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# Sums involving the binomial coefficients, Bernoulli numbers of the second kind and harmonic numbers

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**Abstract:** We offer a number of various finite and infinite sum identities involving the binomial coefficients, Bernoulli numbers of the second kind and harmonic numbers. For example, among many others, we prove

$$\sum_{k=0}^{n} \frac{(-1)^k h_k}{4^k} \binom{2k}{k} G_{n-k} = \frac{(-1)^{n-1}}{2^{2n-1}} \binom{2n-2}{n-1}$$

and

$$\sum_{k=1}^{\infty} \frac{h_k}{k^2(2k-1)4^k} \binom{2k}{k} = 2\pi + 3\zeta(2)\log 2 - 3\zeta(2) - \frac{7}{2}\zeta(3),$$

where $h_k = H_{2k} - \frac{1}{2}H_k$, $G_k$ are Bernoulli numbers of the second kind, and $\zeta$ is the Riemann zeta function. We also give an alternate proof of the series representations for the constants $\log(2\pi)$ and $\gamma$ given by Blagouchine and Coppo.

**Keywords:** Binomial sums, Harmonic sums, Binomial coefficients, Gregory coefficients, Bernoulli numbers of the second kind, Polygamma functions, Riemann zeta function, Harmonic numbers.
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# 1 Introduction

Let us first recall some tools we need in the following analysis. The gamma and digamma functions denoted by $\Gamma$ and $\psi$, respectively, are defined by

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du \quad \text{and} \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad (x > 0).$$

The functions $\psi, \psi', \psi'', \ldots$ are known to be the polygamma functions and $\psi^{(0)}(x) = \psi(x)$. The polygamma functions satisfy the following fundamental functional equation:

$$\psi^{(n)}(x+1) - \psi^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

and

$$\psi^{(n)}(1) = (-1)^{n-1} n! \zeta(n+1) \quad \text{and} \quad \psi^{(n)}(1/2) = (-1)^{n+1} n! \left(2^{n+1} - 1\right) \zeta(n+1).$$

The digamma and trigamma functions also satisfy

$$\psi(n+1/2) = 2h_n - \log 4 - \gamma \quad \text{and} \quad \psi'(n+1/2) = 4\psi'(2n+1) - \psi'(n+1), \tag{1.2}$$

where $h_n = H_{2n} - \frac{1}{2}H_n$, $\zeta(s) = \sum_{k=1}^\infty \frac{1}{k^s}$ $(\Re(s) > 1)$ is the Riemann zeta function and $\gamma$ is the Euler–Mascheroni constant. The beta function $B$ is defined, for $\Re(t), \Re(s) > 0$, by

$$B(s,t) = \int_0^1 u^{t-1}(1-u)^{s-1}du = \frac{\Gamma(t)\Gamma(s)}{\Gamma(s+t)}, \tag{1.3}$$

and

$$\psi(1-x) - \psi(x) = \pi\cot(\pi x) \quad \text{and} \quad \psi'(x) - \psi'(1-x) = \pi^2\csc^2(\pi x). \tag{1.4}$$

For the basic properties of the polygamma functions we refer to [31, pp. 24-35]. A generalized binomial coefficient $\binom{s}{t}$ $(s, t \in \mathbb{C})$ is defined, in terms of the classical gamma function, by

$$\binom{s}{t} = \frac{\Gamma(s+1)}{\Gamma(t+1)\Gamma(s-t+1)}, \quad (s, t \in \mathbb{C}). \tag{1.5}$$

Particularly, we have

$$\binom{k-1/2}{k} = \frac{1}{4^k}\binom{2k}{k} \quad \text{and} \quad \binom{1/2}{k} = \frac{(-1)^{k-1}}{4^k(2k-1)}\binom{2k}{k}. \tag{1.6}$$

Binomial coefficients satisfy the following simple relations

$$\frac{s+1}{t+1}\binom{s}{t} = \binom{s+1}{t+1} \quad \text{and} \quad \binom{s+1}{t} = \binom{s}{t} + \binom{s}{t-1}. \tag{1.7}$$

For $s \in \mathbb{C}$, a generalized harmonic number $H_n^{(s)}$ of order $s$ is defined by

$$H_n^{(s)} = \sum_{k=1}^n \frac{1}{k^s}, \quad \text{and} \quad H_n^{(1)} = H_n, \quad H_0^n = 0; \tag{1.8}$$

see [18, 28, 29] or [30]. For any complex number $x$, which is not a negative integer, the digamma function and harmonic numbers are related with

$$H_x^{(1)} = H_x = \gamma + \psi(x+1), \tag{1.9}$$

where $\gamma$ is the familiar Euler–Mascheroni constant. In the case of non-integer values of the argument $x$, which is not a negative integer, we may write the generalized harmonic numbers $H_\alpha^{(m+1)}$, in terms of the polygamma functions:

$$H_x^{(m+1)} = \zeta(m+1) + \frac{(-1)^m}{m!}\psi^{(m)}(x+1), \quad n \in \mathbb{N}. \tag{1.10}$$

see [18]. Some special values include

$$H^{(1)}_{-\frac{1}{2}} = -2\log 2, \quad H^{(2)}_{-\frac{1}{2}} = -2\zeta(2), \quad H^{(1)}_{-\frac{3}{4}} = -\frac{\pi}{2} - 3\log 2,$$
$$\text{and} \quad H^{(2)}_{-\frac{3}{4}} = -8G - 5\zeta(2),$$

where $G = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$ is Catalan's constant. Here and throughout, an empty sum is understood to be nil and so $H_0(s) = 0$. The Stirling numbers of the first kind are defined by the generating function of the falling factorial $(x)_n = x(x-1)(x-2)\cdots(x-(n-1))$ $(n \geq 0)$, that is,

$$\sum_{k=0}^{\infty} s(n,k)x^k = (x)_n.$$

Some special values include

$$\begin{aligned} &s(n,0) = \delta_{n0}, \quad s(n,1) = (-1)^{n-1}(n-1)!, \quad s(n,2) = (-1)^n(n-1)!H_{n-1} \\ &s(n,3) = \frac{1}{2}(-1)^{n-1}(n-1)!\left(H_{n-1}^2 - H_{n-1}^{(2)}\right) \\ &s(n,4) = \frac{1}{6}(-1)^n(n-1)!\left(H_{n-1}^3 - 3H_{n-1}H_{n-1}^{(2)} + 2H_{n-1}^{(3)}\right). \end{aligned} \tag{1.11}$$

Here $\delta_{ij}$ is the Kronecker delta.

The Bernoulli numbers of the second kind, denoted by $G_n$, which are also known as the Gregory coefficients, are defined by the generating function

$$\frac{x}{\log(x+1)} = \sum_{k=0}^{\infty} G_k x^k, \quad |x| < 1$$

or recursively

$$G_1 = \frac{1}{2} \quad \text{and} \quad G_n = \frac{(-1)^{n+1}}{n+1} + \sum_{k=1}^{n-1} \frac{(-1)^{n+1-k}G_k}{n+1-k}, \quad n \geq 2 \tag{1.12}$$

or by an integral representation

$$G_n = \int_0^1 \binom{\alpha}{n} d\alpha. \tag{1.13}$$

These numbers have various applications in number theory and numerical analysis (see [27] and [7]). For example, we have

$$\gamma = \sum_{k=1}^{\infty} \frac{|G_k|}{k}, \tag{1.14}$$

which is known in the literature as the Fontana–Mascheroni series, and is the first known series representation for the Euler–Mascheroni constant $\gamma$ having rational terms only; see [8, pp. 406, 413, 429] and [11, p. 379]. The first few terms are:

$$G_0 = 1,\ G_1 = \frac{1}{2},\ G_2 = -\frac{1}{12},\ G_3 = \frac{1}{24},\ G_4 = -\frac{19}{720}.$$

Full asymptotic expansions of these numbers can be found in Nemes [23] and Blagouchine [7, Eq. (52)], [16, p.379] and [14]. The Bernoulli numbers of the second kind are related to some special numbers such as $s(n,k)$, the Stirling numbers of the first kind, and $c_n$, the Cauchy numbers of the first kind, by the following formulas

$$G_n = \frac{1}{n!}\sum_{k=1}^{n}\frac{s(n,k)}{k+1} \quad \text{and} \quad G_n = \frac{c_n}{n!};$$

see [7, Section 2.1]. A detailed study of these numbers, and many finite and infinite sums involving these numbers can be found in Boyadzhiev [13] and Merlini et al. [22]. So many results we obtained for $G_n$ can be easily convert to identities involving $G_n$. For example, we obtain from Example 3.1 that

$$\sum_{k=0}^{n}\binom{n}{k}c_k c_{n-k} = (2n-n^2)c_{n-1} + (1-n)c_n. \tag{1.15}$$

In the literature there are many interesting infinite series involving the Bernoulli numbers of the second kind. We can recall the following series:

$$\sum_{n=2}^{\infty}\frac{|G_n|}{n-1} = -\frac{1}{2} + \frac{\log(2\pi)}{2} - \frac{\gamma}{2} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{|G_n|}{n+1} = 1 - \log 2;$$

see [1] and [2], and

$$\sum_{n=1}^{\infty}\frac{|G_n|H_n}{n} = \zeta(2) - 1;$$

see [14, p. 307].

Finite sums involving Bernoulli numbers of the second kind arise in number theory. For example, in [11] the authors discovered the following interesting series for the Euler–Mascheroni constants $\gamma$ and $\log(2\pi)$:

$$\gamma = 2\log(2\pi) - 3 - 2\sum_{n=1}^{\infty}\frac{1}{n+1}\sum_{k=1}^{n}|G_k G_{n+2-k}|, \tag{1.16}$$

and

$$\log(2\pi) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}|G_k G_{n+1-k}|. \tag{1.17}$$

Here we provide alternative proofs of these two identities. We believe that finite sums involving Bernoulli numbers of the second kind were not investigated sufficiently, and as far as we know there are no combinatorial identities involving them in the literature. One of our aims in this work is to derive some finite and infinite sums involving these important numbers.

The Bernoulli polynomials of the second kind $\Psi_n(x)$, also known as the Fontana–Bessel polynomials, are the polynomials defined by the following generating function: For $|x| < 1$

$$\frac{x(1+x)^t}{\log(1+x)} = \sum_{n=0}^{\infty}\Psi_n(t)x^n. \tag{1.18}$$

The first four terms are:

$$\Psi_0(t) = 1,\ \Psi_1(t) = t + \frac{1}{2},\ \Psi_2(t) = \frac{t^2}{2} - \frac{1}{12},\ \Psi_3(t) = \frac{t^3}{6} - \frac{t^2}{4} + \frac{1}{24}.$$

These polynomials may be represented by the integrals

$$\Psi_n(t) = \int_t^{t+1} \binom{u}{n} du. \tag{1.19}$$

Note that $\Psi_n(0) = G_n$. For many other properties of these polynomials see [7, 19, 20].

Our other objective in this work is to investigate the Bernoulli numbers of the second kind and Bernoulli numbers of the second kind polynomials, and their relations with other quantities such as harmonic numbers, binomial coefficients and the Stirling number of the first kind. As can be seen in the second and third sections, our work covers many interesting finite sums that include these numbers. We also present many new finite and infinite binomial sums.

In Jordan's book [19, pp. 166, 194 -195] the following identity, which establishes an interesting relation between $\zeta(n)$ and $s(n,k)$, was recorded:

$$\zeta(n+1) = \sum_{k=n}^{\infty} \frac{s(k,n)}{kk!}, \quad n = 1, 2, 3, \ldots. \tag{1.20}$$

This formula was rediscovered by some authors recently; see, [7, 25, 26]. The Newton series for the digamma function, sometimes referred to as Stern's series [7, 24], reads

$$\psi(z+1) = -\gamma - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \binom{z}{k}, \quad \Re(z) > -1. \tag{1.21}$$

Integrating both sides of this equation over $(x, x+1)$, one obtains

$$\log(x+1) = -\gamma - \int_x^{x+1} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \binom{z}{k} dz \tag{1.22}$$

and to proceed we need to reverse the order of integration and summation. Since, for $k \in \mathbb{N}$, $f_k(z) = \sum_{j=1}^{k} \binom{z}{j}$ converges to $2^z - 1$ as $k \to \infty$, $\left| \sum_{j=1}^{k} \frac{(-1)^j}{j} \binom{z}{j} \right| \leq \sum_{j=1}^{k} \binom{z}{j} \leq 2^z - 1$ and $2^z - 1$ is integrable over $(x, x+1)$, Dominant Convergence Theorem justifies the interchanging of the order of integration and summation. Thus, we have

$$\log(x+1) = -\gamma - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \int_x^{x+1} \binom{z}{k} dz = -\gamma - \sum_{k=1}^{\infty} \frac{(-1)^k \Psi_k(x)}{k}. \tag{1.23}$$

Clearly, (1.23) reduses to (1.14) when $x = 0$. This highlights how important expressions of the form (1.21) are. This is a known result; see [21, p.280, Eq. 10,17]. Inspired by the Newton series for the digamma function given in (1.21), it is natural to consider the following general form of it:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^m} \binom{x}{k}, \quad \Re(x) > -1,\ m \in \mathbb{N}.$$

Our final aim in this study is to provide a closed form evaluation for this sum. We note that a finite form of these sums has been investigated by the first author in [4] and many interesting finite and infinite sums involving generalized harmonic numbers have been established.

We continue with the following lemma.

**Lemma 1.1.** *Let $k$ be a non-negative integer. Then*

$$\lim_{x\to -k}\frac{\psi(x)}{\Gamma(x)}=(-1)^{k-1}k!.$$

*So, we can define*

$$\frac{\psi(x)}{\Gamma(x)}=\begin{cases}\frac{\psi(x)}{\Gamma(x)} & \text{If } x\in\mathbb{C}\backslash(\mathbb{Z}^-\cup\{0\})\\ (-1)^{k-1}k! & \text{If } x=-k,\ k\in(\mathbb{Z}^+\cup\{0\})\end{cases}$$

*Proof.* It is well known that the residue of the gamma function $\Gamma(z)$ at $z=-k$, $k=0,-1,-2,\dots$ is $\frac{(-1)^k}{k!}$. Thus,

$$\text{Res}\,(\Gamma(z),-k)=\lim_{z\to -k}(z+k)\Gamma(z)=\frac{(-1)^k}{k!}.$$

This leads to

$$\lim_{z\to -k}\frac{\psi(x)}{\Gamma(x)}=\lim_{z\to -k}\frac{\Gamma'(z)}{\Gamma^2(z)}=\lim_{z\to -k}\frac{(z+k)^2\Gamma'(z)}{[(z+k)\Gamma(z)]^2}=(k!)^2\lim_{z\to -k}\frac{\Gamma'(z)}{\frac{1}{(z+k)^2}}$$
$$=-(k!)^2\lim_{z\to -k}\frac{\Gamma(z)}{\frac{1}{z+k}}=-(k!)^2\lim_{z\to -k}(z+k)\Gamma(z)=(-1)^{k-1}k!.$$

which is exactly Lemma 1.1. □

# 2 Main results

In this section we collect our main results. Our first theorem provides a generalization of (1.21).

**Theorem 2.1.** *Let $x$ be a real number, which is not a negative integer, and $m\in\mathbb{N}$. Then*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{x}{k}=\frac{(-1)^m\Gamma(x+1)}{m!}\frac{\partial^m}{\partial z^m}G(x,z)\bigg|_{z=1},\tag{2.1}$$

*where*

$$G(x,z)=\frac{\Gamma(z)}{\Gamma(z+x)}.$$

*Proof.* Using the simple fact $\frac{1}{k^m}=\frac{1}{(m-1)!}\int_0^\infty e^{-ku}du$, we get

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^m}\binom{x}{k}=\frac{1}{(m-1)!}\sum_{k=1}^{\infty}(-1)^k\binom{x}{k}\int_0^\infty u^{m-1}e^{-ku}du.$$

Reversing the order of summation and integration, which is justified by a similar argument given for (1.21), we get

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^m}\binom{x}{k}=\frac{1}{(m-1)!}\int_0^\infty u^{m-1}\sum_{k=1}^{\infty}(-1)^k\binom{x}{k}e^{-ku}du$$
$$=\frac{1}{(m-1)!}\int_0^\infty u^{m-1}\left\{(1-e^{-u})^x-1\right\}du.$$

Making the change of variable $t=1-e^{-u}$, we get

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^m}\binom{x}{k}=\frac{(-1)^m}{(m-1)!}\int_0^1\frac{1-t^x}{1-t}\log^{m-1}(1-t)dt.$$

Now integration by parts gives

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{(-1)^k}{k^m}\binom{x}{k} &= \frac{(-1)^{m-1}x}{m!}\int_0^1 t^{x-1}\log^m(1-t)dt \\
&= \frac{(-1)^{m-1}x}{m!}\int_0^1 t^{x-1}\frac{\partial^m}{\partial\nu^m}(1-t)^{\nu-1}\Big|_{\nu=1}dt \\
&= \frac{(-1)^{m-1}x}{m!}\frac{\partial^m}{\partial\nu^m}\int_0^1 t^{x-1}(1-t)^{\nu-1}dt\Big|_{\nu=1} \\
&= \frac{(-1)^{m-1}x}{m!}\frac{\partial^m}{\partial\nu^m}B(x,\nu)\Big|_{\nu=1} = \frac{(-1)^{m-1}x}{m!}\frac{\partial^m}{\partial\nu^m}\frac{\Gamma(x)\Gamma(\nu)}{\Gamma(x+\nu)}\Big|_{\nu=1} \\
&= \frac{(-1)^{m-1}\Gamma(x+1)}{m!}\frac{\partial^m}{\partial\nu^m}G(x,\nu)\Big|_{\nu=1}. \qquad \square
\end{aligned}
$$

**Remark 2.1.** *The following identity is well known and due to Euler [17]; see also [4, 15].*

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k} = H_n. \tag{2.2}$$

*This identity is a particular case of (2.1). Indeed for $m = 1$ and $x = n$ in (2.1) we get (2.2).*

**Theorem 2.2.** *Let $\alpha$ be a complex number, which is not a negative integer, and $n \in \mathbb{N}$. Then, we have*

$$\sum_{k=0}^{n-1}\binom{\alpha}{k}\frac{(-1)^{k-1}}{n-k} = (-1)^n\binom{\alpha}{n}\{\psi(\alpha+1)-\psi(\alpha+1-n)\}. \tag{2.3}$$

*Proof.* We apply mathematical induction. When $n = 1$ (2.3) is easily verified. Now we assume that (2.3) has been verified for $n$ and consider the $n+1$ case. Then, we, clearly, have:

$$
\begin{aligned}
\sum_{k=0}^{n}\binom{\alpha}{k}\frac{(-1)^{k-1}}{n+1-k} &= -\frac{1}{n+1}+\sum_{k=1}^{n-1}\binom{\alpha}{k}\frac{(-1)^{k-1}}{n+1-k} \\
&= -\frac{1}{n+1}+\sum_{k=0}^{n-1}\binom{\alpha}{k+1}\frac{(-1)^k}{n-k} \\
&\quad -\frac{1}{n+1}+\alpha\sum_{k=0}^{n-1}\binom{\alpha-1}{k}\frac{(-1)^k}{(k+1)(n-k)}.
\end{aligned}
$$

By partial fraction decomposition we get by using (1.7)

$$
\begin{aligned}
\sum_{k=0}^{n}\binom{\alpha}{k}\frac{(-1)^{k-1}}{n+1-k} &= -\frac{1}{n+1}+\frac{\alpha}{n+1}\sum_{k=0}^{n-1}\binom{\alpha-1}{k}\frac{(-1)^k}{k+1}+\frac{\alpha}{n+1}\sum_{k=0}^{n-1}\binom{\alpha-1}{k}\frac{(-1)^k}{n-k} \\
&= -\frac{1}{n+1}-\frac{1}{n+1}\sum_{k=1}^{n}\binom{\alpha}{k}(-1)^k+\frac{\alpha}{n+1}\sum_{k=0}^{n-1}\binom{\alpha-1}{k}\frac{(-1)^k}{n-k}
\end{aligned}
$$

In [5] it was proved that

$$\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k} = (-1)^n\binom{\alpha-1}{n}.$$

Using this identity and the induction assumption, we get after a simple calculation

$$\sum_{k=0}^{n}\binom{\alpha}{k}\frac{(-1)^{k-1}}{n+1-k}=\frac{(-1)^{n-1}}{n+1}\binom{\alpha-1}{n}-(-1)^{n}\binom{\alpha}{n+1}\{\psi(\alpha)-\psi(\alpha-n)\}.$$

Using $\psi(\alpha)=\psi(\alpha+1)-\dfrac{1}{\alpha}$, and simplifying the result, we find

$$\sum_{k=0}^{n}\binom{\alpha}{k}\frac{(-1)^{k-1}}{n+1-k}=(-1)^{n+1}\binom{\alpha}{n+1}\{\psi(\alpha+1)-\psi(\alpha-n)\},$$

which shows that (2.3) also holds for $n+1$. This completes the proof. □

**Remark 2.2.** *Clearly for $\alpha=n\in\mathbb{N}$ in (2.3) we have (2.2). So, (2.3) provides another generalization of the identity (2.2).*

**Theorem 2.3.** *Let $\alpha$ be a complex number, which is not zero and a negative integer, and $n\in\mathbb{N}$. Then, we have*

$$\sum_{k=0}^{n}\frac{\binom{\alpha+k-1}{k}}{n-k+1}=\binom{\alpha+n}{1+n}\{\psi(\alpha+n+1)-\psi(\alpha)\}.$$

*Proof.* Let $n\in\mathbb{N}$, $\alpha\in\mathbb{C}\backslash\mathbb{Z}^{-}$ and $x\in\mathbb{C}$. In [5, Theorem 1] the first author proved that

$$\sum_{k=0}^{n}\binom{\alpha+n}{k}x^{k}=(1+x)^{n}\left[1+\alpha\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{\alpha+k}{k}\left(\frac{x}{x+1}\right)^{k+1}\right]. \tag{2.4}$$

By (1.7) we have

$$\frac{\alpha\binom{\alpha+k}{k}}{k+1}=\frac{(\alpha+k+1-(k+1))\binom{\alpha+k}{k}}{k+1}=\frac{\alpha+k+1}{k+1}\binom{\alpha+k}{k}-\binom{\alpha+k}{k}$$
$$=\binom{\alpha+k+1}{k+1}-\binom{\alpha+k}{k}=\binom{\alpha+k}{k+1}.$$

Therefore, (2.4) can be simplified to

$$\sum_{k=0}^{n}\binom{\alpha+n}{k}x^{k}=(1+x)^{n}\sum_{k=0}^{n}\binom{\alpha+k-1}{k}\left(\frac{x}{x+1}\right)^{k}.$$

Replacing $x$ by $1/x$ and then multiplying both sides by $x^{n}$ gives

$$\sum_{k=0}^{n}\binom{\alpha+n}{k}x^{n-k}=\sum_{k=0}^{n}\binom{\alpha+k-1}{k}(1+x)^{n-k}.$$

Integrating both sides with respect to $x$ on $(-1,0)$, we get

$$\sum_{k=0}^{n}\binom{\alpha+n}{k}\frac{(-1)^{n-k}}{n-k+1}=\sum_{k=0}^{n}\frac{\binom{\alpha+k-1}{k}}{n-k+1}.$$

By Theorem 2.2 the left-hand side is equal to

$$\binom{\alpha+n}{1+n}\{\psi(\alpha+n+1)-\psi(\alpha)\},$$

completing the proof. □

**Theorem 2.4.** *Let $n \in \mathbb{N}$, and $\alpha \in \mathbb{C}\backslash\mathbb{Z}^-$. Then it holds that*

$$\sum_{k=1}^{n}(-1)^k\binom{\alpha}{n-k}\frac{H_{k-1}}{k} = \frac{(-1)^n}{2}\binom{\alpha}{n}\{(\psi(\alpha+1)-\psi(\alpha+1-n))^2 + \psi'(\alpha+1)-\psi'(\alpha+1-n)\}. \tag{2.5}$$

*Proof.* We begin with

$$(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k, \quad |x|<1. \tag{2.6}$$

Since the series is uniformly convergent, we can differentiate both sides, with respect to $\alpha$ twice, we get

$$\log^2(1+x)(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}\{(\psi(\alpha+1)-\psi(\alpha+1-k))^2 + \psi'(\alpha+1)-\psi'(\alpha+1-k)\}x^k. \tag{2.7}$$

We know that for $|x|<1$

$$\log^m(1+x) = m!\sum_{k=m}^{\infty}\frac{s(k,m)x^k}{k!} = m!\sum_{k=0}^{\infty}\frac{s(k+m,m)}{(m+k)!}x^{m+k}. \tag{2.8}$$

Using this identity with $m=2$ in (2.7), we get

$$2\Bigg(\sum_{k=0}^{\infty}\frac{(-1)^{k+1}H_k}{k+1}x^{k+1}\Bigg)\Bigg(\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k\Bigg) = \sum_{k=0}^{\infty}\binom{\alpha}{n}\{(\psi(\alpha+1)-\psi(\alpha+1-n))^2+\psi'(\alpha+1)-\psi'(\alpha+1-n)\}x^n.$$

Applying the Cauchy product rule for two convergent series we obtain

$$2\sum_{n=0}^{\infty}\Bigg(\sum_{k=0}^{n}\frac{(-1)^{n-k+1}H_{n-k}}{n-k+1}\binom{\alpha}{k}\Bigg)x^{n+1} = \sum_{n=0}^{\infty}\binom{\alpha}{n}\{(\psi(\alpha+1)-\psi(\alpha+1-n))^2+\psi'(\alpha+1)-\psi'(\alpha+1-n)\}x^n.$$

Equating the coefficients of $x^n$ in both sides and making a change of summation index ($n-k=k'$, then $k'=k$), we get the desired result. □

The following theorem establishes a nice link between the Bernoulli numbers of the second kind and the Stirling numbers of the first kind.

**Theorem 2.5.** *For $m \in \mathbb{N}$ we have*

$$\sum_{k=0}^{n}\frac{s(k+m+1,m+1)G_{n-k}}{(m+k+1)!} = \frac{s(n+m,m)}{(m+1)(m+n)!}, \tag{2.9}$$

*where $s(m,k)$ are Stirling numbers of the first kind.*

*Proof.* We have

$$x\log^m(1+x)=\frac{x}{\log(1+x)}\log^{m+1}(1+x).$$

Thus, by (2.8) we get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{s(n+m,m)x^n}{(m+n)!}&=(m+1)\left(\sum_{n=0}^{\infty}G_nx^n\right)\left(\sum_{n=0}^{\infty}\frac{s(n+m+1,m+1)x^n}{(m+n+1)!}\right)\\&=(m+1)\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{s(k+m+1,m+1)G_{n-k}}{(m+k+1)!}\right)x^n.\end{aligned}$$

Equating the coefficients of $x^n$ in both sides, the proof follows. □

The next theorem, providing a nice relation between the Bernoulli numbers of the second kind and and the Bernoulli polynomials of the second kind, recovers [19, p. 265, Eq. 2.17].

**Theorem 2.6.** *For any non-negative integer $n$ and real number $t$, which is not a negative integer, we have*

$$\Psi_n(t)=\sum_{k=0}^{n}\binom{t}{k}G_{n-k},\tag{2.10}$$

*where $\Psi_n(t)$ are Bernoulli polynomials of the second kind; see (1.18) and (1.19).*

*Proof.* We start with

$$\sum_{k=0}^{\infty}\binom{t}{k}x^k=(1+x)^t\quad(|x|<1).\tag{2.11}$$

If we multiply both sides by $x$ and divide by $\log(x+1)$ we, then, can write

$$\frac{x(1+x)^t}{\log(1+x)}=\frac{x}{\log(1+x)}\sum_{k=0}^{\infty}\binom{t}{k}x^k.$$

Using (1.18) and the power series of the series on the right-hand side we get

$$\begin{aligned}\sum_{n=0}^{\infty}\Psi_n(t)x^n&=\left(\sum_{n=0}^{\infty}G_nx^n\right)\left(\sum_{n=0}^{\infty}\binom{t}{n}x^n\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}G_{n-k}\binom{t}{k}\right)x^n.\end{aligned}$$

Equating the coefficients of $x^n$ in both sides, we arrive at the desired result. □

Differentiating both sides of (2.10) with respect to $t$, and using (1.7) we arrive at

$$\binom{t}{n-1}=\sum_{k=0}^{n}G_{n-k}\binom{t}{k}\big[\psi(t+1)-\psi(t+1-k)\big].\tag{2.12}$$

# 3 Examples

In this section we offer many finite and infinite sum formulas involving the binomial coefficients, the Bernoulli numbers of the second kind and the harmonic numbers.

**Example 3.1.** *For* $n \geq 2$

$$\sum_{k=0}^{n} G_k G_{n-k} = (2-n)G_{n-1} + (1-n)G_n. \tag{3.1}$$

*Proof.* From

$$\left(\sum_{k=0}^{\infty} G_k x^k\right)^2 = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} G_k G_{n-k}\right)x^n = \frac{x^2}{\log^2(x+1)},$$

we write

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} G_k G_{n-k}\right)x^n &= -x^2(1+x)\frac{d}{dx}(\log(x+1))^{-1}\\
&= -(1+x)\sum_{n=0}^{\infty}(n-1)G_n x^n\\
&= -\sum_{n=0}^{\infty}(n-1)G_n x^n - \sum_{n=0}^{\infty}(n-1)G_n x^{n+1}\\
&= -\sum_{n=0}^{\infty}(n-1)G_n x^n - \sum_{n=0}^{\infty}(n-2)G_{n-1} x^n,
\end{aligned}$$

where in the last step we assumed $G_{-1} = 0$, thus this equation can be rearranged as follows:

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} G_k G_{n-k} + (n-1)G_n + (n-2)G_{n-1}\right)x^n = 0,$$

from which we arrive at (3.1). □

**Example 3.2.** *For* $n \geq 1$ *we have*

$$\sum_{k=0}^{n}(-1)^k G_k H_{n-k} = 1.$$

*Proof.* It is well known that

$$\frac{\log(1+x)}{1+x} = \sum_{n=0}^{\infty}(-1)^{n-1}H_n x^n.$$

From this identity we get

$$\frac{x}{1+x} = \frac{x}{\log(1+x)}\sum_{n=0}^{\infty}(-1)^{n-1}H_n x^n$$

or, using their power series at $x = 0$,

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^{n+1}x^{n+1} &= \left(\sum_{n=0}^{\infty} G_n x^n\right)\left(\sum_{n=0}^{\infty}(-1)^n H_n\right)x^n\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^{n-k}H_{n-k}G_k\right)x^n.
\end{aligned}$$

Comparing the coefficients of $x^n$ in each side, we complete the proof. □

**Example 3.3.** *For* $n = 2, 3, 4, \ldots$ *we have*

$$\sum_{k=0}^{n} \binom{n}{k} [H_n - H_k] G_k = n.$$

*Proof.* The proof immediately follows from (2.12) with $t = n$. $\square$

In the next two examples we give different proofs of Blagouchine and Coppo's series for the constants $\log(2\pi)$ and $\gamma$.

$$\gamma = \sum_{n=1}^{\infty} \frac{|G_n|}{n} \quad \text{and} \quad \frac{1}{2}(\log(2\pi) - \gamma - 1) = \sum_{n=2}^{\infty} \frac{|G_n|}{n-1}, \tag{3.2}$$

(see [16, Eq. (2.39)]).

**Example 3.4.** *The following identity holds.*

$$\frac{3}{2} + \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} |G_k G_{n+1-k}| = \log(2\pi). \tag{3.3}$$

*Proof.* Using (3.1) and replacing $n$ by $n-1$, and noting that $|G_k| = (-1)^{k-1} G_k$, we find that:

$$\begin{aligned}
\frac{3}{2} + \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} |G_k G_{n+1-k}| &= \frac{3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} \sum_{k=1}^{n-1} G_k G_{n-k} \\
&= \frac{3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} \left( \sum_{k=0}^{n} G_k G_{n-k} - 2G_n \right) \\
&= \frac{3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} \left[ (2-n) G_{n-1} + (1-n) G_n - 2G_n \right] \\
&= \frac{3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} \left[ (2-n) G_{n-1} - (n+1) G_n \right] \\
&= \frac{3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} (1-n) G_n - \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n-1} (n+1) G_n \\
&= \frac{3}{2} + \sum_{n=2}^{\infty} \frac{|G_n|}{n} + 2 \sum_{n=2}^{\infty} \frac{|G_n|}{n-1} \\
&= 1 + \gamma + (\log(2\pi) - \gamma - 1) = \log(2\pi).
\end{aligned}$$

$\square$

**Example 3.5.** *The Euler–Mascheroni constant has the following series representation:*

$$2\log(2\pi) - 3 - 2\sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{k=1}^{n} |G_k G_{n+2-k}| = \gamma.$$

*Proof.* Replacing $n$ by $n-2$, we get

$$\begin{aligned}
&2\log(2\pi)-3-2\sum_{n=1}^{\infty}\frac{1}{n+1}\sum_{k=1}^{n}|G_kG_{n+2-k}|\\
&=2\log(2\pi)-3-2\sum_{n=3}^{\infty}\frac{(-1)^n}{n-1}\sum_{k=1}^{n-2}G_kG_{n-k}\\
&=2\log(2\pi)-3-2\sum_{n=3}^{\infty}\frac{(-1)^n}{n-1}\left(\sum_{k=0}^{n}G_kG_{n-k}-2G_n-\frac{1}{2}G_{n-1}\right).
\end{aligned}$$

By (3.1) this is

$$\begin{aligned}
&2\log(2\pi)-3-2\sum_{n=1}^{\infty}\frac{1}{n+1}\sum_{k=1}^{n}|G_kG_{n+2-k}|\\
&=2\log(2\pi)-3-2\sum_{n=3}^{\infty}\frac{(-1)^n}{n-1}\left((2-n)G_{n-1}+(1-n)G_n-2G_n-\frac{1}{2}G_{n-1}\right)\\
&=2\log(2\pi)-3-\sum_{n=3}^{\infty}\frac{(-1)^n}{n-1}\left((3-2n)G_{n-1}-2(n+1)G_n\right)\\
&=2\log(2\pi)-3-\sum_{n=2}^{\infty}\frac{(-1)^{n-1}(1-2n)}{n}G_n+\sum_{n=3}^{\infty}\frac{(-1)^{n-1}(n+1)G_n}{n-1}\\
&=2\log(2\pi)-3-\frac{1}{8}-\sum_{n=3}^{\infty}|G_n|\left[\frac{1}{n}+\frac{4}{n-1}\right].
\end{aligned}$$

Using (3.2), we get

$$\begin{aligned}
&2\log(2\pi)-3-2\sum_{n=1}^{\infty}\frac{1}{n+1}\sum_{k=1}^{n}|G_kG_{n+2-k}|\\
&=2\log(2\pi)-3+\underbrace{\left(\frac{1}{8}+\frac{13}{24}+\frac{1}{3}\right)}_{1}-\sum_{n=1}^{\infty}\frac{|G_n|}{n}-4\sum_{n=2}^{\infty}\frac{|G_n|}{n-1}\\
&=2\log(2\pi)-2-\sum_{n=1}^{\infty}\frac{|G_n|}{n}-2\log(2\pi)+2\gamma+2=\gamma.
\end{aligned}$$

□

**Example 3.6.** *Setting $m=2$ in (2.1), we get*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}\binom{x}{k}=\frac{1}{2}\left((\psi(1)-\psi(x+1))^2+\psi'(1)-\psi'(x+1)\right).$$

Integrating both sides over the unit interval, reversing the order of summation and integration is justified by a similar argument as given for (1.21), yields after a simple calculation

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{|G_k|}{k^2}&=\frac{1}{2}\left(\psi^2(1)+\psi'(1)-2\psi(1)\underbrace{\int_0^1\psi(x+1)dx}_{0}-\underbrace{\int_0^1\psi'(x+1)dx}_{1}+\int_0^1\psi^2(x+1)dx\right)\\
&=\frac{\gamma^2+\zeta(2)-1}{2}+\frac{1}{2}\int_0^1\psi^2(x+1)dx.
\end{aligned}$$

In a similar way, for $m=3$ we get

$$\sum_{k=1}^{\infty} \frac{|G_k|}{k^3} = \frac{1}{12}\left(2\gamma^3 - 5 + 6\gamma\zeta(2) + 4\zeta(3)\right) + \frac{1}{6}\int_0^1 \left(3\gamma\psi^2(x+1) + \psi^3(x+1)\right) dx,$$

where $\zeta$ is the Riemann zeta function.

**Remark 3.1.** *These two identities are known and due to Coffey; see [16, p.23].*

**Example 3.7.** *Let $m, n \in \mathbb{N}$. Then we have*

$$(-1)^{n-1}\sum_{k=0}^{\infty} \frac{1}{(n+k+1)^{m+1}\binom{n+k}{k}} = H_n \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^m}\binom{n}{k} - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^m}\binom{n}{k} H_{n-k} - \frac{(-1)^m}{m!}\frac{\partial}{\partial x}\left[\Gamma(x+1)\frac{\partial^m}{\partial z^m}G(x,z)\right]_{(x,z)=(n,1)}, \tag{3.4}$$

*where* $G(x,z) = \frac{\Gamma(z)}{\Gamma(x+z)}$.

*Proof.* Differentiating both sides of (2.1) with respect to $x$, which is permissible since the series is uniformly convergent, we get

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^m}\binom{x}{k}\{\psi(x+1) - \psi(x-k+1)\} = \frac{(-1)^m}{m!}\frac{\partial}{\partial x}\left[\Gamma(x+1)\frac{\partial^m}{\partial z^m}G(x,z)\right]_{z=1}.$$

Replacing $x = n \in \mathbb{N}$ here, we get

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^m}\binom{n}{k}\psi(n+1) - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^m}\binom{n}{k}\psi(n+1-k) = \frac{(-1)^m}{m!}\frac{\partial}{\partial x}\left[\Gamma(x+1)\frac{\partial^m}{\partial z^m}G(x,z)\right]_{(x,z)=(n,1)}. \tag{3.5}$$

Obviously, by (1.9) we have

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^m}\binom{n}{k}\psi(n+1) = (-\gamma + H_n)\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^m}\binom{n}{k}. \tag{3.6}$$

Splitting the sum into two parts yields

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^m}\binom{n}{k}\psi(n+1-k) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^m}\binom{n}{k}\psi(n+1-k) + \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k^m}\binom{n}{k}\psi(n+1-k). \tag{3.7}$$

Substituting $k+n+1$ for $k$, and using Lemma 1.2, we get

$$\begin{aligned}\sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k^m}\binom{n}{k}\psi(n-k+1) &= \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k^m}\frac{n!}{k!}\frac{\psi(n-k+1)}{\Gamma(n-k+1)} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{n-k}}{(n+k+1)^m}\frac{n!}{(n+k+1)!}\underbrace{\lim_{t\to -k}\frac{\psi(t)}{\Gamma(t)}}_{(-1)^{k-1}k!} \\ &= (-1)^{n-1}\sum_{k=1}^{\infty} \frac{1}{(k+n+1)^{m+1}\binom{n+k}{k}}.\end{aligned} \tag{3.8}$$

By (1.9) we obtain

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\psi(n-k+1)=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}(-\gamma+H_{n-k}). \tag{3.9}$$

Combining (3.7), (3.8) and (3.9) we find that

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\psi(n-k+1)\\&=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}(H_{n-k}-\gamma)-(-1)^n\sum_{k=0}^{\infty}\frac{1}{(n+k+1)^{m+1}\binom{n+k}{k}}.\end{aligned} \tag{3.10}$$

Now the proof follows from (3.5), (3.6) and (3.10). □

**Example 3.8.** *Letting $m=1$ and $2$ in (3.4) and using [4, Eq. (3.4)] we obtain, respectively*

$$(-1)^{n-1}\sum_{k=0}^{\infty}\frac{1}{(n+k+1)^2\binom{n+k}{k}}=H_n^2+H_n^{(2)}+\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{n}{k}H_{n-k}-\zeta(2)$$

*and*

$$\begin{aligned}(-1)^{n-1}\sum_{k=0}^{\infty}\frac{1}{(n+k+1)^3\binom{n+k}{k}}=&\frac{H_n^3}{2}+\frac{3}{2}H_nH_n^{(2)}-\zeta(2)H_n+H_n^{(3)}\\&+\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\binom{n}{k}H_{n-k}-\zeta(3).\end{aligned}$$

Integrating both sides of (2.10) with respect to $t$ over the unit interval and using (3.1), we get

**Example 3.9.** *For any non-negative integer $n$ we have*

$$\int_0^1\Psi_n(t)dt=(2-n)G_{n-1}+(1-n)G_n.$$

**Example 3.10.** *Let $t$ be any complex number, which is not an integer, and $n\in\mathbb{N}$. Then*

$$\sum_{k=1}^{n}\binom{-t}{n-k}\binom{t}{k}\{\psi(t+1)-\psi(t+1-k)\}=\frac{(-1)^{n-1}}{n}. \tag{3.11}$$

*Proof.* Differentiating (2.11) with respect to $t$, which is permissible since the series is uniformly convergent, we get

$$(x+1)^t\log(x+1)=\sum_{k=0}^{\infty}\binom{t}{k}\{\psi(t+1)-\psi(t+1-k)\}x^k$$

or

$$\log(x+1)=(x+1)^{-t}\sum_{k=0}^{\infty}\binom{t}{k}\{\psi(t+1)-\psi(t+1-k)\}x^k.$$

Using the power series of $\log(x+1)$ and $(1+x)^{-t}$ yields

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}=\left(\sum_{n=0}^{\infty}\binom{-t}{n}x^n\right)\left(\sum_{n=0}^{\infty}\binom{t}{n}\{\psi(t+1)-\psi(t+1-n)\}x^n\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{-t}{n-k}\binom{t}{k}\{\psi(t+1)-\psi(t+1-k)\}\right)x^n.$$

Equating the coefficients of $x^n$ in both sides yields (3.9). If we let $t=-1/2$ in (3.11), we get for $n\in\mathbb{N}$,

**Example 3.11.** *For any* $n\in\mathbb{N}$ *we have*

$$\sum_{k=0}^{n}\frac{h_{n-k}}{2k-1}\binom{2k}{k}\binom{2n-2k}{n-k}=-\frac{4^n}{2n},$$

*where* $h_k=H_{2k}-\frac{1}{2}H_k$.

□

**Example 3.12.** *For* $\alpha=-1/2$ *in Theorem 2.2, we get from (1.5) with* $s=-1/2$ *and* $t=k$

$$\sum_{k=0}^{n-1}\frac{1}{(n-k)4^k}\binom{2k}{k}=\frac{2h_n}{4^n}\binom{2n}{n}.$$

**Example 3.13.** *We have*

$$\sum_{k=0}^{n-1}\frac{1}{4^k(2k-1)}\binom{2k}{k}=-\frac{2n}{4^n(2n-1)}\binom{2n}{n}.$$

*Proof.* Putting $\alpha=1/2$ in Theorem 2.2, we get

$$\sum_{k=0}^{n-1}\frac{\binom{2k}{k}}{4^k(2k-1)(n-k)}=\frac{\binom{2n}{n}}{(2n-1)4^n}\left[2h_n-\frac{4n}{2n-1}\right].$$

By the partial fraction decomposition this leads to

$$\sum_{k=0}^{n-1}\frac{\binom{2k}{k}}{4^k(2k-1)(n-k)}=\frac{2}{2n-1}\sum_{k=0}^{n-1}\frac{\binom{2k}{k}}{4^k(2k-1)}-\frac{1}{2n-1}\sum_{k=0}^{n-1}\frac{\binom{2k}{k}}{4^k(n-k)}.$$

Now the proof follows immediately from Example 3.12. □

**Example 3.14.** *Setting* $\alpha=n\in\mathbb{N}$ *in Theorem 2.3, we obtain*

$$\sum_{k=0}^{n-1}\frac{1}{n-k}\binom{n+k}{1+k}=\binom{2n}{n+1}(H_{2n}-H_{n-1})-\frac{1}{n+1}.$$

**Example 3.15.** *Putting* $\alpha=-\frac{1}{2}$ *in (2.5) we get*

$$\sum_{k=1}^{n}\frac{4^k}{k}\binom{2n-2k}{n-k}H_{k-1}=\binom{2n}{n}\left[2h_n^2-2h_n^{(2)}-\frac{1}{2}H_n^{(2)}\right],$$

*where* $h_n=H_{2n}-\frac{1}{2}H_n$ *and* $h_n^{(2)}=H_{2n}^{(2)}-\frac{1}{2}H_n^{(2)}$.

**Remark 3.2.** *Please refer to [3] and [6] for many new and interesting finite sums involving the central binomial coefficients* $B_n = \binom{2n}{n}$ *and Catalan numbers* $C_n = \frac{1}{n+1}\binom{2n}{n}$.

**Example 3.16.** *Setting* $m = 0, 1$ *and* $2$ *in (2.9) we get, for* $n \geq 0$*, respectively*

$$\sum_{k=0}^{n} \frac{(-1)^k G_{n-k}}{k+1} = \frac{\delta_{n0}}{n!}, \tag{3.12}$$

$$\sum_{k=0}^{n} \frac{(-1)^k H_{k+1} G_{n-k}}{k+2} = \frac{(-1)^n}{2(n+1)},$$

$$\text{and} \quad \sum_{k=0}^{n} \frac{(-1)^k \left(H_{k+2}^2 - H_{k+2}^{(2)}\right) G_{n-k}}{k+3} = \frac{2(-1)^n H_{n+1}}{3(n+2)}.$$

**Remark 3.3.** *It is obvious that (3.12) is equivalent to (1.12).*

**Example 3.17.** *Setting* $t = -1/2$ *in (2.12), we get by using (1.6)*

$$\sum_{k=0}^{n} \frac{(-1)^k (2h_k)}{4^k} \binom{2k}{k} G_{n-k} = \frac{(-1)^{n-1}}{4^{n-1}} \binom{2n-2}{n-1}.$$

**Example 3.18.** *Setting* $\alpha = n \in \mathbb{N}$ *in (2.5) and using Theorem 2.1 of [4], we get*

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k}{k} = H_n^{(2)},$$

*which is a well known result; see [4, Eq. (1.5)].*

Differentiating both sides of (2.5) with respect to $\alpha$, and then setting $\alpha = n$, we get

**Example 3.19.** *For any non-negative integer* $n$ *we have*

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{H_k H_{k-1}}{k} = H_n^{(3)} - H_n H_n^{(2)}.$$

The binomial inversion theorem states

$$a_n = \sum_{k=1}^{n} (-1)^k \binom{n}{k} b_k \iff b_n = \sum_{k=1}^{n} (-1)^k \binom{n}{k} a_k$$

so that,

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} \left(H_k H_k^{(2)} - H_k^{(3)}\right) = \frac{H_n H_{n-1}}{n}.$$

**Example 3.20.** *Differentiating both sides of (2.1) and then setting* $x = -\frac{1}{2}$*, we get*

$$\sum_{k=1}^{\infty} \frac{h_k}{2^{2k} k^m} \binom{2k}{k} = \frac{(-1)^m}{m!} \frac{d}{dx} \left(\Gamma(x+1) \frac{\partial^m}{\partial z^m} G(x,z)\right) \bigg|_{(z,x)=(1,-1/2)} \tag{3.13}$$

**Example 3.21.** *Putting $m = 3$ in (3.13), we get*

$$\sum_{k=1}^{\infty} \frac{h_k}{2^{2k}k^3}\binom{2k}{k} = \frac{15}{4}\zeta(4) + 3\zeta(2)\log^2 2 - 7\zeta(3)\log 2. \tag{3.14}$$

**Remark 3.4.** *Examples 3.20 and 3.21 are not new, and these and many similar series containing harmonic numbers have been proved by Wang and Xu in a different way; see [32, Theorems 2.3, 3.7 and Corollary 2.5]. Example 3.19 can be easily deduced by combining the Equations (9,16) and (9.5) included in [12].*

**Example 3.22.** *Differentiating both sides of (2.1) and then setting $x = \frac{1}{2}$ in we get*

$$\sum_{k=1}^{\infty} \frac{h_k}{k^3(2k-1)4^k}\binom{2k}{k} = 7\zeta(3)\log 2 - \frac{15}{4}\zeta(4) - 3\zeta(2)\log^2 2 - 6\zeta(2) + 6\zeta(2)\log 2 - 7\zeta(3) + 4\pi, \tag{3.15}$$

*where we have used*

$$\sum_{k=1}^{\infty} \frac{1}{k^3(2k-1)4^k}\binom{2k}{k}\left(2 + \frac{2}{2k-1}\right) = 4\zeta(2) + 8\pi - 48 - 8\log^2 2 + 32\log 2.$$

*From (3.14) and (3.15) we conclude that*

$$\sum_{k=1}^{\infty} \frac{h_k}{k^3(2k-1)4^k}\binom{2k}{k} = -\sum_{k=1}^{\infty} \frac{h_k}{k^3 4^k}\binom{2k}{k} + 4\pi + 6\zeta(2)\log 2 - 6\zeta(2) - 7\zeta(3),$$

*and upon re-arrangement we prove the following example:*

**Example 3.23.**

$$\sum_{k=1}^{\infty} \frac{h_k}{k^2(2k-1)4^k}\binom{2k}{k} = 2\pi + 3\zeta(2)\log 2 - 3\zeta(2) - \frac{7}{2}\zeta(3).$$

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