)^2(r+s)^2}}\right) - 9 \\ &= t^3 + 9\left(\sqrt[3]{\frac{rs}{(r+s)^2}} - 1\right)t - 9\left(-2\sqrt[3]{\frac{rs}{(r+s)^2}} + 1\right) = 0 \\ &= t^3 + 9\left(\sqrt[3]{\frac{|r|}{4\xi^2}} - 1\right)t + 9\left(\sqrt[3]{\frac{2|r|}{\xi^2}} - 1\right). \end{aligned}$$

Using (8), the discriminant of $f(x) = 0$ is

$$\begin{aligned} \frac{D(f)}{C^2} &= \frac{1}{(rs)^2(r+s)^2}\left(108(rs)^3 - 27(rs)^2(r+s)^2\right) \\ &= \frac{1}{4\xi^2|r|^4}\left(108|r|^6 - 108\xi^2|r|^4\right) \\ &= \frac{27}{\xi^2}\left(|r|^2 - \xi^2\right) = \frac{27\eta^2}{\xi^2}. \end{aligned}$$

Next, taking the square root allows us to substitute into (7). Hence,

$$\frac{\sqrt{D(f)}}{C} = 3\sqrt{3}\left(\frac{\eta}{\xi}\right). \tag{24}$$

By (7) in Theorem 2.3, the real root t is

$$t = \sqrt[3]{\frac{9\left(1 - \sqrt[3]{\frac{2|r|}{\xi^2}}\right) + 3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}} + \sqrt[3]{\frac{9\left(1 - \sqrt[3]{\frac{2|r|}{\xi^2}}\right) - 3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}}.$$

Then, by Theorem 2.3,

$$\begin{aligned} &\sqrt[3]{-2|r|\cos\left(\frac{\theta}{3}\right)} + \sqrt[3]{-2|r|\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)} + \sqrt[3]{-2|r|\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right)} \\ &\qquad = -\sqrt[9]{2\xi|r|^2}\sqrt[3]{6 + 3t}. \end{aligned}$$

Thus,

$$\sqrt[3]{\cos\left(\frac{\theta}{3}\right)} + \sqrt[3]{\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)} + \sqrt[3]{\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right)} = \sqrt[9]{\frac{2\xi}{|r|}}\sqrt[3]{3 + \frac{3t}{2}}.$$

On the other hand,

$$\begin{aligned} &\frac{1}{\sqrt[3]{-2|r|\cos\left(\frac{\theta}{3}\right)}} + \frac{1}{\sqrt[3]{-2|r|\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)}} + \frac{1}{\sqrt[3]{-2|r|\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right)}} \\ &\qquad = -\frac{1}{\sqrt[3]{2\xi|r|^2}}\sqrt[3]{-3|r|^2 + 6\sqrt[3]{4|r|^4\xi^2} + 3t\sqrt[3]{4|r|^4\xi^2}}. \end{aligned}$$

Thus,

$$\frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3}\right)}} + \frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}} + \frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}} = \sqrt[3]{\frac{3}{\xi}}\sqrt[3]{-|r| + 2\sqrt[3]{4\xi^2|r|} + t\sqrt[3]{4\xi^2|r|}}. \quad \square$$

Lemma 2.5.1. *Let $f(x) = x^3 + Ax^2 + Bx + C = 0$, where $A, B, C \in \mathbb{R}$, $A^2 \neq 3B$, and $C \neq 0$. Let its depressed cubic form be $x^3 - 3rsx + rs(r+s) = 0$. Denote $D(f)$ as the discriminant of $f(x)$.*

(a) If $D(f) > 0$, then r and s are complex conjugates.

(b) If $D(f) = 0$, then $r = s$ is a real number.

(c) If $D(f) < 0$, then r and s are distinct real numbers.

Proof. By substituting $x = y - \frac{A}{3}$ into $f(x)$, we get

$$\left(y - \frac{A}{3}\right)^3 + A\left(y - \frac{A}{3}\right)^2 + B\left(y - \frac{A}{3}\right) + C = 0,$$

which simplifies to the depressed cubic

$$y^3 - \frac{1}{3}\left(A^2 - 3B\right)y + \frac{1}{27}\left(2A^3 - 9AB + 27C\right) = 0. \tag{25}$$

Using the result from Theorem 1.2, we have

$$\begin{aligned} -3rs &= -\frac{1}{3}(A^2 - 3B), \\ rs(r+s) &= \frac{1}{27}\left(2A^3 - 9AB + 27C\right). \end{aligned}$$

Hence,

$$\begin{aligned} rs &= \frac{A^2 - 3B}{9}, \\ r + s &= \frac{(2A^3 - 9AB + 27C)}{3(A^2 - 3B)}. \end{aligned}$$

Then r and s are roots of the quadratic equation $h(z) = z^2 - \frac{\left(2A^3 - 9AB + 27C\right)}{3(A^2-3B)}z + \frac{A^2-3B}{9}$. The discriminant of this quadratic equation is

$$\begin{aligned} D(h) &= \frac{(2A^3 - 9AB + 27C)^2}{9(A^2 - 3B)^2} - \frac{4(A^2 - 3B)}{9} \\ &= \frac{1}{9(A^2 - 3B)^2}\left(\left(2A^3 - 9AB + 27C\right)^2 - 4(A^2 - 3B)^3\right) \\ &= \frac{1}{9(A^2 - 3B)^2}\left(4A^6 - 36A^4B + 108A^3C + 81A^2B^2 - 486ABC + 729C^2\right) \\ &\quad - \frac{4}{9(A^2 - 3B)^2}\left(A^6 - 9A^4B + 27A^2B^2 - 27B^3\right) \\ &= -\frac{27}{9(A^2 - 3B)^2}\left(A^2B^2 - 4(A^3C + B^3) + 18ABC - 27C^2\right) = -\frac{3D(f)}{(A^2 - 3B)^2}, \end{aligned}$$

where $D(f)$ is the discriminant of $f(x)$. Therefore, if $D(f) > 0$, then $D(h) < 0$. Hence, if the discriminant of $f(x)$ is positive, i.e., it has three distinct real roots, then r and s are complex conjugates.

Similarly, if $D(f) = 0$, then $D(h) = 0$. Hence, if the discriminant of $f(x)$ is zero, then $r = s$ is a real number. If $D(f) < 0$, then $D(h) > 0$. Hence, if the discriminant of $f(x)$ is negative, then r and s are distinct real numbers. $\square$

Remark 2.6. *If $A^2 = 3B$, then (25) becomes $y^3 - \frac{A^3}{27} + C = 0$. The roots of this cubic equation are $\alpha_y = \frac{1}{3}\sqrt[3]{A^3 - 27C}$, $\beta_y = \omega\alpha_y$, and $\gamma_y = \omega^2\alpha_y$, where $\omega = \frac{-1+\sqrt{3}}{2}$. Hence, the roots for $f(x)$ are $\alpha = \frac{1}{3}\sqrt[3]{A^3 - 27C} - \frac{A}{3}$, $\beta = \frac{\omega}{3}\sqrt[3]{A^3 - 27C} - \frac{A}{3}$, and $\gamma = \frac{\omega^2}{3}\sqrt[3]{A^3 - 27C} - \frac{A}{3}$.*

Given a general cubic equation, Lemma 2.5.1 gives the condition of when r and$s$ are complex conjugates. Corollary 2.7 uses Theorem 2.5 to construct cosine Ramanujan-type identity given a particular form of cubic equation.

Corollary 2.7. *Let $f(x) = x^3 + Ax^2 + \frac{A^2-9}{3}x + \frac{A^3}{27} - A + 2C$, where $A \in \mathbb{R}$ and $C \in (-1, 1)$. Then the discriminant of $f(x)$, $D(f)$, is greater than 0. Moreover, let the distinct roots of $f(x) = 0$ be $\alpha,\ \beta$, and γ. Then the cosine Ramanujan-type identities are*

$$\sqrt[3]{\alpha + \frac{A}{3}} + \sqrt[3]{\beta + \frac{A}{3}} + \sqrt[3]{\gamma + \frac{A}{3}} = -\sqrt[3]{6\sqrt[3]{2C} + 3t\sqrt[3]{2C}}, \tag{26}$$

and

$$\frac{1}{\sqrt[3]{\alpha + \frac{A}{3}}} + \frac{1}{\sqrt[3]{\beta + \frac{A}{3}}} + \frac{1}{\sqrt[3]{\gamma + \frac{A}{3}}} = -\frac{1}{\sqrt[3]{2C}}\sqrt[3]{-3 + 6\sqrt[3]{4C^2} + 3t\sqrt[3]{4C^2}}, \tag{27}$$

where

$$t = \sqrt[3]{\frac{9\left(1 - \sqrt[3]{\frac{2}{C^2}}\right) + 3\sqrt{3}\left(\frac{\sqrt{1-C^2}}{C}\right)}{2}} + \sqrt[3]{\frac{9\left(1 - \sqrt[3]{\frac{2}{C^2}}\right) - 3\sqrt{3}\left(\frac{\sqrt{1-C^2}}{C}\right)}{2}}. \tag{28}$$

Proof. Let $f(x) = x^3 + Ax^2 + \frac{A^2-9}{3}x + \frac{A^3}{27} - A + 2C = 0$, with $A \in \mathbb{R}$ and $C \in (-1, 1)$. The discriminant $D(f)$ is

$$\begin{aligned} D(f) &= A^2\left(\frac{A^2-9}{3}\right) - 4\left(A^3\left(\frac{A^3}{27} - A + 2C\right) + \left(\frac{A^2-9}{3}\right)^3\right) \\ &\qquad + 18A\left(\frac{A^2-9}{3}\right)\left(\frac{A^3}{27} - A + 2C\right) - 27\left(\frac{A^3}{27} - A + 2C\right)^2 \\ &= 108(1 - C^2), \end{aligned}$$

which is greater than zero for $C \in (-1, 1)$.

Next, by substituting $x = y - \frac{A}{3}$ into $f(x)$, we get

$$\left(y - \frac{A}{3}\right)^3 + A\left(y - \frac{A}{3}\right)^2 + \left(\frac{A^2-9}{3}\right)\left(y - \frac{A}{3}\right) + \frac{A^3}{27} - A + 2C = 0,$$

which simplifies to the depressed cubic

$$y^3 - 3y + 2C = 0. \tag{29}$$

Denote the roots of (29) as α_y, β_y, and γ_y. Next, using Theorem 2.5, we have

$$-3rs = -3,$$
$$rs(r+s) = 2C.$$

Then

$$rs = 1,$$
$$r+s = 2C.$$

Then r, s are roots of the quadratic equation

$$z^2 - 2Cz + 1 = 0.$$

Using the quadratic formula,

$$r = C + \sqrt{C^2-1} = C + i\sqrt{1-C^2},$$
$$s = C - i\sqrt{1-C^2}.$$

To find the terms inside (23) in Theorem 2.5, we first find

$$\frac{\sqrt{D(f)}}{C} = 3\sqrt{3}\left(\frac{\sqrt{1-C^2}}{C}\right).$$

Then

$$t = \sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{C^2}}\right)+3\sqrt{3}\left(\frac{\sqrt{1-C^2}}{C}\right)}{2}} + \sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{C^2}}\right)-3\sqrt{3}\left(\frac{\sqrt{1-C^2}}{C}\right)}{2}}.$$

By the result of Theorem 2.3, we have

$$\sqrt[3]{\alpha_y} + \sqrt[3]{\beta_y} + \sqrt[3]{\gamma_y} = -\sqrt[3]{6\sqrt[3]{2C} + 3t\sqrt[3]{2C}}$$

and

$$\frac{1}{\sqrt[3]{\alpha_y}} + \frac{1}{\sqrt[3]{\beta_y}} + \frac{1}{\sqrt[3]{\gamma_y}} = -\sqrt[3]{-3 + 6\sqrt[3]{4C^2} + 3t\sqrt[3]{4C^2}}.$$

Finally,

$$\sqrt[3]{\alpha + \frac{A}{3}} + \sqrt[3]{\beta + \frac{A}{3}} + \sqrt[3]{\gamma + \frac{A}{3}} = -\sqrt[3]{6\sqrt[3]{2C} + 3t\sqrt[3]{2C}}$$

and

$$\frac{1}{\sqrt[3]{\alpha + \frac{A}{3}}} + \frac{1}{\sqrt[3]{\beta + \frac{A}{3}}} + \frac{1}{\sqrt[3]{\gamma + \frac{A}{3}}} = -\frac{1}{\sqrt[3]{2C}}\sqrt[3]{-3 + 6\sqrt[3]{4C^2} + 3t\sqrt[3]{4C^2}}. \qquad \square$$

3 Ramanujan cubic polynomials

Shevelev [11] defined the Ramanujan cubic polynomials (RCP) as follows: Let $A, B, C \in \mathbb{R}$, $C \neq 0$. The cubic polynomial $f(x) = x^3 + Ax^2 + Bx + C$ is RCP if it has three real roots and satisfies the condition $A\sqrt[3]{C} + B + 3\sqrt[3]{C^2} = 0$.

For the case of RCP, $f(x) = x^3 + Ax^2 + Bx + C = 0$ has three distinct roots [11]. We have the following theorem.

Theorem 3.1. *Let $f(x) = x^3 + Ax^2 + Bx + C = 0$, $A, B, C \in \mathbb{R}$, $C \neq 0$, be an RCP. The root t of its associated Ramanujan equation, $t^3 - \left(\frac{AB}{C} - 9\right) = 0$, is*

$$t = \sqrt[3]{\frac{AB}{C} - 9}, \tag{30}$$

and

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{-A - 6\sqrt[3]{C} + 3\sqrt[3]{9C - AB}}, \tag{31}$$

$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = \frac{1}{\sqrt[3]{C}}\sqrt[3]{-B - 6\sqrt[3]{C^2} + 3\sqrt[3]{9C^2 - ABC}}. \tag{32}$$

Proof. Let $f(x) = x^3 + Ax^2 + Bx + C = 0$, $A, B, C \in \mathbb{R}$, $C \neq 0$, and $A\sqrt[3]{C} + B + 3\sqrt[3]{C^2} = 0$. From Theorem 2.3, we have

$$\begin{aligned}
D(f) &= A^2B^2 - 4\left(\left(A\sqrt[3]{C}\right)^3 + B^3\right) + 18ABC - 27C^2 \\
&= A^2B^2 - 4\left(A\sqrt[3]{C} + B\right)\left(A^2\sqrt[3]{C^2} - AB\sqrt[3]{C} + B^2\right) + 18ABC - 27C^2 \\
&= A^2B^2 - 4\left(-3\sqrt[3]{C^2}\right)\left(A^2\sqrt[3]{C^2} - AB\sqrt[3]{C} + B^2\right) + 18ABC - 27C^2 \\
&= A^2B^2 + 12\sqrt[3]{C^2}\left(A^2\sqrt[3]{C^2} + B^2\right) + 6ABC - 27C^2 \\
&= A^2B^2 + 12\sqrt[3]{C^2}\left(\left(A\sqrt[3]{C} + B\right)^2 - 2AB\sqrt[3]{C}\right) + 6ABC - 27C^2 \\
&= A^2B^2 + 12\sqrt[3]{C^2}\left(9C^{\frac{4}{3}} - 2AB\sqrt[3]{C}\right) + 6ABC - 27C^2 \\
&= (AB)^2 + 81C^2 - 18ABC = (AB - 9C)^2 \geq 0.
\end{aligned}$$

From the definition of RCP, $A\sqrt[3]{C} + B + 3\sqrt[3]{C^2} = 0$, we have

$$\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3 = 0.$$

Therefore, the associated Ramanujan equation of $f(x)$ is

$$t^3 - \left(\frac{AB}{C} - 9\right) = 0.$$

Hence, the real root t to the associated Ramanujan equation is

$$t = \sqrt[3]{\frac{AB}{C} - 9}.$$

By Theorem 2.3,

$$\begin{aligned}
\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} &= -\sqrt[3]{A + 6\sqrt[3]{C} + 3t\sqrt[3]{C}} \\
&= \sqrt[3]{-A - 6\sqrt[3]{C} + 3\sqrt[3]{9C - AB}},
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} &= -\frac{1}{\sqrt[3]{C}}\sqrt[3]{B + 6\sqrt[3]{C^2} + 3t\sqrt[3]{C^2}} \\
&= \frac{1}{\sqrt[3]{C}}\sqrt[3]{-B - 6\sqrt[3]{C^2} + 3\sqrt[3]{9C^2 - ABC}}.
\end{aligned}$$

□

Remark 3.2. *Theorem 3.1 was proved by Shevelev [11] with a different proof.*

Dresden et al. [4] investigated the following cubic equation.

Corollary 3.3. *Let $f(x) = x^3 - \frac{3+B}{2}x^2 - \frac{3-B}{2}x + 1 = 0$. Then $f(x)$ is an RCP. Moreover, the Ramanujan-type identities are*

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{\frac{3+B}{2} - 6 + 3\sqrt[3]{\frac{27+B^2}{4}}}.$$

and

$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = \sqrt[3]{\frac{3-B}{2} - 6 + 3\sqrt[3]{\frac{27+B^2}{4}}}.$$

Proof. Substituting $A = -\frac{3+B}{2}$, $B = -\frac{3-B}{2}$, and $C = 1$ in the result of Theorem 3.1 yields the result. $\square$

Remark 3.4. *The results are proved differently in their paper.*

Similarly, Wituła [17] defined the cubic polynomials of the form $f(x) = x^3 + Ax^2 + Bx + C$ with real roots and satisfying $B^3 + A^3C + 27C^2 = 0$ as Ramanujan cubic polynomials of the second kind (RCP2). For the case of RCP2, we have the following result.

Theorem 3.5. *Let $f(x) = x^3 + Ax^2 + Bx + C = 0$, $A, B, C \in \mathbb{R}$, $C \neq 0$, be an RCP2. Then the root t of the associated Ramanujan equation,*

$$t^3 - 3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right)t - \left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = 0,$$

is

$$t = \sqrt[3]{\frac{1}{C}\left(A + 3\sqrt[3]{C}\right)\left(B + 3\sqrt[3]{C^2}\right)} + \sqrt[3]{\frac{3}{\sqrt[3]{C^2}}\left(A\sqrt[3]{C} + B\right)}, \tag{33}$$

and

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{-A - 6\sqrt[3]{C} - 3\left(\sqrt[3]{\left(A + 3\sqrt[3]{C}\right)\left(B + 3\sqrt[3]{C^2}\right)} + \sqrt[3]{3\sqrt[3]{C}\left(A\sqrt[3]{C} + B\right)}\right)}, \tag{34}$$

$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = \sqrt[3]{-\frac{B}{C} - \frac{6}{\sqrt[3]{C}} - \frac{3}{\sqrt[3]{C^2}}\left(\sqrt[3]{\left(A + 3\sqrt[3]{C}\right)\left(B + 3\sqrt[3]{C^2}\right)} + \sqrt[3]{3\sqrt[3]{C}\left(A\sqrt[3]{C} + B\right)}\right)}. \tag{35}$$

Proof. From the condition of RCP2, we have

$$B^3 + A^3C + 27C^2 = 0.$$

From Theorem 2.3, we have

$$\begin{aligned} D(f) &= A^2B^2 - 4B^3 - 4A^3C + 18ABC - 27C^2 \\ &= A^2B^2 + 4C(A^3 + 27C) - 4A^3C + 18ABC - 27C^2 \\ &= A^2B^2 + 18ABC + 81C^2 = (AB + 9C)^2 \geq 0. \end{aligned}$$

Then

$$\frac{D(f)}{C^2} = \left(\frac{AB}{C} + 9\right)^2.$$

Taking the square root allows us to substitute into (7):

$$\frac{\sqrt{D(f)}}{C} = \frac{AB}{C} + 9. \tag{36}$$

The associated Ramanujan equation of $f(x)$ is

$$t^3 - 3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right)t - \left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = 0.$$

From (12), the real root t to the associated Ramanujan equation is

$$t = \sqrt[3]{\frac{\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9 + \frac{\sqrt{D(f)}}{C}}{2}} + \sqrt[3]{\frac{\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9 - \frac{\sqrt{D(f)}}{C}}{2}}.$$

Computing the quantities under the cube roots give:

$$\begin{aligned} \frac{1}{2}\left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9 + \frac{AB}{C} + 9\right) &= \frac{AB}{C} + 9 + 3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) \\ &= \frac{1}{C}\left(AB + 9C + \left(A\sqrt[3]{C^2} + B\sqrt[3]{C}\right)\right) \\ &= \frac{1}{C}\left(A + 3\sqrt[3]{C}\right)\left(B + 3\sqrt[3]{C^2}\right) \end{aligned}$$

and

$$\frac{1}{2}\left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9 - \left(\frac{AB}{C} + 9\right)\right) = \frac{3}{\sqrt[3]{C^2}}\left(A\sqrt[3]{C} + B\right).$$

Denote the roots of $f(x)$ by α, β, and γ. By Theorem 2.3, we have

$$\begin{aligned} \sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} &= -\sqrt[3]{A + 6\sqrt[3]{C} + 3t\sqrt[3]{C}} \\ &= \sqrt[3]{-A - 6\sqrt[3]{C} - 3\left(\sqrt[3]{\left(A + 3\sqrt[3]{C}\right)\left(B + 3\sqrt[3]{C^2}\right)} + \sqrt[3]{3\sqrt[3]{C}\left(A\sqrt[3]{C} + B\right)}\right)}, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} &= -\frac{1}{\sqrt[3]{C}}\sqrt[3]{B + 6\sqrt[3]{C^2} + 3t\sqrt[3]{C^2}} \\ &= \sqrt[3]{-\frac{B}{C} - \frac{6}{\sqrt[3]{C}} - \frac{3}{\sqrt[3]{C^2}}\left(\sqrt[3]{\left(A + 3\sqrt[3]{C}\right)\left(B + 3\sqrt[3]{C^2}\right)} + \sqrt[3]{3\sqrt[3]{C}\left(A\sqrt[3]{C} + B\right)}\right)}. \square \end{aligned}$$

Remark 3.6. *The first identity was proved by Wituła [16] but not the second identity.*

4 Cubic Shevelev sum

Shevelev ([11]) defined the sum,

$$\sqrt[3]{\frac{\alpha}{\beta}}+\sqrt[3]{\frac{\beta}{\gamma}}+\sqrt[3]{\frac{\gamma}{\alpha}}+\sqrt[3]{\frac{\beta}{\alpha}}+\sqrt[3]{\frac{\gamma}{\beta}}+\sqrt[3]{\frac{\alpha}{\gamma}},$$

as the cubic Shevelev sum, where α, β, and γ are roots of a cubic equation $f(x)=x^3+Ax^2+Bx+C=0$.

Remark 4.1. *Wang ([14]) investigated the following quantity similar to the cubic Shevelev sum by showing that the discriminant of $f(x)=x^3-ax^2+bx-1=0$, $D(f)$, is*

$$\left(\left(\frac{\alpha}{\beta}+\frac{\beta}{\gamma}+\frac{\gamma}{\alpha}\right)-\left(\frac{\beta}{\alpha}+\frac{\gamma}{\beta}+\frac{\alpha}{\gamma}\right)\right)^2,$$

where α, β, and γ, with each not equal to zero, are roots of $f(x)=0$.

Using Theorem 2.3, we have the following generalized cubic Shevelev sum.

Theorem 4.2. *Let $f(x)=x^3+Ax^2+Bx+C=0$, $A,B,C\in\mathbb{R}$, A,B not both 0, $C\neq 0$. Let $f(x)$ have three distinct real roots α, β, and γ, with $\alpha,\ \beta,\ \gamma\neq 0$. Then*

$$\sqrt[3]{\frac{\alpha}{\beta}}+\sqrt[3]{\frac{\beta}{\gamma}}+\sqrt[3]{\frac{\gamma}{\alpha}}+\sqrt[3]{\frac{\beta}{\alpha}}+\sqrt[3]{\frac{\gamma}{\beta}}+\sqrt[3]{\frac{\alpha}{\gamma}}=t, \tag{37}$$

where t is shown in (7).

Proof. First note that

$$\begin{aligned}&\left(\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}\right)\left(\frac{1}{\sqrt[3]{\alpha}}+\frac{1}{\sqrt[3]{\beta}}+\frac{1}{\sqrt[3]{\gamma}}\right)\\&\qquad=\sqrt[3]{\frac{\alpha}{\beta}}+\sqrt[3]{\frac{\beta}{\gamma}}+\sqrt[3]{\frac{\gamma}{\alpha}}+\sqrt[3]{\frac{\beta}{\alpha}}+\sqrt[3]{\frac{\gamma}{\beta}}+\sqrt[3]{\frac{\alpha}{\gamma}}+3.\end{aligned} \tag{38}$$

On the other hand, using (9) and (10), we have

$$\begin{aligned}&\left(\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}\right)\left(\frac{1}{\sqrt[3]{\alpha}}+\frac{1}{\sqrt[3]{\beta}}+\frac{1}{\sqrt[3]{\gamma}}\right)\\&\qquad=\left(-\sqrt[3]{A+6\sqrt[3]{C}+3t\sqrt[3]{C}}\right)\left(-\sqrt[3]{\frac{B}{C}+\frac{6}{\sqrt[3]{C}}+\frac{3}{\sqrt[3]{C}}t}\right)\\&\qquad=\sqrt[3]{\left(A+6\sqrt[3]{C}+3t\sqrt[3]{C}\right)\left(\frac{B}{C}+\frac{6}{\sqrt[3]{C}}+\frac{3}{\sqrt[3]{C}}t\right)}.\end{aligned}$$

Simplifying the expression under the cube root, we have

$$\begin{aligned}&\left(A+6\sqrt[3]{C}+3t\sqrt[3]{C}\right)\left(\frac{B}{C}+\frac{6}{\sqrt[3]{C}}+\frac{3}{\sqrt[3]{C}}t\right)=\left(\frac{A}{\sqrt[3]{C}}+6+3t\right)\left(\frac{B}{\sqrt[3]{C^2}}+6+3t\right)\\&=\frac{AB}{C}+6\frac{A}{\sqrt[3]{C}}+3\frac{A}{\sqrt[3]{C}}t+6\frac{B}{\sqrt[3]{C^2}}+36+18t+3\frac{B}{\sqrt[3]{C^2}}t+18t+9t^2\\&=\frac{AB}{C}+\frac{6}{\sqrt[3]{C^2}}\left(A\sqrt[3]{C}+B\right)+36+36t+\frac{3}{\sqrt[3]{C^2}}\left(A\sqrt[3]{C}+B\right)t+9t^2.\end{aligned}$$

Recall the associated Ramanujan equation $t^3+pt+q=0$, where $p=-3\left(\frac{A}{\sqrt[3]{C}}+\frac{B}{\sqrt[3]{C^2}}+3\right)$ and $q=-\left(\frac{AB}{C}+6\left(\frac{A}{\sqrt[3]{C}}+\frac{B}{\sqrt[3]{C^2}}\right)+9\right)$. Then we have

$$\begin{aligned}&\left(A+6\sqrt[3]{C}+3t\sqrt[3]{C}\right)\left(\frac{B}{C}+\frac{6}{\sqrt[3]{C}}+\frac{3}{\sqrt[3]{C}}t\right)\\&\qquad=-q+27-pt+27t+9t^2=t^3+9t^2+27t+27=(t+3)^3.\end{aligned}$$

Hence,

$$\left(\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}\right)\left(\frac{1}{\sqrt[3]{\alpha}}+\frac{1}{\sqrt[3]{\beta}}+\frac{1}{\sqrt[3]{\gamma}}\right)=\sqrt[3]{(t+3)^3}=t+3. \tag{39}$$

Comparing (38) and (39) gives

$$\sqrt[3]{\frac{\alpha}{\beta}}+\sqrt[3]{\frac{\beta}{\gamma}}+\sqrt[3]{\frac{\gamma}{\alpha}}+\sqrt[3]{\frac{\beta}{\alpha}}+\sqrt[3]{\frac{\gamma}{\beta}}+\sqrt[3]{\frac{\alpha}{\gamma}}=t.$$

□

Applying Theorem 4.2, we have the following three corollaries.

Corollary 4.3 (Shevelev [11])**.** *If $f(x)$ is an RCP, then the cubic Shevelev sum is*

$$\sqrt[3]{\frac{\alpha}{\beta}}+\sqrt[3]{\frac{\beta}{\gamma}}+\sqrt[3]{\frac{\gamma}{\alpha}}+\sqrt[3]{\frac{\beta}{\alpha}}+\sqrt[3]{\frac{\gamma}{\beta}}+\sqrt[3]{\frac{\alpha}{\gamma}}=\sqrt[3]{\frac{AB}{C}-9}. \tag{40}$$

Proof. Let $f(x)=x^3+Ax^2+Bx+C=0$ be a cubic equation satisfying Theorem 3.1. Then $t=\sqrt[3]{\frac{AB}{C}-9}$. From Theorem 4.2, the cubic Shevelev sum is

$$\sqrt[3]{\frac{\alpha}{\beta}}+\sqrt[3]{\frac{\beta}{\gamma}}+\sqrt[3]{\frac{\gamma}{\alpha}}+\sqrt[3]{\frac{\beta}{\alpha}}+\sqrt[3]{\frac{\gamma}{\beta}}+\sqrt[3]{\frac{\alpha}{\gamma}}=t=\sqrt[3]{\frac{AB}{C}-9}.$$ □

Remark 4.4. *This result was originally due to Shevelev [11], where he provided a different proof.*

Corollary 4.5 (Wituła [16])**.** *If $f(x)$ is an $RCP2$, then the cubic Shevelev sum is*

$$\sqrt[3]{\frac{\alpha}{\beta}}+\sqrt[3]{\frac{\beta}{\gamma}}+\sqrt[3]{\frac{\gamma}{\alpha}}+\sqrt[3]{\frac{\beta}{\alpha}}+\sqrt[3]{\frac{\gamma}{\beta}}+\sqrt[3]{\frac{\alpha}{\gamma}}=\sqrt[3]{\frac{1}{C}\left(A+3\sqrt[3]{C}\right)\left(B+3\sqrt[3]{C^2}\right)}+\sqrt[3]{\frac{3}{\sqrt[3]{C^2}}\left(A\sqrt[3]{C}+B\right)}. \tag{41}$$

Proof. Let $f(x)=x^3+Ax^2+Bx+C=0$ be a cubic equation satisfying Theorem 3.5. Then $t=\sqrt[3]{\frac{1}{C}\left(A+3\sqrt[3]{C}\right)\left(B+3\sqrt[3]{C^2}\right)}+\sqrt[3]{\frac{3}{\sqrt[3]{C^2}}\left(A\sqrt[3]{C}+B\right)}$. From Theorem 4.2, the cubic Shevelev sum is

$$\begin{aligned}&\sqrt[3]{\frac{\alpha}{\beta}}+\sqrt[3]{\frac{\beta}{\gamma}}+\sqrt[3]{\frac{\gamma}{\alpha}}+\sqrt[3]{\frac{\beta}{\alpha}}+\sqrt[3]{\frac{\gamma}{\beta}}+\sqrt[3]{\frac{\alpha}{\gamma}}=t\\&\qquad=\sqrt[3]{\frac{1}{C}\left(A+3\sqrt[3]{C}\right)\left(B+3\sqrt[3]{C^2}\right)}+\sqrt[3]{\frac{3}{\sqrt[3]{C^2}}\left(A\sqrt[3]{C}+B\right)}.\end{aligned}$$ □

Remark 4.6. *This result was due to Wituła [16] in which he provided a different proof.*

Corollary 4.7. *Let $f(x) = x^3 - 3rsx + rs(r+s) = 0$ as described in Theorem 2.5. Then the cubic Shevelev sum is*

$$\sqrt[3]{\frac{\cos\left(\frac{\theta}{3}\right)}{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}{\cos\left(\frac{\theta}{3}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{\theta}{3}\right)}{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}} + \sqrt[3]{\frac{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}{\cos\left(\frac{\theta}{3}\right)}}$$
$$= \sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{\xi^2}}\right)+3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}} + \sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{\xi^2}}\right)-3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}}. \tag{42}$$

Proof. From Theorem 2.5, $t = \sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{\xi^2}}\right)+3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}} + \sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{\xi^2}}\right)-3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}}$. From Theorem 4.2, the cubic Shevelev sum is

$$\sqrt[3]{\frac{\alpha}{\beta}} + \sqrt[3]{\frac{\beta}{\gamma}} + \sqrt[3]{\frac{\gamma}{\alpha}} + \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{\gamma}{\beta}} + \sqrt[3]{\frac{\alpha}{\gamma}} = t$$
$$= \sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{\xi^2}}\right)+3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}} + \sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{\xi^2}}\right)-3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}}. \quad \square$$

5 Computation procedure

In this section, two computation procedures are given. The first is based on Theorems 1.2 and 2.3. This is to construct a Ramanujan-type identity given a general cubic equation $f(x) = x^3 + Ax^2 + Bx + C = 0$, with $A, B, C \in \mathbb{R}, C \neq 0$.

Computation Procedure 1:

1: $M \leftarrow \frac{A^2-3B}{9}$.
2: $N \leftarrow \frac{2A^3-9AB+27C}{3(A^2-3B)}$
3: Find the roots z_1, z_2 of $z^2 - Nz + M = 0$
4: $\theta \leftarrow \mathrm{Arg}(z_1)$
5: $\alpha \leftarrow -2\sqrt{M}\cos\frac{\theta}{3} - \frac{A}{3}$
6: $\beta \leftarrow -2\sqrt{M}\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right) - \frac{A}{3}$
7: $\gamma \leftarrow -2\sqrt{M}\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right) - \frac{A}{3}$
8: $q \leftarrow \sqrt{\frac{1}{C^2}\left((AB)^2 - 4(A^3C + B^3) + 18ABC - 27C^2\right)}$
9: $p \leftarrow \frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9$
10: $t \leftarrow \sqrt[3]{\frac{p+q}{2}} + \sqrt[3]{\frac{p-q}{2}}$
11: $G_1 \leftarrow -\left(A + 6\sqrt[3]{C} + 3t\sqrt[3]{C}\right)$
12: $G_2 \leftarrow -\frac{1}{C}\left(B + 6\sqrt[3]{C^2} + 3t\sqrt[3]{C^2}\right)$
13: Return the Ramanujan-type identities $\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{G_1}$ and $\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = \sqrt[3]{G_2}$.

Given a cubic equation, we first find the roots with the first 7 steps. Steps 11–12 give the right-hand sides of (9) and (10), respectively. Finally, Step 13 returns the Ramanujan-type identities.

The second computation procedure gives the approach based on Theorem 2.5 for constructing a cosine Ramanujan-type identity. Given a complex number, we can the cubic equation and the Ramanujan-type identities.

Computation Procedure 2:

1: Determine the cubic equation $x^3 - 3rsx + rs(r+s) = 0$.
2: $\alpha \leftarrow \cos\left(\frac{\theta}{3}\right)$
3: $\beta \leftarrow \cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)$
4: $\gamma \leftarrow \cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right)$
5: $p \leftarrow \frac{3\sqrt{3}\eta}{\xi}$
6: $q \leftarrow 9\left(1 - \sqrt[3]{\frac{2|r|}{\xi^2}}\right)$
7: $t \leftarrow \sqrt[3]{\frac{q+p}{2}} + \sqrt[3]{\frac{q-p}{2}}$
8: $G_1 \leftarrow \sqrt[3]{\frac{2\xi}{|r|}}\left(3 + \frac{3t}{2}\right)$
9: $G_2 \leftarrow \left(\frac{3}{\xi}\right)\left(-|r| + 2\sqrt[3]{4\xi^2|r|} + t\sqrt[3]{4\xi^2|r|}\right)$
10: Return the cosine Ramanujan-type identities $\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{G_1}$ and $\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = \sqrt[3]{G_2}$.

6 Examples

In this section, we present examples about Ramanujan-type identities and cubic Shevelev sums. In the last subsection, we give examples on constructing cosine Ramanujan-type identities.

6.1 RCP

The following cubic equation is from Wituła [15].

Example 6.1.1. *Let $f(x) = x^3 + 7x^2 - 98x - 343 = 0$. Denote $A = -7$, $B = -98$, and $C = 343$. The roots of $x^3 + 7x^2 - 98x - 343 = 0$ are $\alpha = 14\cos\frac{2\pi}{7}$, $\beta = 14\cos\frac{4\pi}{7}$, and $\gamma = 14\cos\frac{8\pi}{7}$ [15]. Note that $f(x)$ is an RCP. Using* (8),

$$\frac{D(f)}{C^2} = \left(\frac{7\cdot 98}{343} - 9\right)^2 = (2-9)^2.$$

Then $\frac{\sqrt{D(f)}}{C} = 7$. Next, by (30), $t = -\sqrt[3]{7}$. *Thus, by* (31),

$$\sqrt[3]{14\cos\frac{2\pi}{7}} + \sqrt[3]{14\cos\frac{4\pi}{7}} + \sqrt[3]{14\cos\frac{8\pi}{7}} = 343^{\frac{1}{9}}\sqrt[3]{-1+6-3\sqrt[3]{7}},$$

which simplifies to

$$\sqrt[3]{\cos\frac{2\pi}{7}} + \sqrt[3]{\cos\frac{4\pi}{7}} + \sqrt[3]{\cos\frac{8\pi}{7}} = \sqrt[3]{\frac{5-3\sqrt[3]{7}}{2}}.$$

By (32)*,*

$$\frac{1}{\sqrt[3]{14\cos\frac{2\pi}{7}}}+\frac{1}{\sqrt[3]{14\cos\frac{4\pi}{7}}}+\frac{1}{\sqrt[3]{14\cos\frac{8\pi}{7}}}=\frac{1}{343^{\frac{1}{9}}}\sqrt[3]{-2+6-3\sqrt[3]{7}},$$

which simplifies to

$$\frac{1}{\sqrt[3]{\cos\frac{2\pi}{7}}}+\frac{1}{\sqrt[3]{\cos\frac{4\pi}{7}}}+\frac{1}{\sqrt[3]{\cos\frac{8\pi}{7}}}=\sqrt[3]{8-6\sqrt[3]{7}}.$$

Using (40)*, the cubic Shevelev sum is*

$$\sqrt[3]{\frac{\cos\frac{2\pi}{7}}{\cos\frac{4\pi}{7}}}+\sqrt[3]{\frac{\cos\frac{4\pi}{7}}{\cos\frac{2\pi}{7}}}+\sqrt[3]{\frac{\cos\frac{2\pi}{7}}{\cos\frac{8\pi}{7}}}+\sqrt[3]{\frac{\cos\frac{8\pi}{7}}{\cos\frac{2\pi}{7}}}+\sqrt[3]{\frac{\cos\frac{4\pi}{7}}{\cos\frac{8\pi}{7}}}+\sqrt[3]{\frac{\cos\frac{8\pi}{7}}{\cos\frac{4\pi}{7}}}=-\sqrt[3]{7}. \qquad (43)$$

Example 6.1.2. ([11])*.Let $f(x)=x^3-\frac{3}{4}x+\frac{1}{8}=0$. Denote $A=0$, $B=-\frac{3}{4}$, and $C=\frac{1}{8}$. It is known that the roots are $\alpha=\cos\frac{2\pi}{9}$, $\beta=\cos\frac{4\pi}{9}$, and $\gamma=\cos\frac{8\pi}{9}$. Note that $f(x)$ is an RCP. Using* (30)*,*

$$t=\sqrt[3]{\frac{AB}{C}-9}=-\sqrt[3]{9}.$$

The Ramanujan-type identities are obtained by using (31) *and* (32)*:*

$$\sqrt[3]{-2\cos\frac{2\pi}{9}}+\sqrt[3]{-2\cos\frac{4\pi}{9}}+\sqrt[3]{-2\cos\frac{8\pi}{9}}=\sqrt[3]{6-3\sqrt[3]{9}},$$

which simplifies to

$$\sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}}=-\sqrt[3]{\frac{6-3\sqrt[3]{9}}{2}}$$

On the other hand,

$$\frac{1}{\sqrt[3]{-2\cos\frac{2\pi}{9}}}+\frac{1}{\sqrt[3]{-2\cos\frac{4\pi}{9}}}+\frac{1}{\sqrt[3]{-2\cos\frac{8\pi}{9}}}=\sqrt[3]{3-3\sqrt[3]{9}},$$

which simplifies to

$$\frac{1}{\sqrt[3]{\cos\frac{2\pi}{9}}}+\frac{1}{\sqrt[3]{\cos\frac{4\pi}{9}}}+\frac{1}{\sqrt[3]{\cos\frac{8\pi}{9}}}=-\sqrt[3]{6-6\sqrt[3]{9}}.$$

By (40)*, the cubic Shevelev sum is*

$$\sqrt[3]{\frac{\cos\frac{2\pi}{9}}{\cos\frac{4\pi}{9}}}+\sqrt[3]{\frac{\cos\frac{4\pi}{9}}{\cos\frac{2\pi}{9}}}+\sqrt[3]{\frac{\cos\frac{2\pi}{9}}{\cos\frac{8\pi}{9}}}+\sqrt[3]{\frac{\cos\frac{8\pi}{9}}{\cos\frac{2\pi}{9}}}+\sqrt[3]{\frac{\cos\frac{4\pi}{9}}{\cos\frac{8\pi}{9}}}+\sqrt[3]{\frac{\cos\frac{8\pi}{9}}{\cos\frac{4\pi}{9}}}=-\sqrt[3]{9}.$$

Example 6.1.3. *Let* $f(x) = x^3+x^2-(3n^2+n)x+n^3 = 0$, $n \in \mathbb{N}$. *Denote* $A = 1$, $B = -(3n^2+n)$, *and* $C = n^3$. *Note that this is an RCP. By Theorem 3.1, we have*

$$9C - AB = 9n^3 + 3n^2 + n = n(9n^2 + 3n + 1).$$

Denote the roots of $f(x)$ *as* α, β, *and* γ, *then by* (9),

$$\begin{aligned}\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} &= \sqrt[3]{-\left(A + 6\sqrt[3]{C}\right) + 3\sqrt[3]{9C - AB}} \\ &= \sqrt[3]{-(6n+1) + 3\sqrt[3]{n(9n^2 + 3n + 1)}}.\end{aligned}$$

Then with (10),

$$\begin{aligned}\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} &= \frac{1}{\sqrt[3]{C}}\sqrt[3]{-B - 6\sqrt[3]{C^2} + 3\sqrt[3]{9C^2 - ABC}} \\ &= \frac{1}{n}\sqrt[3]{3n^2 + n - 6n^2 + 3\sqrt[3]{n^4\left(9n^2 + 3n + 1\right)}} \\ &= \frac{1}{\sqrt[3]{n^2}}\sqrt[3]{-3n + 1 + 3\sqrt[3]{n(9n^2 + 3n + 1)}}.\end{aligned}$$

Since it is an RCP, using (40), *the cubic Shevelev sum is*

$$\sqrt[3]{\frac{\alpha}{\beta}} + \sqrt[3]{\frac{\beta}{\gamma}} + \sqrt[3]{\frac{\gamma}{\alpha}} + \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{\gamma}{\beta}} + \sqrt[3]{\frac{\alpha}{\gamma}} = -\sqrt[3]{\frac{9n^2 + 3n + 1}{n^2}}.$$

6.2 RCP2

In this part, we give an example on Ramanujan cubic polynomials of the second kind.

Example 6.2.1. *Let* $f(x) = x^3 - \frac{3}{\sqrt[3]{2}}x^2 - \frac{3}{\sqrt[3]{2}}x + 1 = 0$. *Denote* $A = -\frac{3}{\sqrt[3]{2}}$, $B = -\frac{3}{\sqrt[3]{2}}$, *and* $C = 1$. *To show that it is an RCP2:*

$$A^3C + B^3 + 27C^2 = -\frac{27}{2} - \frac{27}{2} + 27 = 0.$$

By using the rational roots theorem, we have $\alpha = -1$ *as a root of* $f(x) = 0$. *Dividing* $f(x)$ *by* $x + 1$, *we obtain*

$$2x^2 - \left(2 + 3\sqrt[3]{4}\right)x + 2 = 0.$$

The remaining two roots are

$$\beta = \frac{1}{4}\left(2 + 3\sqrt[3]{4} + \sqrt{\left(2 + 3\sqrt[3]{4}\right)^2 - 16}\right) = \frac{1}{4}\left(2 + 3\sqrt[3]{4} + 2 + 4\sqrt[3]{2} - \sqrt[3]{4}\right) = 1 + \frac{1}{\sqrt[3]{2}} + \sqrt[3]{2},$$

$$\gamma = \frac{1}{4}\left(2 + 3\sqrt[3]{4} - \sqrt{\left(2 + 3\sqrt[3]{4}\right)^2 - 16}\right) = \frac{1}{4}\left(2 + 3\sqrt[3]{4} - 2 - 4\sqrt[3]{2} + \sqrt[3]{4}\right) = \sqrt[3]{4} - \sqrt[3]{2}.$$

Since it is an RCP2, by (33)*, the root* t *of the associated Ramanujan is*

$$\begin{aligned}t &= \sqrt[3]{\frac{1}{C}\left(A+3\sqrt[3]{C}\right)\left(B+3\sqrt[3]{C^2}\right)} + \sqrt[3]{\frac{3}{\sqrt[3]{C^2}}\left(A\sqrt[3]{C}+B\right)} \\ &= \sqrt[3]{\left(-\frac{3}{\sqrt[3]{2}}+3\right)\left(-\frac{3}{\sqrt[3]{2}}+3\right)} + \sqrt[3]{3\left(-\frac{3}{\sqrt[3]{2}}-\frac{3}{\sqrt[3]{2}}\right)} \\ &= \sqrt[3]{\left(3-\frac{3}{\sqrt[3]{2}}\right)^2} - \sqrt[3]{9}\sqrt[9]{4}.\end{aligned}$$

By (34)*, the first Ramanujan-type identity is*

$$-1+\sqrt[3]{1+\frac{1}{\sqrt[3]{2}}+\sqrt[3]{2}}+\sqrt[3]{\sqrt[3]{4}-\sqrt[3]{2}} = -\sqrt[3]{-\frac{3}{\sqrt[3]{2}}+6+3\left(\sqrt[3]{\left(3-\frac{3}{\sqrt[3]{2}}\right)^2}-\sqrt[3]{9}\sqrt[9]{4}\right)}. \quad (44)$$

By (35)*, the second Ramanujan-type identity is*

$$-1+\frac{1}{\sqrt[3]{1+\frac{1}{\sqrt[3]{2}}+\sqrt[3]{2}}}+\frac{1}{\sqrt[3]{\sqrt[3]{4}-\sqrt[3]{2}}} = -\sqrt[3]{-\frac{3}{\sqrt[3]{2}}+6+3\left(\sqrt[3]{\left(3-\frac{3}{\sqrt[3]{2}}\right)^2}-\sqrt[3]{9}\sqrt[9]{4}\right)}. \quad (45)$$

A byproduct from this example is

$$\sqrt[3]{1+\frac{1}{\sqrt[3]{2}}+\sqrt[3]{2}}+\sqrt[3]{\sqrt[3]{4}-\sqrt[3]{2}} = \frac{1}{\sqrt[3]{1+\frac{1}{\sqrt[3]{2}}+\sqrt[3]{2}}}+\frac{1}{\sqrt[3]{\sqrt[3]{4}-\sqrt[3]{2}}}. \quad (46)$$

The cubic Shevelev sum is

$$\begin{aligned}&-\sqrt[3]{\frac{1}{1+\frac{1}{\sqrt[3]{2}}+\sqrt[3]{2}}}-\sqrt[3]{1+\frac{1}{\sqrt[3]{2}}+\sqrt[3]{2}}+\sqrt[3]{\frac{1+\frac{1}{\sqrt[3]{2}}+\sqrt[3]{2}}{\sqrt[3]{4}-\sqrt[3]{2}}} \\ &\quad+\sqrt[3]{\frac{\sqrt[3]{4}-\sqrt[3]{2}}{1+\frac{1}{\sqrt[3]{2}}+\sqrt[3]{2}}}-\sqrt[3]{\frac{1}{\sqrt[3]{4}-\sqrt[3]{2}}}-\sqrt[3]{\sqrt[3]{4}-\sqrt[3]{2}} = \sqrt[3]{\left(3-\frac{3}{\sqrt[3]{2}}\right)^2}-\sqrt[3]{9}\sqrt[9]{4}. \quad (47)\end{aligned}$$

6.3 General case

In this part, we discuss the cubic equations that are neither an RCP nor RCP2.

Example 6.3.1. *Let* $\alpha = 1$, $\beta = 8$*, and* $\gamma = 27$*. Then*

$$A = \alpha+\beta+\gamma = 36,\ \alpha\beta+\beta\gamma+\gamma\alpha = 8+216+27 = 251,\ \alpha\beta\gamma = 216.$$

The cubic equation with these roots is

$$f(x) = x^3-36x^2+251x-216 = 0.$$

Denote $A = -36$, $B = 251$, *and* $C = -216$. *It is not an RCP since*

$$A\sqrt[3]{C} + B + 3\sqrt[3]{C^2} = 36 \cdot 6 + 251 + 36 = 503 \neq 0.$$

It neither an RCP2 since

$$A^3C + B^3 + 27C^2 = 36^3 \cdot 216 + 251^3 + 27 \cdot 216^2 \neq 0.$$

The associated Ramanujan equation of $f(x)$ *is*

$$t^3 - 3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right)t - \left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = 0.$$

Now,

$$-3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right) = -3\left(6 + \frac{251}{36} + 3\right) = -\frac{575}{12},$$
$$-\left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = -\left(\frac{251}{6} + 6\left(6 + \frac{251}{36}\right) + 9\right) = -\frac{386}{3}.$$

Then the associated Ramanujan equation is $t^3 - \frac{575}{12}t - \frac{386}{3} = 0$.
By (8),

$$\frac{D(f)}{C^2} = \frac{1}{216^2}\left((36 \cdot 251)^2 - 4\left(36^3 \cdot 216 + 251^3\right) + 18 \cdot 36 \cdot 251 \cdot 216 - 27 \cdot 216^2\right)$$

Hence

$$\frac{\sqrt{D(f)}}{C} = \frac{1729}{108}.$$

By (7), *the real root* t *is*

$$t = \sqrt[3]{\frac{\frac{386}{3} + \frac{1729}{108}}{2}} + \sqrt[3]{\frac{\frac{386}{3} - \frac{1729}{108}}{2}} = \sqrt[3]{\frac{15625}{216}} + \sqrt[3]{\frac{12167}{216}} = \frac{25}{6} + \frac{23}{6} = 8.$$

Hence by (9),

$$\sqrt[3]{1} + \sqrt[3]{8} + \sqrt[3]{27} = 1 + 2 = 3 = 6,$$
$$-\sqrt[3]{-36 - 6 \cdot 6 - 3 \cdot 6 \cdot 8} = 6.$$

On the other hand, by (10),

$$\frac{1}{\sqrt[3]{1}} + \frac{1}{\sqrt[3]{8}} + \frac{1}{\sqrt[3]{27}} = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6},$$
$$\frac{1}{6}\sqrt[3]{251 + 6 \cdot 36 + 3 \cdot 8 \cdot 36} = \frac{11}{6}.$$

By (37), *the cubic Shevelev sum is*

$$\sqrt[3]{\frac{1}{8}} + \sqrt[3]{\frac{8}{1}} + \sqrt[3]{\frac{1}{27}} + \sqrt[3]{\frac{27}{1}} + \sqrt[3]{\frac{27}{8}} + \sqrt[3]{\frac{8}{27}} = \frac{1}{2} + 2 + \frac{1}{3} + 3 + \frac{3}{2} + \frac{2}{3} = 8.$$

Recall that $t = 8$. *Thus, the cubic Shevelev sum formula holds.*

Example 6.3.2. *Let $f(x) = x^3 - 3x^2 - 6x + 18 = 0$. Denote $A = -3$, $B = -6$, and $C = 18$. It is not an RCP since*

$$A\sqrt[3]{C} + B + 3\sqrt[3]{C^2} = -3\sqrt[3]{18} - 6 + 3\sqrt[3]{18^2} \neq 0.$$

It is neither an RCP2 since

$$A^3C + B^3 + 27C^2 = -27(18) - 216 + 27(18^2) = 8046 \neq 0.$$

To find the roots, α, β, and γ, of $f(x)$, we first depress it by substituting $x = y + 1$ to obtain

$$g(y) = y^3 - 9y + 10 = 0.$$

The roots of $g(y)$ are $\alpha_y = 2$, $\beta_y = -1 + \sqrt{6}$, and $\gamma_y = -1 - \sqrt{6}$. Then the roots of $f(x)$ are $\alpha = 3$, $\beta = \sqrt{6}$, and $\gamma = -\sqrt{6}$. Then the Ramanujan-type identities are

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{3} + \sqrt[3]{\sqrt{6}} + \sqrt[3]{-\sqrt{6}} = \sqrt[3]{3}$$

and

$$\frac{1}{\sqrt[3]{\alpha}} + \frac{1}{\sqrt[3]{\beta}} + \frac{1}{\sqrt[3]{\gamma}} = \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{\sqrt{6}}} + \frac{1}{\sqrt[3]{-\sqrt{6}}} = \frac{1}{\sqrt[3]{3}}.$$

The associated Ramanujan equation of $f(x)$ is

$$t^3 - 3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right)t - \left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = 0.$$

Now,

$$-3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right) = -3\left(\frac{-3}{\sqrt[3]{18}} + \frac{-6}{\sqrt[3]{18^2}} + 3\right) = -3\left(3 - \sqrt[3]{\frac{2}{3}} - \sqrt[3]{\frac{3}{2}}\right),$$

$$-\left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = -\left(10 + 6\left(-\sqrt[3]{\frac{2}{3}} - \sqrt[3]{\frac{3}{2}}\right)\right).$$

Then the associated Ramanujan equation is

$$t^3 - 3(3 + K)t - (6K + 10) = 0,$$

where $K = -\sqrt[3]{\frac{2}{3}} - \sqrt[3]{\frac{3}{2}}$. By (8)*, the discriminant is*

$$\frac{D(f)}{C^2} = \frac{1}{18^2}\left(9 \cdot 36 - 4\left(-27(18) - 216\right) + 18(-3)(-6)(18) - 27(18^2)\right) = \frac{2}{3}.$$

By (7)*, the only real root t is*

$$t = \sqrt[3]{\frac{10 + 6\left(-\sqrt[3]{\frac{2}{3}} - \sqrt[3]{\frac{3}{2}}\right) + \sqrt{\frac{2}{3}}}{2}} + \sqrt[3]{\frac{10 + 6\left(-\sqrt[3]{\frac{2}{3}} - \sqrt[3]{\frac{3}{2}}\right) - \sqrt{\frac{2}{3}}}{2}}.$$

We can simplify this quantity by manipulating the associated Ramanujan equation

$$\begin{aligned}0 &= t^3 - 3(3+K)t - (6K+10)\\ &= t^3 - (3K+9)t - 2(3K+5)\\ &= t^3 - (3K+5)t - 4t - 2(3K+5)\\ &= (t^3-4t) - (t+2)(3K+5)\\ &= t(t^2-4) - (t+2)(3K+5)\\ &= t(t+2)(t-2) - (t+2)(3K+5)\\ &= (t+2)\left[t(t-2) - 3K+5\right].\end{aligned}$$

So $t=-2$ is the real root. Since by Theorem 2.3, there is only one real root t, we can conclude that

$$\sqrt[3]{\frac{10+6\left(-\sqrt[3]{\frac{2}{3}}-\sqrt[3]{\frac{3}{2}}\right)+\sqrt{\frac{2}{3}}}{2}}+\sqrt[3]{\frac{10+6\left(-\sqrt[3]{\frac{2}{3}}-\sqrt[3]{\frac{3}{2}}\right)-\sqrt{\frac{2}{3}}}{2}}=-2.$$

Thus, the Ramanujan-type identities are

$$\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}=-\sqrt[3]{A+6\sqrt[3]{C}+3t\sqrt[3]{C}}=-\sqrt[3]{-3+6\sqrt[3]{18}-6\sqrt[3]{18}}=\sqrt[3]{3}$$

and

$$\frac{1}{\sqrt[3]{\alpha}}+\frac{1}{\sqrt[3]{\beta}}+\frac{1}{\sqrt[3]{\gamma}}=-\frac{1}{\sqrt[3]{C}}\sqrt[3]{B+6\sqrt[3]{C^2}+3t\sqrt[3]{C^2}}=-\frac{1}{\sqrt[3]{18}}\sqrt[3]{-6+6\sqrt[3]{18^2}-6\sqrt[3]{18^2}}=\frac{1}{\sqrt[3]{3}}.$$

Example 6.3.3 (Liao et al. [7])**.** *Let $f(x)=x^3-48x-64\sqrt{2}=0$. Denote $A=0$, $B=-48$, and $C=-64\sqrt{2}$. It is not an RCP since*

$$A\sqrt[3]{C}+B+3\sqrt[3]{C^2}=-48+48\sqrt[3]{2}\neq 0.$$

It is also not an RCP2 since

$$A^3C+B^3+27C^2=-48^3+27\cdot 64^2\cdot 2\neq 0.$$

The associated Ramanujan equation of $f(x)$ is

$$t^3+\left(\frac{9}{\sqrt[3]{2}}-9\right)t+\frac{18}{\sqrt[3]{2}}-9=0.$$

By (8)*,*

$$\frac{D(f)}{C^2}=\frac{1}{2\cdot 64^2}\left(-4\cdot(-48)^3-27\cdot 2\cdot 64^2\right)=27.$$

Hence,

$$\frac{\sqrt{D(f)}}{C}=3\sqrt{3}.$$

By (7)*, the real root t is*

$$t=\sqrt[3]{\frac{9-\frac{18}{\sqrt[3]{2}}+3\sqrt{3}}{2}}+\sqrt[3]{\frac{9-\frac{18}{\sqrt[3]{2}}-3\sqrt{3}}{2}}=\sqrt[3]{\frac{3}{2}}\left(\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}+\sqrt{3}}+\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}-\sqrt{3}}\right).$$

From [7], the roots of $f(x)=0$ are:

$$\alpha=-4\sqrt{2},\ \beta=2\sqrt{2}+2\sqrt{6},\ \gamma=2\sqrt{2}-2\sqrt{6}.$$

By (9)*, the first Ramanujan-type identity is*

$$\begin{aligned}&\sqrt[3]{-4\sqrt{2}}+\sqrt[3]{2\sqrt{2}+2\sqrt{6}}+\sqrt[3]{2\sqrt{2}-2\sqrt{6}}\\&\quad=-\sqrt[3]{-6\cdot 4\sqrt[6]{2}-3\left(4\sqrt[6]{2}\right)\sqrt[3]{\frac{3}{2}}\left(\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}+\sqrt{3}}+\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}-\sqrt{3}}\right)}\\&\quad=\sqrt[3]{12}\,\sqrt[18]{2}\sqrt[3]{2+\sqrt[3]{\frac{3}{2}}\left(\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}+\sqrt{3}}+\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}-\sqrt{3}}\right)}.\end{aligned}$$

Hence,

$$-\sqrt[3]{2}+\sqrt[3]{1+\sqrt{3}}+\sqrt[3]{1-\sqrt{3}}=\sqrt[3]{6\sqrt[3]{4}+3\sqrt[3]{6}\left(\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}+\sqrt{3}}+\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}-\sqrt{3}}\right)}. \tag{48}$$

By (10)*, the second Ramanujan-type identity is*

$$\begin{aligned}&\frac{1}{\sqrt[3]{-4\sqrt{2}}}+\frac{1}{\sqrt[3]{2\sqrt{2}+2\sqrt{6}}}+\frac{1}{\sqrt[3]{2\sqrt{2}-2\sqrt{6}}}\\&\quad=\frac{1}{4\sqrt[6]{2}}\sqrt[3]{-48+6\cdot 16\sqrt[3]{2}+3\left(16\sqrt[3]{2}\right)\sqrt[3]{\frac{3}{2}}\left(\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}+\sqrt{3}}+\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}-\sqrt{3}}\right)}\\&\quad=\frac{2\sqrt[3]{6}}{4\sqrt[6]{2}}\sqrt[3]{-1+2\sqrt[3]{2}+\sqrt[3]{3}\left(\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}+\sqrt{3}}+\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}-\sqrt{3}}\right)}.\end{aligned}$$

Hence,

$$-\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{1+\sqrt{3}}}+\frac{1}{\sqrt[3]{1-\sqrt{3}}}=\frac{\sqrt[3]{12}}{2}\sqrt[3]{-1+2\sqrt[3]{2}+\sqrt[3]{3}\left(\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}+\sqrt{3}}+\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}-\sqrt{3}}\right)}. \tag{49}$$

By (37)*, the cubic Shevelev sum is*

$$\begin{aligned}&\sqrt[3]{\frac{-4\sqrt{2}}{2\sqrt{2}+2\sqrt{6}}}+\sqrt[3]{\frac{2\sqrt{2}+2\sqrt{6}}{-4\sqrt{2}}}+\sqrt[3]{\frac{2\sqrt{2}+2\sqrt{6}}{2\sqrt{2}-2\sqrt{6}}}+\sqrt[3]{\frac{2\sqrt{2}-2\sqrt{6}}{2\sqrt{2}+2\sqrt{6}}}+\sqrt[3]{\frac{-4\sqrt{2}}{2\sqrt{2}-2\sqrt{6}}}+\sqrt[3]{\frac{2\sqrt{2}-2\sqrt{6}}{-4\sqrt{2}}}\\&=\sqrt[3]{\frac{3}{2}}\left(\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}+\sqrt{3}}+\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}-\sqrt{3}}\right).\end{aligned}$$

Simplification gives

$$\sqrt[3]{\frac{1+\sqrt{3}}{1-\sqrt{3}}}+\sqrt[3]{\frac{1-\sqrt{3}}{1+\sqrt{3}}}-\sqrt[3]{\frac{2}{1+\sqrt{3}}}-\sqrt[3]{\frac{1+\sqrt{3}}{2}}-\sqrt[3]{\frac{2}{1-\sqrt{3}}}-\sqrt[3]{\frac{1-\sqrt{3}}{2}}$$
$$=\sqrt[3]{\frac{3}{2}}\left(\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}+\sqrt{3}}+\sqrt[3]{3-\frac{6}{\sqrt[3]{2}}-\sqrt{3}}\right). \tag{50}$$

Example 6.3.4. *Let $f(x)=x^3-3x+\sqrt{3}=0$. Denote $A=0$, $B=-3$, and $C=\sqrt{3}$. It is not an RCP since*

$$A\sqrt[3]{C}+B+3\sqrt[3]{C^2}=-3+3\sqrt[3]{3}\neq 0.$$

It is also not an RCP2 since

$$A^3C+B^3+27C^2=-27+27\cdot 3\neq 0.$$

The associated Ramanujan equation of $f(x)$ is $t^3+pt+q=0$, with

$$p=-3\left(\frac{A}{\sqrt[3]{C}}+\frac{B}{\sqrt[3]{C^2}}+3\right)=-3\left(\frac{-3}{\sqrt[3]{3}}+3\right)=3\sqrt[3]{9}-9,$$
$$q=-\left(\frac{AB}{C}+6\left(\frac{A}{\sqrt[3]{C}}+\frac{B}{\sqrt[3]{C^2}}\right)+9\right)=-\left(\frac{-18}{\sqrt[3]{3}}+9\right)=6\sqrt[3]{9}-9.$$

By (8)*,*

$$\frac{D(f)}{C^2}=\frac{1}{3}\left(-4\left(-27\right)-27\cdot 3\right)=\frac{1}{3}(27)=9.$$

Then

$$\frac{\sqrt{D(f)}}{C}=3.$$

By (7)*, the real root t is*

$$t=\sqrt[3]{\frac{-6\sqrt[3]{9}+9+3}{2}}+\sqrt[3]{\frac{-6\sqrt[3]{9}+9-3}{2}}=\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}.$$

From [16], the roots of $f(x)=0$ are:

$$\alpha=2\sin\frac{2\pi}{9},\ \beta=-2\sin\frac{4\pi}{9},\ \gamma=2\sin\frac{8\pi}{9}.$$

By (9)*,*

$$\sqrt[3]{2\sin\frac{2\pi}{9}}+\sqrt[3]{-2\sin\frac{4\pi}{9}}+\sqrt[3]{2\sin\frac{8\pi}{9}}=-\sqrt[3]{6\sqrt[6]{3}+3\sqrt[6]{3}\left(\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}\right)}.$$

Then the first Ramanujan-type identity is

$$\sqrt[3]{\sin\frac{2\pi}{9}}-\sqrt[3]{\sin\frac{4\pi}{9}}+\sqrt[3]{\sin\frac{8\pi}{9}}=-\frac{1}{\sqrt[3]{2}}\sqrt[3]{6\sqrt[6]{3}+3\sqrt[6]{3}\left(\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}\right)}. \tag{51}$$

By (10),

$$\frac{1}{\sqrt[3]{2\sin\frac{2\pi}{9}}}+\frac{1}{\sqrt[3]{-2\sin\frac{4\pi}{9}}}+\frac{1}{\sqrt[3]{2\sin\frac{8\pi}{9}}}=-\frac{1}{\sqrt[6]{3}}\sqrt[3]{-3+6\sqrt[3]{3}+3\sqrt[3]{3}\left(\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}\right)}.$$

Then the second Ramanujan-type identity is

$$\frac{1}{\sqrt[3]{\sin\frac{2\pi}{9}}}-\frac{1}{\sqrt[3]{\sin\frac{4\pi}{9}}}+\frac{1}{\sqrt[3]{\sin\frac{8\pi}{9}}}=\sqrt[3]{2}\,\sqrt[18]{243}\sqrt[3]{\frac{1}{\sqrt[3]{3}}-2-\left(\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}\right)}. \tag{52}$$

These Ramanujan-type identities can also be found in [12, 14]. By (37)*, the cubic Shevelev sum is:*

$$-\sqrt[3]{\frac{\sin\frac{2\pi}{9}}{\sin\frac{4\pi}{9}}}-\sqrt[3]{\frac{\sin\frac{4\pi}{9}}{\sin\frac{2\pi}{9}}}-\sqrt[3]{\frac{\sin\frac{4\pi}{9}}{\sin\frac{8\pi}{9}}}-\sqrt[3]{\frac{\sin\frac{8\pi}{9}}{\sin\frac{4\pi}{9}}}+\sqrt[3]{\frac{\sin\frac{2\pi}{9}}{\sin\frac{8\pi}{9}}}+\sqrt[3]{\frac{\sin\frac{8\pi}{9}}{\sin\frac{2\pi}{9}}}=\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}. \tag{53}$$

Example 6.3.5. *Let* $f(x)=x^3-3\sqrt[3]{2}x^2-3\sqrt[3]{2}x+1=0$*. Denote* $A=-3\sqrt[3]{2}=B$ *and* $C=1$*. It is not an RCP since*

$$A\sqrt[3]{C}+B+3\sqrt[3]{C^2}=-3\sqrt[3]{2}+-3\sqrt[3]{2}+3\neq 0.$$

It is not an RCP2 since

$$A^3C+B^3+27C^2=-54-54+27\neq 0.$$

The associated Ramanujan equation of $f(x)$ *is* $t^3+pt+=0$*, with*

$$\begin{aligned}p&=-3\left(\frac{A}{\sqrt[3]{C}}+\frac{B}{\sqrt[3]{C^2}}+3\right)=-3\left(-3\sqrt[3]{2}-3\sqrt[3]{2}+3\right)=18\sqrt[3]{2}-9,\\ q&=-\left(\frac{AB}{C}+6\left(\frac{A}{\sqrt[3]{C}}+\frac{B}{\sqrt[3]{C^2}}\right)+9\right)=-\left(9\sqrt[3]{4}+6\left(-3\sqrt[3]{2}-3\sqrt[3]{2}\right)+9\right)\\ &=36\sqrt[3]{2}-9\sqrt[3]{4}-9.\end{aligned}$$

By (8)*,*

$$\frac{D(f)}{C^2}=162\sqrt[3]{2}-4\left(-54-54\right)+162\sqrt[3]{4}-27=162\sqrt[3]{4}+162\sqrt[3]{2}+405.$$

Hence,

$$\sqrt{D(f)}=9\sqrt{5+2\sqrt[3]{2}+2\sqrt[3]{4}}.$$

By (7)*, the real root t is*

$$t=\sqrt[3]{\frac{9+9\sqrt[3]{4}-36\sqrt[3]{2}+9\sqrt{5+2\sqrt[3]{2}+2\sqrt[3]{4}}}{2}}+\sqrt[3]{\frac{9+9\sqrt[3]{4}-36\sqrt[3]{2}-9\sqrt{5+2\sqrt[3]{2}+2\sqrt[3]{4}}}{2}}.$$

To find the roots of $f(x)=0$*, we first use the rational roots theorem to obtain one root* $\alpha=-1$*.*

Divide $f(x)$ by $x+1$ yields

$$x^2-\left(1+3\sqrt[3]{2}\right)x+1=0.$$

The remaining two roots are

$$\beta=\frac{1}{2}\left(1+3\sqrt[3]{2}+\sqrt{\left(1+3\sqrt[3]{2}\right)^2-4}\right)=\frac{1}{2}\left(1+3\sqrt[3]{2}+\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right),$$
$$\gamma=\frac{1}{2}\left(1+3\sqrt[3]{2}-\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right).$$

By (9)*, the first Ramanujan-type identity is*

$$-1+\sqrt[3]{\frac{1}{2}\left(1+3\sqrt[3]{2}+\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right)}+\sqrt[3]{\frac{1}{2}\left(1+3\sqrt[3]{2}-\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right)}$$
$$=\sqrt[3]{3\sqrt[3]{2}-6-3\left(\sqrt[3]{\frac{9+9\sqrt[3]{4}-36\sqrt[3]{2}+9\sqrt{5+2\sqrt[3]{2}+2\sqrt[3]{4}}}{2}}+\sqrt[3]{\frac{9+9\sqrt[3]{4}-36\sqrt[3]{2}-9\sqrt{5+2\sqrt[3]{2}+2\sqrt[3]{4}}}{2}}\right)}. \tag{54}$$

By (10)*, the second Ramanujan-type identity is*

$$-1+\frac{1}{\sqrt[3]{\frac{1}{2}\left(1+3\sqrt[3]{2}+\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right)}}+\frac{1}{\sqrt[3]{\frac{1}{2}\left(1+3\sqrt[3]{2}-\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right)}}$$
$$=\sqrt[3]{3\sqrt[3]{2}-6-3\left(\sqrt[3]{\frac{9+9\sqrt[3]{4}-36\sqrt[3]{2}+9\sqrt{5+2\sqrt[3]{2}+2\sqrt[3]{4}}}{2}}+\sqrt[3]{\frac{9+9\sqrt[3]{4}-36\sqrt[3]{2}-9\sqrt{5+2\sqrt[3]{2}+2\sqrt[3]{4}}}{2}}\right)}. \tag{55}$$

A byproduct of this example is

$$\sqrt[3]{\frac{1}{2}\left(1+3\sqrt[3]{2}+\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right)}+\sqrt[3]{\frac{1}{2}\left(1+3\sqrt[3]{2}-\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right)}$$
$$=\frac{1}{\sqrt[3]{\frac{1}{2}\left(1+3\sqrt[3]{2}+\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right)}}+\frac{1}{\sqrt[3]{\frac{1}{2}\left(1+3\sqrt[3]{2}-\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right)}}. \tag{56}$$

By (37)*, the cubic Shevelev sum is*

$$-\sqrt[3]{\frac{2}{1+3\sqrt[3]{2}+\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}}}-\sqrt[3]{\frac{1}{2}\left(1+3\sqrt[3]{2}+\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right)}$$
$$-\sqrt[3]{\frac{2}{1+3\sqrt[3]{2}-\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}}}-\sqrt[3]{\frac{1}{2}\left(1+3\sqrt[3]{2}-\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}\right)}$$
$$+\sqrt[3]{\frac{1+3\sqrt[3]{2}+\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}}{1+3\sqrt[3]{2}-\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}}}+\sqrt[3]{\frac{1+3\sqrt[3]{2}-\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}}{1+3\sqrt[3]{2}+\sqrt{6\sqrt[3]{2}+9\sqrt[3]{4}-3}}}$$
$$=\sqrt[3]{\frac{9+9\sqrt[3]{4}-36\sqrt[3]{2}+9\sqrt{5+2\sqrt[3]{2}+2\sqrt[3]{4}}}{2}}+\sqrt[3]{\frac{9+9\sqrt[3]{4}-36\sqrt[3]{2}-9\sqrt{5+2\sqrt[3]{2}+2\sqrt[3]{4}}}{2}}. \tag{57}$$

Example 6.3.6. *Let $f(x) = x^3+mx^2+mx+1 = 0$, where $m \in \mathbb{R}$ and $m > 3$ or $m < -1$. Denote $A = B = m$ and $C = 1$. This is an RCP if $m = -\frac{3}{2}$, since $A\sqrt[3]{C}+B+3\sqrt[3]{C^2} = -\frac{3}{2} - \frac{3}{2}+3 = 0$. This is an RCP2 if $m = -\frac{3}{\sqrt[3]{2}}$, since $A^3 + B^3 + 27C^2 = -\frac{27}{2} - \frac{27}{2} + 27 = 0$.*

For the cases of $m \neq -\frac{3}{2}$ and $m \neq -\frac{3}{\sqrt[3]{2}}$, $f(x) = 0$ is neither an RCP nor RCP2. Note that $f(x) = 0$ will always have a root $\alpha = -1$. The quadratic factor of $f(x)$ is $x^2 + (m-1)x + 1$. Then the remaining two roots are

$$\beta = \frac{1}{2}\left(1 - m + \sqrt{m^2 - 2m - 3}\right),$$
$$\gamma = \frac{1}{2}\left(1 - m - \sqrt{m^2 - 2m - 3}\right),$$

which are distinct.

The associated Ramanujan equation of $f(x)$ is $t^3 + pt + q = 0$, with

$$p = -3\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}} + 3\right) = -3(2m+3) = -(6m+9),$$
$$q = -\left(\frac{AB}{C} + 6\left(\frac{A}{\sqrt[3]{C}} + \frac{B}{\sqrt[3]{C^2}}\right) + 9\right) = -(m^2 + 6(2m) + 9) = -(m^2 + 12m + 9).$$

By (8)*, the discriminant of $f(x)$ is*

$$D(f) = m^4 - 4(m^3 + m^3) + 18m^2 - 27 = m^4 - 8m^3 + 18m^2 - 27 = (m-3)^3(m+1).$$

Hence,

$$\sqrt{D(f)} = \sqrt{(m-3)^3(m+1)},$$

since $m > 3$ or $m < -1$. By (7)*, the real root t is*

$$t = \sqrt[3]{\frac{m^2 + 12m + 9 + \sqrt{(m-3)^3(m+1)}}{2}} + \sqrt[3]{\frac{m^2 + 12m + 9 - \sqrt{(m-3)^3(m+1)}}{2}}.$$

By (9)*, the first Ramanujan-type identity is*

$$-1 + \sqrt[3]{\frac{1}{2}\left(1 - m + \sqrt{m^2 - 2m - 3}\right)} + \sqrt[3]{\frac{1}{2}\left(1 - m - \sqrt{m^2 - 2m - 3}\right)}$$
$$= -\sqrt[3]{m + 6 + 3\left(\sqrt[3]{\frac{m^2 + 12m + 9 + \sqrt{(m-3)^3(m+1)}}{2}} + \sqrt[3]{\frac{m^2 + 12m + 9 - \sqrt{(m-3)^3(m+1)}}{2}}\right)}. \quad (58)$$

By (10)*, the second Ramanujan-type identity is*

$$-1 + \frac{1}{\sqrt[3]{\frac{1}{2}\left(1 - m + \sqrt{m^2 - 2m - 3}\right)}} + \frac{1}{\sqrt[3]{\frac{1}{2}\left(1 - m - \sqrt{m^2 - 2m - 3}\right)}}$$
$$= -\sqrt[3]{m + 6 + 3\left(\sqrt[3]{\frac{m^2 + 12m + 9 + \sqrt{(m-3)^3(m+1)}}{2}} + \sqrt[3]{\frac{m^2 + 12m + 9 - \sqrt{(m-3)^3(m+1)}}{2}}\right)}. \quad (59)$$

A byproduct from this example is

$$\sqrt[3]{\frac{1}{2}\left(1-m+\sqrt{m^2-2m-3}\right)}+\sqrt[3]{\frac{1}{2}\left(1-m-\sqrt{m^2-2m-3}\right)}$$
$$=\frac{1}{\sqrt[3]{\frac{1}{2}\left(1-m+\sqrt{m^2-2m-3}\right)}}+\frac{1}{\sqrt[3]{\frac{1}{2}\left(1-m-\sqrt{m^2-2m-3}\right)}}. \tag{60}$$

Recall β and γ are two roots of the quadratic equation $x^2+(m-1)x+1=0$. This gives $\beta\gamma=1$. Hence, the byproduct is a result of

$$\sqrt[3]{\beta}+\sqrt[3]{\gamma}=\frac{1}{\sqrt[3]{\beta}}+\frac{1}{\sqrt[3]{\gamma}}.$$

We also have $\sqrt[3]{\beta}+\frac{1}{\sqrt[3]{\beta}}=\sqrt[3]{\gamma}+\frac{1}{\sqrt[3]{\gamma}}$. Hence, the Ramanujan-type identities are exactly the same. In addition, the left hand side of the cubic Shevelev sum (37) *can be simplified to*

$$\sqrt[3]{\frac{\alpha}{\beta}}+\sqrt[3]{\frac{\beta}{\alpha}}+\sqrt[3]{\frac{\alpha}{\gamma}}+\sqrt[3]{\frac{\gamma}{\alpha}}+\sqrt[3]{\frac{\beta}{\gamma}}+\sqrt[3]{\frac{\gamma}{\beta}}=-\sqrt[3]{\gamma}-\sqrt[3]{\beta}-\sqrt[3]{\beta}-\sqrt[3]{\gamma}+\sqrt[3]{\beta^2}+\sqrt[3]{\gamma^2}$$
$$=\sqrt[3]{\beta^2}+\sqrt[3]{\gamma^2}-2\sqrt[3]{\beta}-2\sqrt[3]{\gamma}.$$

Then

$$\sqrt[3]{\beta^2}+\sqrt[3]{\gamma^2}-2\sqrt[3]{\beta}-2\sqrt[3]{\gamma}=t.$$

Thus, the cubic Shevelev sum becomes

$$\sqrt[3]{\frac{\left(1-m+\sqrt{m^2-2m-3}\right)^2}{4}}+\sqrt[3]{\frac{\left(1-m-\sqrt{m^2-2m-3}\right)^2}{4}}$$
$$-2\sqrt[3]{\frac{1-m+\sqrt{m^2-2m-3}}{2}}-2\sqrt[3]{\frac{1-m-\sqrt{m^2-2m-3}}{2}}$$
$$=\sqrt[3]{\frac{m^2+12m+9+\sqrt{(m-3)^3(m+1)}}{2}}+\sqrt[3]{\frac{m^2+12m+9-\sqrt{(m-3)^3(m+1)}}{2}}. \tag{61}$$

6.4 Constructing cosine Ramanujan-type identities

To construct cosine Ramanujan-type identities by using Theorem 2.5, we choose a suitable $r\in\mathbb{C}$. For a more general result, we can use Corollary 2.7 and choose any $r\in\mathbb{C}$ and $A\in\mathbb{R}$.

Dresden et al. [4] gave an example similar to the following example. We use our approach in the following example.

Example 6.4.1. *Let $r=\frac{\sqrt{3}}{2}+\frac{1}{2}i$. Then $s=\frac{\sqrt{3}}{2}-\frac{1}{2}i$, $r+s=\sqrt{3}$, $rs=|r|^2=1$, and $\theta=\operatorname{Arg}(r)=\frac{\pi}{6}$. By Theorem 2.5, the cubic equation is $x^3-3x+\sqrt{3}=0$. The roots are*

$$\alpha=-2\sqrt{rs}\cos\left(\frac{\theta}{3}\right),\ \beta=-2\sqrt{rs}\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right),\ \gamma=-2\sqrt{rs}\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right).$$

Then

$$\alpha = -2\cos\frac{\pi}{18},\ \beta = -2\cos\frac{13\pi}{18},\ \gamma = -2\cos\frac{25\pi}{18}.$$

The real root t to the associated Ramanujan equation is

$$\begin{aligned} t &= \sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{\xi^2}}\right)+3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}} + \sqrt[3]{\frac{9\left(1-\sqrt[3]{\frac{2}{\xi^2}}\right)-3\sqrt{3}\left(\frac{\eta}{\xi}\right)}{2}} \\ &= \sqrt[3]{\frac{9\left(1-2\sqrt[3]{\frac{1}{3}}\right)+3\sqrt{3}\left(\frac{1}{\sqrt{3}}\right)}{2}} + \sqrt[3]{\frac{9\left(1-2\sqrt[3]{\frac{1}{3}}\right)-3\sqrt{3}\left(\frac{1}{\sqrt{3}}\right)}{2}} \\ &= \sqrt[3]{6-\frac{9}{\sqrt[3]{3}}} + \sqrt[3]{3-\frac{9}{\sqrt[3]{3}}} = \sqrt[3]{6-3\sqrt[3]{9}} + \sqrt[3]{3-3\sqrt[3]{9}} \end{aligned}$$

By (21)*,*

$$\sqrt[3]{\cos\left(\frac{\theta}{3}\right)} + \sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)} + \sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)} = \sqrt[9]{2\xi}\sqrt[3]{\frac{6+3t}{2}}.$$

Then the first Ramanujan-type identity is

$$\sqrt[3]{\cos\frac{\pi}{18}} + \sqrt[3]{\cos\frac{13\pi}{18}} + \sqrt[3]{\cos\frac{25\pi}{18}} = \frac{\sqrt[18]{3}}{\sqrt[3]{2}}\sqrt[3]{6+3\left(\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}\right)}$$

By (22)*,*

$$\frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3}\right)}} + \frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}} + \frac{1}{\sqrt[3]{\cos\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}} = \sqrt[3]{\frac{3}{\xi}}\sqrt[3]{-1+2\sqrt[3]{4\xi^2}+t\sqrt[3]{4\xi^2}}.$$

Then the second Ramanujan-type identity is

$$\frac{1}{\sqrt[3]{\cos\frac{\pi}{18}}} + \frac{1}{\sqrt[3]{\cos\frac{13\pi}{18}}} + \frac{1}{\sqrt[3]{\cos\frac{25\pi}{18}}} = \sqrt[3]{2}\sqrt[6]{3}\sqrt[3]{-1+2\sqrt[3]{3}+\sqrt[3]{3}\left(\sqrt[3]{6-3\sqrt[3]{9}}+\sqrt[3]{3-3\sqrt[3]{9}}\right)}.$$

7 Conclusion

In this paper, we have related the Ramanujan, Shevelev and Cardano cubic equations with a generalized computation procedure for constructing related identities, including cosine Ramanujan-type identities. This work connects some sixteenth century and twentieth and twenty-first century advances in understanding cubic equations and their inter-related numerical treatment. Ramanujan's work too responds to speculation about the generalization on Newton's formula for finding the root of a non-linear function when applied to cubic polynomials [3].

It also opens up extensions for further development with computation procedures and equations of higher degree. For instance, with computation procedures such as extensions of the Bernoulli iteration [10], or with polynomials of higher degree; for example, the advances of Ludovico Ferrari of Bologna (1522–1565), a student of Cardano, who discovered the general method for solving quartics [6]. Likewise, Paolo Ruffini (1765–1822) anticipated the general idea of a group and claimed to have proved the insolvability of the quintic by radicals, though it was Niels Henrik Abel (1802–1829) who finally settled the issue to the satisfaction of the mathematical community of the time. This, in turn, opens up further connections of the worlds of algebra and related Diophantine equations more generally [13], since the lack of integer roots in the case of some cubics is a consequence of Fermat's Last Theorem.

References

[1] Boyer, C. B., & Merzbach, U. C. (2011). *A History of Mathematics*. John Wiley & Sons.

[2] Chen, W. Y. C. (2022). Cubic equations through the looking glass of Sylvester. *The College Mathematics Journal*, 53(5), 396–398.

[3] De Pillis, L. G. (1998). Newton's cubic roots. *Gazette of the Australian Mathematical Society*, 25(5), 236–241.

[4] Dresden, G., Panthi, P., Shrestha, A., & Zhang, J. (2019). Cubic polynomials, linear shifts, and Ramanujan simple cubics. *Mathematics Magazine*, 92(5), 374–381.

[5] Gilbert, L., & Gilbert, J. (2014). *Elements of Modern Algebra* (8th ed.). Cengage Learning.

[6] Hillman, A. P., & Alexanderson, G. L. (1988). *A First Undergraduate Course in Abstract Algebra*. Brooks/Cole.

[7] Liao, H.-C., Saul, M., & Shiue, P. J.-S. (in press). Revisiting the general cubic: A simplification of Cardano's solution. *The Mathematical Gazette*.

[8] McLeish, J. (1994). *The Story of Numbers*. Ballantine Books.

[9] Ramanujan, S. (1957). *Notebooks of Srinivasa Ramanujan* (2 volumes). Tata Institute of Fundamental Research, Bombay.

[10] Shannon, A. G. (1974). The Jacobi–Perron algorithm and Bernoulli's iteration. *The Mathematics Student*, 42, 52–56.

[11] Shevelev, V. (2007). On Ramanujan cubic polynomials. *South East Asian Mathematics and Mathematical Sciences*, 8, 113–122.

[12] Shiue, P. J.-S., Shannon, A. G., Huang, S. C., & Reyes, J. E. (2022). Notes on efficient computation of Ramanujan cubic equations. *Notes on Number Theory and Discrete Mathematics*, 28(2), 350–375.

[13] Van der Poorten, A. (1996). Notes on Fermat's last theorem. *Computers & Mathematics with Applications*, 31(11), 139–139.

[14] Wang, K. (2021). On Ramanujan type identities and Cardano formula. *Notes on Number Theory and Discrete Mathematics*, 27(3), 155–174.

[15] Wituła, R. (2010). Full description of Ramanujan cubic polynomials. *Journal of Integer Sequences*, 13, Article 10.5.7.

[16] Wituła, R. (2010). Ramanujan cubic polynomials of the second kind. *Journal of Integer Sequences*, 13, Article 10.7.5.

[17] Wituła, R. (2012). Ramanujan type trigonometric formulae. *Demonstratio Mathematica*, 45(4), 779–796.