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Transcendental properties of the certain mix infinite products

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Abstract: Let k and l be two multiplicatively independent positive integers and b be an integer with $b \geq 2$. Let S be a finite set of integers. Nishioka proved that for any algebraic number α with $0 < |\alpha| < 1$ the infinite products $\prod_{y=0}^{\infty}(1-\alpha^{d^y})$ $(d = 2, 3, \ldots)$ are algebraically independent over $\mathbb{Q}$. As her result, for example, the transcendence of $\prod_{y=0}^{\infty}(1 - \frac{1}{b^{2^y}}) \prod_{y=0}^{\infty}(1 - \frac{1}{b^{3^y}})$ is deduced. On the other hand, Tachiya, Amou–Väänänen investigated the certain infinite products which satisfy infinite chains of Mahler functional equation. The special case of the result of Tachiya shows that the infinite product $\prod_{y\geq 0}(1 + \sum_{i=1}^{k-1} \frac{\tau(i,y)}{b^{ik^y}})$ with $\tau(i, y) \in S$ $(1 \leq i \leq k-1, y \geq 0)$ is either rational or transcendental.
In this paper, we prove that the infinite product $\prod_{y\geq 0}(1 + \sum_{i=1}^{k-1} \frac{\tau(i,y)}{b^{ik^y}}) \prod_{y\geq 0}(1 + \sum_{j=1}^{l-1} \frac{\delta(j,y)}{b^{jl^y}})$ with $\tau(i, y), \delta(j, y) \in S$ $(1 \leq i \leq k-1, 1 \leq j \leq l-1, y \geq 0)$ is either rational or transcendental. Moreover, we give sufficient conditions that $\prod_{y\geq 0}(1 + \sum_{i=1}^{k-1} \frac{\tau(i,y)}{b^{ik^y}}) \prod_{y\geq 0}(1 + \sum_{j=1}^{l-1} \frac{\delta(j,y)}{b^{jl^y}})$ is transcendental.
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1 Introduction

Let k and l be two multiplicatively independent positive integers and b be an integer with $b \geq 2$. Let S be a finite set of integers. Let z, w be two complex variables. In 1994, Nishioka [11] proved that for any algebraic number α with $0 < |\alpha| < 1$ the infinite products $\prod_{y=0}^{\infty}(1-\alpha^{d^y})$ $(d = 2, 3, \ldots)$ are algebraically independent over $\mathbb{Q}$. As her result, for example, the transcendence of $\prod_{y=0}^{\infty}(1-\frac{1}{b^{2^y}})\prod_{y=0}^{\infty}(1-\frac{1}{b^{3^y}})$ is deduced. On the other hand, Tachiya [13], Amou–Väänänen [2, 3] investigated the following infinite products. For any non-negative integers e and h, we define two infinite products $f_e(z)$, $g_h(w)$ and two sequences $(a_e(n))_{n\geq 0}$, $(b_h(m))_{m\geq 0}$ by

$$f_e(z) := \prod_{y\geq e}(1+\sum_{i=1}^{k-1}\tau(i,y)z^{ik^{y-e}}) = \sum_{n=0}^{\infty}a_e(n)z^n$$

and

$$g_h(w) := \prod_{y\geq h}(1+\sum_{j=1}^{l-1}\delta(j,y)w^{jl^{y-h}}) = \sum_{m=0}^{\infty}b_h(m)w^m$$

with $\tau(i,y), \delta(j,y) \in S$ ($1 \leq i \leq k-1, 1 \leq j \leq l-1, y \geq 0$). By the definitions of $f_e(z)$ and $g_h(w)$, for any non-negative integers e and h, we get the following infinite chains of Mahler functional equations

$$f_0(z) = f_e(z^{k^e})\prod_{y=0}^{e-1}(1+\sum_{i=1}^{k-1}\tau(i,y)z^{ik^y})$$

and

$$g_0(w) = g_h(w^{l^h})\prod_{y=0}^{h-1}(1+\sum_{j=1}^{l-1}\delta(j,y)w^{jl^y}). \tag{1}$$

Tachiya [13], Amou–Väänänen [2, 3] investigated the arithmetical properties of the infinite products $f_0(z)$ which satisfy infinite chains of Mahler functional equation (1). The special case of Theorem 1 in [13] shows that the infinite product $f_0(\frac{1}{b})$ is either rational or transcendental.

In this paper, we prove the following theorem.

Theorem 1.1. *The infinite product $f_0(\frac{1}{b})g_0(\frac{1}{b})$ is either rational or transcendental.*

The proof of Theorem 1.1 relies on the method of the proof of Theorem 2.5.1 in [6] (see also [4]), Nishioka's asymptotic lower bounds (see Lemma 2.2) and the infinite chains of Mahler functional equations (1).

Remark 1.1. The sequences $(a_0(n))_{n\geq 0}$ and $(b_0(n))_{n\geq 0}$ are in the set of q-multiplicative sequences introduced by Gel'fond [9] and Delange [7] (see also 1p in [14]). In this paper, for the analysis of arithmetical properties of $f_0(\frac{1}{b})g_0(\frac{1}{b})$, we assumed that $(a_0(n))_{n\geq 0}$ and $(b_0(n))_{n\geq 0}$ lie in the set of integers and the set of $\tau(i,y), \delta(j,y)$ ($1 \leq i \leq k-1, 1 \leq j \leq l-1, y \geq 0$) lie in S .

Moreover, we give sufficient conditions that $f_0(\frac{1}{b})g_0(\frac{1}{b})$ is transcendental as follows.

Amou–Väänänen [2, 3] introduce the following notion of the irrationality measure of formal power series. Let X be a complex variable. The irrationality measure $\mu(F)$ of $F(X) \in \mathbb{Q}[[X]]$ is defined to be the infimum of μ such that

$$\mathrm{ord}(A(X)F(X) - B(X)) \leq \mu M$$

holds for all $A(X), B(X) \in \mathbb{Q}[X]$, not both zero, satisfying $\max(\deg A, \deg B) \leq M$ provided that $M \geq M_0$ with some sufficiently large M_0 depending only on $F(X)$, where for $g(X) \in \mathbb{Q}[[X]]$ we denote by $\operatorname{ord} g(X)$ the zero order of $g(X)$ at $X = 0$. If there does not exist such a μ, we define $\mu(F) := \infty$. We give a sufficient condition that $f_0(\frac{1}{b})g_0(\frac{1}{b})$ is transcendental by using the irrationality measure of formal power series as follows.

Theorem 1.2. *Assume that* $(1+\sum_{i=1}^{k-1}\frac{\tau(i,y)}{b^{ik^y}})(1+\sum_{j=1}^{l-1}\frac{\delta(j,y)}{b^{jl^y}}) \neq 0$ *for all* $y \geq 0$. *If the irrationality measures of* $f_0(z)$ *and* $g_0(w)$ *are finite, then the infinite product* $f_0(\frac{1}{b})g_0(\frac{1}{b})$ *is transcendental.*

The proof of Theorem 1.2 relies on Theorem 1.1, Nishioka's the asymptotic lower bounds and the most classical Mahler method (see p. 20 in [10]). By Remark 2 in [2], Amou–Väänänen proved that if $\mu(f_0) < \infty$ and $(1+\sum_{i=1}^{k-1}\frac{\tau(i,y)}{b^{ik^y}}) \neq 0$ for all $y \geq 0$, then $f_0(\frac{1}{b})$ is either a Mahler's S-number or a T-number. Moreover, if $f_0(z)$ is irrational with $\mu(f_0) = \infty$ and $(1+\sum_{i=1}^{k-1}\frac{\tau(i,y)}{b^{ik^y}}) \neq 0$ for all $y \geq 0$, then $f_0(\frac{1}{b})$ is a Mahler's U-number. Therefore, from the property of Mahler's classification of real numbers and Theorem 1.2, we also get the following theorem.

Theorem 1.3. *Assume that* $(1+\sum_{i=1}^{k-1}\frac{\tau(i,y)}{b^{ik^y}})(1+\sum_{j=1}^{l-1}\frac{\delta(j,y)}{b^{jl^y}}) \neq 0$ *for all* $y \geq 0$. *If the irrationality measure of* $f_0(z)$ *is finite, then the infinite product* $f_0(\frac{1}{b})g_0(\frac{1}{b})$ *is transcendental.*

From Lemma 9 in [2], one can give concrete examples of Theorem 1.3. Moreover, by Theorem 4.3 in [12] and Theorem 1.3, we also get the following corollary.

Corollary 1.1. *Assume that* $(1+\sum_{i=1}^{k-1}\frac{\tau(i,y)}{b^{ik^y}})(1+\sum_{j=1}^{l-1}\frac{\delta(j,y)}{b^{jl^y}}) \neq 0$ *and* $\tau(i,y) = \tau(i,y+1)$ *for all* $y \geq 0$ *and all* i *with* $0 \leq i \leq k-1$. *If* $f_0(z)$ ($= f_e(z)$ *for all* $e \geq 0$) *is irrational, then the infinite product* $f_0(\frac{1}{b})g_0(\frac{1}{b})$ *is transcendental.*

Remark 1.2. Assume that $(1+\sum_{i=1}^{k-1}\frac{\tau(i,y)}{b^{ik^y}})(1+\sum_{j=1}^{l-1}\frac{\delta(j,y)}{b^{jl^y}}) \neq 0$ and $\tau(i,y) = \tau(i,y+1)$ for all $y \geq 0$ and all i with $0 \leq i \leq k-1$. From Theorem 1, Theorem 6-(ii) in [13] and Example 1.3.1 in [12], one can give the necessary-sufficient conditions that $f_0(\frac{1}{b})g_0(\frac{1}{b})$ is rational. Especially, $f_0(\frac{1}{b})g_0(\frac{1}{b})$ is rational if and only if $f_0(z)$ and $g_0(w)$ are rational.

From Example 1.3.1 in [12], one can give concrete examples of Corollary 1.1. Indeed, we can find the following example of Corollary 1.1.

Corollary 1.2. *Assume that* $\delta(1,y) \in \{1,-1\}$ *for all* $y \geq 0$. *Then* $\prod_{y\geq 0}(1-\frac{1}{b^{2^y}})\prod_{y\geq 0}(1+\frac{\delta(1,y)}{b^{3^y}})$ *is transcendental.*

Finally, we propose the following problem.

Problem 1.1. *Assume that* $(1+\sum_{i=1}^{k-1}\frac{\tau(i,y)}{b^{ik^y}})(1+\sum_{j=1}^{l-1}\frac{\delta(j,y)}{b^{jl^y}}) \neq 0$ *for all* $y \geq 0$. *Do there exist two irrational infinite products* $f_0(z)$, $g_0(w)$ *with* $\mu(f_0) = \infty, \mu(g_0) = \infty$ *and an integer* b *with* $b \geq 2$ *such that the infinite product* $f_0(\frac{1}{b})g_0(\frac{1}{b})$ *is rational?*

This paper is organized as follows. In Section 2, we gather lemmas for the proof of theorems. In Section 3, we give the proof of Theorem 1.1. In Section 4, we give the proof of Theorem 1.2.

2 Preliminaries

The following lemma is known as Siegel's lemma (see Lemma 1.4.1 in [12]).

Lemma 2.1. *Consider the m equations in n unknowns*

$$a_{i1}x_1 + \cdots + a_{in}x_n = 0, \quad i = 1, 2, \ldots, m \tag{2}$$

with integer coefficients a_{ij} and $0 < m < n$. Let A be a positive integer such that $A \geq |a_{ij}|$, for all i and j. Then there is a nontrivial solution $x_1, x_2, \ldots, x_n$ in integers of equations (2) *such that*

$$|x_j| < 1 + (nA)^{\frac{n}{(n-m)}}, \quad j = 1, 2, \ldots, n.$$

The following lemma is a special case of Lemma 2 in [11].

Lemma 2.2. *Let $(e_i(n))_{n\geq 0}(1 \leq i \leq 2)$ be sequences of positive integer with $\lim_{n\to\infty} e_i(n) = +\infty$. Moreover, assume that k and l are two multiplicatively independent positive integers with $k, l \geq 2$ and $\lim_{n\to\infty} \frac{e_1(n)}{e_2(n)}$ converges to an irrational number as $n \to +\infty$. Let a_0, a_1 be two integers with $a_0 \neq 0$. Let $0 < \gamma < 1$. Then, we have*

$$|a_0 k^{e_1(n)} + a_1 l^{e_2(n)}| > k^{e_1(n)} \gamma^{e_1(n)}$$

for all large integer n.

The following lemma is in Lemma 2.5.9 in [1]

Lemma 2.3. *If k and l are two multiplicatively independent positive integers with $k, l \geq 2$, then the set $\{\frac{k^e}{l^f} \mid e, f \geq 0\}$ is dense in the positive reals.*

The following lemma is known as p-adic Schmidt subspace theorem (see Theorem E.10 in [5] or Theorem 2.5.4 in [6]).

Lemma 2.4. *Let $n \geq 2$, $\epsilon > 0$, and let $p_1, \ldots, p_s$ be distinct prime numbers. Further, let $L_{1,\infty}, \ldots, L_{n,\infty}$ be linearly independent linear forms in $X_1, \ldots, X_n$ with algebraic coefficients in $\mathbb{C}$, and for $j = 1, \ldots, s$, $L_{1,p_j}, \ldots, L_{n,p_j}$ be linearly independent linear forms in $X_1, \ldots, X_n$ with algebraic coefficients in $\bar{\mathbb{Q}}_{p_j}$. Consider the inequality*

$$|L_{1,\infty}(\mathbf{x}) \cdots L_{n,\infty}(\mathbf{x})| \prod_{j=1}^{s} |L_{1,p_j}(\mathbf{x}) \cdots L_{n,p_j}(\mathbf{x})|_{p_j} < \max\{|x_1|, \ldots, |x_n|\}^{-\epsilon} \tag{3}$$

with $\mathbf{x} := (x_1, \ldots, x_n)$ in $\mathbb{Z}^n$. There are a finite number of proper linear subspaces $T_1, \ldots, T_t$ of $\mathbb{Q}^n$ such that all solutions of (3) *lie in $T_1 \cup \cdots \cup T_t$.*

From Lemma 8 in [2] and Theorem 5 in [13] (or Theorem 5 in [8]), we get the following lemma (see also Remark 4 in [2]).

Lemma 2.5. *Assume that $\mu(f_0) < \infty$. Then, for any $P(z), Q(z) \in \mathbb{Q}[z]$ with $\max(\deg P, \deg Q) \leq M$, $Q(z) \neq 0$ and $M \geq 1$, there exists a constant C such that $\mathrm{ord}(Q(z)f_e(z) - P(z)) \leq CM$ for all $e \geq 0$.*

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Let r be a positive integer parameter with $[\sqrt{2}r]-r-6>0$. We define the integer A by $A:=\max\{|a| \mid a\in S\cup\{2\}\}$.

Lemma 3.1. *Notation is the same as above. Then, for any non-negative integers e, h, n and m,*

$$|a_e(n)|\leq A^n \text{ and } |b_h(m)|\leq A^m$$

Proof. For any non-negative integer n, we define the base k-representation of n by

$$n=\sum_{y=0} n_y k^y,$$

where $0\leq n_y\leq k-1$. For any integer i with $0\leq i\leq k-1$ and any non-negative integer y, we define the digital counting function $d(n;ik^y)$ by

$$d(n;ik^y):=\begin{cases}1 & \text{there exists an integer } q \text{ such that } n_q k^q=ik^y,\\ 0 & \text{otherwise.}\end{cases}$$

By the definition of $f_e(z)$, $(a_e(n))_{n\geq 0}$ is decomposed into the following infinite product

$$a_e(n)=\prod_{y=0}^{\infty} a_e(n_y k^y)=\prod_{i=1}^{k-1}\prod_{y=0}^{\infty}\tau(i,y+e)^{d(n;ik^y)} \tag{4}$$

and $a_e(0)=1$. By (4), for any non-negative integer n, we get

$$|a_e(n)|\leq A^{1+\log_k n}\leq A^n.$$

By the same way, for any non-negative integers h and m, we also get

$$|b_h(m)|\leq A^{1+\log_l m}\leq A^m. \qquad \square$$

Lemma 3.2. *Notation is the same as for Section 1. For any non-negative integers e and h, there exist auxiliary functions*

$$Q_{e,h}(z,w)f_e(z)g_h(w)-P_{e,h}(z,w)=\sum_{i+j\geq[\sqrt{2}r]} a_{e,h}(i,j)z^i w^j, \tag{5}$$

with polynomials $Q_{e,h}(z,w)=\sum_{i+j\leq r} q_{e,h}(i,j)z^i w^j\neq 0$, $P_{e,h}(z,w)=\sum_{i+j\leq r} p_{e,h}(i,j)z^i w^j\in\mathbb{Z}[z,w]$ *and*

$$|q_{e,h}(i,j)|\leq 1+\left(\frac{(r+1)(r+2)}{2}A^{2r}\right)^{\frac{(r+1)(r+2)}{2}}=:c_{r,1}, \tag{6}$$

$$|p_{e,h}(i,j)|\leq c_{r,1}(r+1)^2A^r=:c_{r,2}, \tag{7}$$

$$|a_{e,h}(i,j)|\leq c_{r,1}A^{2(i+j)}. \tag{8}$$

Proof. For any non-negative integers e, h, n and m, we define the integer $d_{e,h}(n,m)$ by

$$f_e(z)g_h(w) = (\sum_{n=0}^{\infty} a_e(n)z^n)(\sum_{m=0}^{\infty} b_h(m)w^m) =: \sum_{n+m\geq 0} d_{e,h}(n,m)z^n w^m.$$

Let us denote

$$Q_{e,h}(z,w) = \sum_{i+j\leq r} q_{e,h}(i,j)z^i w^j.$$

By the definition of $d_{e,h}(n,m)$, we get

$$Q_{e,h}(z,w)(\sum_{n+m\leq[\sqrt{2}r]-1} d_{e,h}(n,m)z^n w^m) = \sum_{i+j\geq 0}(\sum_{i_1+i_2=i,j_1+j_2=j} q_{e,h}(i_1,j_1)d_{e,h}(i_2,j_2))z^i w^j.$$

By Lemma 3.1, we consider a system of linear equations

$$\sum_{i_1+i_2=i,j_1+j_2=j} q_{e,h}(i_1,j_1)d_{e,h}(i_2,j_2) = 0, \quad i+j = r+1, r+2, \ldots, [\sqrt{2}r]-1. \tag{9}$$

where $|d_{e,h}(i_2,j_2)| \leq |a_e(i_2)b_h(j_2)| \leq A^{i_2+j_2} \leq A^{[\sqrt{2}r]-1}$. We also have

$$\frac{(r+1)(r+2)}{2} > \frac{[\sqrt{2}r]([\sqrt{2}r]+1)}{2} - \frac{(r+1)(r+2)}{2}. \tag{10}$$

By Lemma 2.1, (9) and (10), there exists a non-zero polynomial $Q_{e,h}(z,w) = \sum_{i+j\leq r} q_{e,h}(i,j)z^i w^j \in \mathbb{Z}[z,w]$, which satisfies (5) and (6), and a polynomial $P_{e,h}(z,w) = \sum_{i+j\leq r} p_{e,h}(i,j)z^i w^j \in \mathbb{Z}[z,w]$, which satisfies (5) and

$$p_{e,h}(i,j) = \sum_{i_1+i_2=i,j_1+j_2=j} q_{e,h}(i_1,j_1)d_{e,h}(i_2,j_2), \quad i+j = 0, 1, \ldots, r. \tag{11}$$

By Lemma 3.1, (6) and (11), we get

$$|p_{e,h}(i,j)| \leq (i+1)(j+1)c_{r,1}A^{i+j} \leq c_{r,1}(r+1)^2 A^r, \quad i+j = 0, 1, \ldots, r.$$

Therefore, the polynomial $P_{e,h}(z,w)$ satisfies (5) and (7). For any non-negative integers e, h, i and j with $i+j \geq [\sqrt{2}r]$, we define the integer $a_{e,h}(i,j)$ by

$$a_{e,h}(i,j) := \sum_{i_1+i_2=i,j_1+j_2=j} q_{e,h}(i_1,j_1)d_{e,h}(i_2,j_2), \quad i+j \geq [\sqrt{2}r].$$

From Lemma 3.1 and (6), we have

$$|a_{e,h}(i,j)| \leq (i+1)(j+1)c_{r,1}A^{i+j} \leq c_{r,1}A^{2(i+j)}, \quad i+j \geq [\sqrt{2}r]. \qquad \square$$

Lemma 3.3. *For any positive integer n, there exists an integer tuple (n_1, n_2) such that*

$$1 < \frac{k^{n_1}}{l^{n_2}} < 1 + \frac{1}{n}.$$

Proof. By Lemma 2.3. $\square$

For any positive integer n, we define the integers $e_1(n), e_2(n)$ by

$$e_1(n) := \min\{n_1 \mid 1 < \frac{k^{n_1}}{l^{n_2}} < 1 + \frac{1}{n}\}, e_2(n) := \min\{n_2 \mid 1 < \frac{k^{e_1(n)}}{l^{n_2}} < 1 + \frac{1}{n}\}.$$

From the definitions of $e_1(n)$ and $e_2(n)$, we get

$$1 < \frac{k^{e_1(n)}}{l^{e_2(n)}} < 1 + \frac{1}{n}. \tag{12}$$

By the definition of $e_1(n)$, we get

$$\lim_{n\to\infty} e_1(n) = +\infty. \tag{13}$$

By (13) and the definition of $e_2(n)$, we have

$$\lim_{n\to\infty} e_2(n) = +\infty. \tag{14}$$

By (12), we have

$$\frac{e_1(n)\log k}{\log l} - \frac{\log(1+\frac{1}{n})}{\log l} < e_2(n) < \frac{e_1(n)\log k}{\log l}. \tag{15}$$

From (15), we also have

$$\frac{\log l}{\log k} < \frac{e_1(n)}{e_2(n)} < \frac{\log l}{\log k} + \frac{\log(1+\frac{1}{n})}{e_2(n)\log k}. \tag{16}$$

By (14), (16) and the assumption that k and l are multiplicatively independent, $\lim_{n\to\infty}\frac{e_1(n)}{e_2(n)}$ converges to an irrational number $\frac{\log l}{\log k}$ as $n \to +\infty$. By the pigeonhole principle, there exist a subset I of indexes of $Q_{e_1(n),e_2(n)}(z,w)$ and an infinite subset N_1 of $\mathbb{N}$ such that

$$N_1 = \{n \in \mathbb{N} \mid \text{for any } (i,j) \in I, q_{e_1(n),e_2(n)}(i,j) \neq 0 \text{ and for any } (i,j) \notin I, q_{e_1(n),e_2(n)}(i,j) = 0\}.$$

By the pigeonhole principle and Lemma 2.2, there exist an index $(I_M, J_M) \in I$ and an infinite subset N_2 of N_1 such that

$$N_2 = \{n \in N_1 \mid \text{for any } (i,j) \in I \setminus \{(I_M, J_M)\}, I_M k^{e_1(n)} + J_M l^{e_2(n)} > i k^{e_1(n)} + j l^{e_2(n)}\}.$$

Lemma 3.4. *Let $Q_{e,h}(z,w)$ be defined in Lemma 3.2. There exists an integer M_1 such that*

$$Q_{e_1(n),e_2(n)}(\frac{1}{b^{k^{e_1(n)}}}, \frac{1}{b^{l^{e_2(n)}}}) \neq 0$$

for all $n \geq M_1$ with $n \in N_2$.

Proof. For any integer n in N_2, we have

$$\begin{aligned} \mathbb{Z} \ni\ & b^{I_M k^{e_1(n)} + J_M l^{e_2(n)}} Q_{e_1(n),e_2(n)}(\frac{1}{b^{k^{e_1(n)}}}, \frac{1}{b^{l^{e_2(n)}}}) \\ &= q_{e_1(n),e_2(n)}(I_M, J_M) + \sum_{(i,j)\in I, (i,j)\neq(I_M,J_M)} q_{e_1(n),e_2(n)}(i,j) b^{(I_M-i)k^{e_1(n)} + (J_M-j)l^{e_2(n)}}. \end{aligned} \tag{17}$$

Let $\frac{1}{2} < \gamma < 1$. By Lemma 2.2, (13), (14) and the irrationality of $\lim_{n\to\infty}\frac{e_1(n)}{e_2(n)}$, for any index (i,j) with $(i,j) \neq (I_M, J_M)$, we have

$$\begin{gathered}(I_M - i)k^{e_1(n)} + (J_M - j)l^{e_2(n)} > k^{e_1(n)}\gamma^{e_1(n)} \\ \text{or} \\ (I_M - i)k^{e_1(n)} + (J_M - j)l^{e_2(n)} > l^{e_2(n)}\gamma^{e_2(n)}\end{gathered} \tag{18}$$

for all large integer n in N_2. We assume that there exist infinitely many integers n with $n \in N_2$ such that

$$Q_{e_1(n),e_2(n)}(\frac{1}{b^{k^{e_1(n)}}}, \frac{1}{b^{l^{e_2(n)}}}) = 0. \tag{19}$$

From (17) and (19), we get

$$-q_{e_1(n),e_2(n)}(I_M, J_M) = \sum_{(i,j)\in I,(i,j)\neq(I_M,J_M)} q_{e_1(n),e_2(n)}(i,j)b^{(I_M-i)k^{e_1(n)}+(J_M-j)l^{e_2(n)}}. \tag{20}$$

Let p be a prime factor of b. By (20), (18), (6) and the p-adic valuation of $q_{e_1(n),e_2(n)}(I_M, J_M)$, for any sufficiently large integer n in N_2 which satisfies (19), we get $q_{e_1(n),e_2(n)}(I_M, J_M) = 0$. This contradicts $q_{e_1(n),e_2(n)}(I_M, J_M) \neq 0$ with $n \in N_2$. Therefore, for any sufficiently large integer n in N_2, we get

$$Q_{e_1(n),e_2(n)}(\frac{1}{b^{k^{e_1(n)}}}, \frac{1}{b^{l^{e_2(n)}}}) \neq 0.$$ □

By the pigeonhole principle and Lemma 2.2, there exist an index (I_m, J_m) in I and an infinite subset N_3 of N_2 such that

$$\begin{aligned}N_3 = \{n \in N_2 \mid &\text{ for any } (i,j) \in I \setminus \{(I_m, J_m)\}, \\ &I_m k^{e_1(n)} + J_m l^{e_2(n)} < ik^{e_1(n)} + jl^{e_2(n)} \text{ and } n \geq M_1\}.\end{aligned}$$

Lemma 3.5. *For any positive integer n,*

$$\begin{aligned}&|Q_{e_1(n),e_2(n)}(\frac{1}{b^{k^{e_1(n)}}}, \frac{1}{b^{l^{e_2(n)}}})f_{e_1(n)}(\frac{1}{b^{k^{e_1(n)}}})g_{e_2(n)}(\frac{1}{b^{l^{e_2(n)}}}) - P_{e_1(n),e_2(n)}(\frac{1}{b^{k^{e_1(n)}}}, \frac{1}{b^{l^{e_2(n)}}})| \\ &\leq \sum_{j\geq[\sqrt{2}r]} \frac{c_{r,1}A^{2j}(j+1)}{b^{jl^{e_2(n)}}} \leq \sum_{j\geq[\sqrt{2}r]} \frac{c_{r,1}(A^3)^{j+1}}{b^{jl^{e_2(n)}}}.\end{aligned} \tag{21}$$

Proof. By Lemma 3.2, (12) and $A \geq 2$. □

Lemma 3.6. *For any integer n in N_3, we define a rational number p_n by*

$$p_n := \prod_{y=0}^{e_1(n)-1} (1 + \sum_{i=1}^{k-1} \frac{\tau(i,y)}{b^{ik^y}}) \cdot \prod_{y=0}^{e_2(n)-1} (1 + \sum_{j=1}^{l-1} \frac{\delta(j,y)}{b^{jl^y}}) \cdot P_{e_1(n),e_2(n)}(\frac{1}{b^{k^{e_1(n)}}}, \frac{1}{b^{l^{e_2(n)}}}).$$

Then

$$\lim_{n\in N_3, n\to\infty} \frac{p_n}{Q_{e_1(n),e_2(n)}(\frac{1}{b^{k^{e_1(n)}}}, \frac{1}{b^{l^{e_2(n)}}})} = \lim_{n\in N_3, n\to\infty} \frac{b^{I_m k^{e_1(n)}+J_m l^{e_2(n)}} p_n}{q_{e_1(n),e_2(n)}(I_m, J_m)} = f_0(\frac{1}{b})g_0(\frac{1}{b}). \tag{22}$$

Proof. From the definition of N_3 and (12), then

$$|Q_{e_1(n),e_2(n)}(\frac{1}{b^{k^{e_1(n)}}},\frac{1}{b^{l^{e_2(n)}}})| \geq \frac{1}{b^{rk^{e_1(n)}}} \tag{23}$$

for all integer n in N_3. By Lemma 3.5, we get

$$\begin{aligned} &|Q_{e_1(n),e_2(n)}(\frac{1}{b^{k^{e_1(n)}}},\frac{1}{b^{l^{e_2(n)}}})f_0(\frac{1}{b})g_0(\frac{1}{b}) - p_n| \\ &\leq |\prod_{y=0}^{e_1(n)-1}(1+\sum_{i=1}^{k-1}\frac{\tau(i,y)}{b^{ik^y}})\prod_{y=0}^{e_2(n)-1}(1+\sum_{j=1}^{l-1}\frac{\delta(j,y)}{b^{jl^y}})| \sum_{j\geq[\sqrt{2}r]} c_{r,1}\frac{(A^3)^{j+1}}{b^{jl^{e_2(n)}}} \\ &\leq A^{e_1(n)}A^{e_2(n)}\frac{b^2}{(b-1)^2}\sum_{j\geq[\sqrt{2}r]}\frac{c_{r,1}(A^3)^{j+1}}{b^{jl^{e_2(n)}}} \\ &\leq \frac{b^2}{(b-1)^2}b^{k^{e_1(n)}}b^{l^{e_2(n)}}\sum_{j\geq[\sqrt{2}r]}\frac{c_{r,1}(A^3)^{j+1}}{b^{jl^{e_2(n)}}}. \end{aligned} \tag{24}$$

By dividing (24) by $Q_{e_1(n),e_2(n)}(\frac{1}{b^{k^{e_1(n)}}},\frac{1}{b^{l^{e_2(n)}}}) \neq 0$, (23) and (12), for any sufficiently large integer n in N_3, we get

$$\begin{aligned} |f_0(\frac{1}{b})g_0(\frac{1}{b}) - \frac{p_n}{Q_{e_1(n),e_2(n)}(\frac{1}{b^{k^{e_1(n)}}},\frac{1}{b^{l^{e_2(n)}}})}| &\leq \frac{b^2}{(b-1)^2}b^{(r+1)k^{e_1(n)}}b^{l^{e_2(n)}}\sum_{j\geq[\sqrt{2}r]}\frac{c_{r,1}(A^3)^{j+1}}{b^{jl^{e_2(n)}}} \\ &\leq \frac{b^2}{(b-1)^2}b^{(r+2)l^{e_2(n)}\frac{1+n}{n}}\sum_{j\geq[\sqrt{2}r]}\frac{c_{r,1}(A^3)^{j+1}}{b^{jl^{e_2(n)}}} \\ &\leq \frac{b^2}{(b-1)^2}b^{(r+3)l^{e_2(n)}}\sum_{j\geq[\sqrt{2}r]}\frac{c_{r,1}(A^3)^{j+1}}{b^{jl^{e_2(n)}}}. \end{aligned} \tag{25}$$

By $[\sqrt{2}r] - r - 6 > 0$, (14) and n tends to infinity in (25), we get (22). □

Now we assume that $f_0(\frac{1}{b})g_0(\frac{1}{b})$ is a non-zero algebraic number. We define the integer s by $s := \#I$. We align elements (i,j) of I as $(i_1,j_1) := (I_m,J_m),\dots,(i_t,j_t),\dots,(i_s,j_s) := (I_M,J_M)$. Let T be the set of prime factors of b. We define the linearly independent linear forms $L_{t,\infty}(\mathbf{x}) := L_{t,\infty}(x_1,x_2,\dots,x_{s+1})$ $(1 \leq t \leq s+1)$ by

$$L_{t,\infty}(\mathbf{x}) := x_t \ (1 \leq t \leq s) \ \text{ and } \ L_{s+1,\infty}(\mathbf{x}) := f_0(\frac{1}{b})g_0(\frac{1}{b})\sum_{t=1}^{s}x_t + x_{s+1}.$$

For any prime p in T, we also define the linearly independent linear forms $L_{t,p}(\mathbf{x})$ $(1 \leq t \leq s+1)$ by

$$L_{t,p}(\mathbf{x}) := x_t \ (1 \leq t \leq s+1).$$

For any integer n in N_3, we define the integer tuple $\mathbf{x}(n) := (x_1(n),\dots x_t(n),\dots,x_{s+1}(n))$ by

$$\begin{aligned} \mathbf{x}(n) :&= (x_1(n),\dots,x_t(n),\dots,x_{s+1}(n)) \\ &= (b^{(r+2-I_m)k^{e_1(n)}-J_m l^{e_2(n)}}q_{e_1(n),e_2(n)}(I_m,J_m),\dots, \\ &\quad b^{(r+2-i_t)k^{e_1(n)}-j_t l^{e_2(n)}}q_{e_1(n),e_2(n)}(i_t,j_t),\dots,-b^{(r+2)k^{e_1(n)}}p_n). \end{aligned}$$

where $(i_t, j_t) \in I$ $(1 \le t \le s)$. By (12) and Lemma 3.2, for any sufficiently large integer n in N_3, we have

$$\begin{aligned}\max\{|x_t(n)| \mid 1 \le t \le s+1\} &\le b^{(r+2)k^{e_1(n)}} c_{r,1} \frac{b^2}{(b-1)^2} A^{e_1(n)} A^{e_2(n)} \frac{(r+1)(r+2)}{2} c_{r,2} \\ &\le b^{(r+3)k^{e_1(n)}} b^{k^{e_1(n)}} b^{l^{e_2(n)}} \\ &\le b^{(r+3)l^{e_2(n)}(1+\frac{1}{n})} b^{l^{e_2(n)}(1+\frac{1}{n})} b^{l^{e_2(n)}} \le b^{(r+6)l^{e_2(n)}}.\end{aligned} \tag{26}$$

For any integer t with $1 \le t \le s$, by the definition of T and $q_{e_1(n),e_2(n)}(i_t, j_t) \in \mathbb{Z}$, we get

$$\begin{aligned}|L_{t,\infty}(\mathbf{x}(n))| \prod_{p \in T} |L_{t,p}(\mathbf{x}(n))|_p &= |q_{e_1(n),e_2(n)}(i_t, j_t)| \prod_{p \in T} |q_{e_1(n),e_2(n)}(i_t, j_t)|_p \\ &\le |q_{e_1(n),e_2(n)}(i_t, j_t)| \le c_{r,1}.\end{aligned} \tag{27}$$

By (24), we have

$$|L_{s+1,\infty}(\mathbf{x}(n))| \le \frac{b^2}{(b-1)^2} b^{(r+3)k^{e_1(n)}} b^{l^{e_2(n)}} \sum_{j \ge [\sqrt{2}r]} \frac{c_{r,1}(A^3)^{j+1}}{b^{jl^{e_2(n)}}}. \tag{28}$$

From $-b^{(r+2)k^{e_1(n)}} p_n \in \mathbb{Z}$, we get

$$\prod_{p \in T} |L_{s+1,p}(\mathbf{x}(n))|_p \le 1. \tag{29}$$

By (26)–(29) and (12), for any sufficiently large integer n in N_3, we have

$$\begin{aligned}&|L_{1,\infty}(\mathbf{x}(n)) \cdots L_{s+1,\infty}(\mathbf{x}(n))| \prod_{p \in T} |L_{1,p}(\mathbf{x}(n)) \cdots L_{s+1,p}(\mathbf{x}(n))|_p \\ &\le c_{r,1}^s \frac{b^2}{(b-1)^2} b^{(r+3)k^{e_1(n)}} b^{l^{e_2(n)}} \sum_{j \ge [\sqrt{2}r]} \frac{c_{r,1}(A^3)^{j+1}}{b^{jl^{e_2(n)}}} \\ &\le c_{r,1}^s \frac{b^2}{(b-1)^2} b^{(r+3)l^{e_2(n)}(1+\frac{1}{n})} b^{2l^{e_2(n)}} \sum_{j \ge [\sqrt{2}r]} \frac{c_{r,1}(A^3)^{j+1}}{b^{jl^{e_2(n)}}} \\ &\le \frac{b^{(r+6)l^{e_2(n)}}}{b^{[\sqrt{2}r]l^{e_2(n)}}} \le \frac{1}{(\max\{|x_t(n)| \mid 1 \le t \le s+1\})^{\frac{[\sqrt{2}r]-r-6}{r+6}}}.\end{aligned} \tag{30}$$

By (30), $[\sqrt{2}r] - r - 6 > 0$ and Lemma 2.4, for any sufficiently large integer n in N_3, $(x_1(n), \ldots, x_{s+1}(n))$ lie in finitely many proper linear subspaces of $\mathbb{Q}^{s+1}$. There exist an infinite subset $N_4 \subset N_3$ and a non-zero integer tuple $(z_1, \ldots, z_{s+1})$ such that, for any n in N_4,

$$\begin{aligned}&z_1 b^{(r+2-I_m)k^{e_1(n)} - J_m l^{e_2(n)}} q_{e_1(n),e_2(n)}(I_m, J_m) \\ &+ \cdots + z_t b^{(r+2-i_t)k^{e_1(n)} - j_t l^{e_2(n)}} q_{e_1(n),e_2(n)}(i_t, j_t) \\ &+ \cdots + z_{s+1} b^{(r+2)k^{e_1(n)}} p_n = 0.\end{aligned} \tag{31}$$

We define the integer u as $u := \min\{i \mid z_i \neq 0\}$. If $1 < u$, we have

$$\begin{aligned} & z_u b^{(r+2-i_u)k^{e_1(n)}-j_u l^{e_2(n)}} q_{e_1(n),e_2(n)}(i_u, j_u) \\ & + \cdots + z_M b^{(r+2-i_M)k^{e_1(n)}-j_M l^{e_2(n)}} q_{e_1(n),e_2(n)}(i_M, j_M) \\ & = -z_{s+1} b^{(r+2)k^{e_1(n)}} p_n. \end{aligned} \tag{32}$$

By $f_0(\frac{1}{b})g_0(\frac{1}{b}) \neq 0$, $|q_{e_1(n),e_2(n)}(I_m, J_m)| \geq 1$ and (22), for any sufficiently large integer n in N_4, we have

$$|b^{I_m k^{e_1(n)}+J_m l^{e_2(n)}} p_n| \geq |\frac{f_0(\frac{1}{b})g_0(\frac{1}{b})}{2}| \neq 0. \tag{33}$$

By (32) and (33), for any sufficiently large integer n in N_4, we have

$$\begin{aligned} & |z_u b^{(r+2-i_u)k^{e_1(n)}-j_u l^{e_2(n)}} q_{e_1(n),e_2(n)}(i_u, j_u) + \cdots + z_M b^{(r+2-i_M)k^{e_1(n)}-j_M l^{e_2(n)}} q_{e_1(n),e_2(n)}(i_M, j_M)| \\ & \geq |z_{s+1} b^{(r+2-I_m)k^{e_1(n)}-J_m l^{e_2(n)}} \frac{f_0(\frac{1}{b})g_0(\frac{1}{b})}{2}|. \end{aligned} \tag{34}$$

By (33), Lemma 2.2, (6), the minimality of (I_m, J_m), dividing (34) by $b^{(r+2-I_m)k^{e_1(n)}-J_m l^{e_2(n)}}$ and n tends to infinity, we get $z_{s+1}\frac{f_0(\frac{1}{b})g_0(\frac{1}{b})}{2} = 0$. From $f_0(\frac{1}{b})g_0(\frac{1}{b}) \neq 0$, we have $z_{s+1} = 0$. We define the integer v by $v := \#\{t \mid z_t \neq 0, \ 1 \leq t \leq s\}$. Let $z_{u_1}, z_{u_2}, \ldots, z_{u_v}$ be v pairwise distinct non-zero coordinates of $(z_1, \ldots, z_{s+1})$. We define the subindex I_{sub} by $I_{sub} := \{(i_{u_1}, j_{u_1}), \ldots, (i_{u_v}, j_{u_v})\}$. By the pigeonhole principle and Lemma 2.2, there exist an index $(i_{u_m}, j_{u_m}) \in I_{sub}$ and an infinite subset N_5 of N_4 such that

$$N_5 = \{n \in N_4 \mid \text{for any } (i, j) \in I_{sub} \setminus \{(i_{u_m}, j_{u_m})\}, i_{u_m} k^{e_1(n)} + j_{u_m} l^{e_2(n)} < i k^{e_1(n)} + j l^{e_2(n)}\}.$$

By (32) and $z_{s+1} = 0$, we have

$$\begin{aligned} & z_{u_1} b^{(r+2-i_{u_1})k^{e_1(n)}-j_{u_1} l^{e_2(n)}} q_{e_1(n),e_2(n)}(i_u, j_u) \\ & + \cdots + z_{u_m} b^{(r+2-i_{u_m})k^{e_1(n)}-j_{u_m} l^{e_2(n)}} q_{e_1(n),e_2(n)}(i_{u_m}, j_{u_m}) \\ & + \cdots + z_{u_v} b^{(r+2-i_{u_v})k^{e_1(n)}-j_{u_v} l^{e_2(n)}} q_{e_1(n),e_2(n)}(i_{u_v}, j_{u_v}) = 0. \end{aligned} \tag{35}$$

By dividing (35) by $b^{(r+2-i_{u_m})k^{e_1(n)}-j_{u_m} l^{e_2(n)}}$, Lemma 2.2, (6) and n tends to infinity, we get $\lim_{n \in N_5, n \to \infty} z_{u_m} q_{e_1(n),e_2(n)}(i_{u_m}, j_{u_m}) = 0$. This contradicts $|z_{u_m} q_{e_1(n),e_2(n)}(i_{u_m}, j_{u_m})| \geq 1$. Therefore $u = 1$. By dividing (31) by $b^{(r+2-I_m)k^{e_1(n)}-J_m l^{e_2(n)}} q_{e_1(n),e_2(n)}(I_m, J_m)$, Lemma 2.2, (22) and n (in N_4) tends to infinity, we get

$$z_1 + z_{s+1} f_0(\frac{1}{b}) g_0(\frac{1}{b}) = 0. \tag{36}$$

By (36), $z_1 \neq 0$ and $f_0(\frac{1}{b})g_0(\frac{1}{b}) \neq 0$, $f_0(\frac{1}{b})g_0(\frac{1}{b})$ is a rational number. This completes the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. By the pigeonhole principle and $Q_{e_1(n),e_2(n)}(z, w) \neq 0$, there exist an infinite subset N_6 of N_3 and an integer u with $0 \leq 2u \leq r$ such that

$$\tilde{Q}_{e_1(n),e_2(n)}(z,w) := \frac{Q_{e_1(n),e_2(n)}(z,w)}{z^u w^u} \in \mathbb{Z}[z,w]$$

and

$$\tilde{Q}_{e_1(n),e_2(n)}(z,0) \neq 0 \text{ or } \tilde{Q}_{e_1(n),e_2(n)}(0,w) \neq 0. \tag{37}$$

for all n in N_6. By (5), we have

$$\begin{aligned} &\tilde{Q}_{e_1(n),e_2(n)}(z,w) f_{e_1(n)}(z) g_{e_2(n)}(w) - \frac{P_{e_1(n),e_2(n)}(z,w)}{z^u w^u} \\ &= \sum_{i+j \geq [\sqrt{2}r]-2u} a_{e_1(n),e_2(n)}(i+u, j+u) z^i w^j = \sum_{i+j \geq [\sqrt{2}r]-2u} \tilde{a}_{e_1(n),e_2(n)}(i,j) z^i w^j \end{aligned} \tag{38}$$

and $\tilde{P}_{e_1(n),e_2(n)}(z,w) := \frac{P_{e_1(n),e_2(n)}(z,w)}{z^u w^u} \in \mathbb{Z}[z,w]$. By Lemma 2.5, (37) and (38), there exists an infinite subset N_7 of N_6, a constant C with $C > [\sqrt{2}r] - 2u$ and an index (i_C, j_C) with $i_C + j_C = C$ such that

$$\tilde{a}_{e_1(n),e_2(n)}(i_C, j_C) \neq 0 \text{ and } \tilde{a}_{e_1(n),e_2(n)}(i,j) = 0 \tag{39}$$

for all (i,j) with $i + j < C$ and all n in N_7. From the pigeonhole principle and Lemma 2.2, there exist an index (I_o, J_o) with $I_o + J_o = C$ and an infinite subset N_8 of N_7 such that

$$\begin{aligned} N_8 = \{ n \in N_7 \mid &\text{ for any } (i,j) \neq (I_o, J_o) \text{ with } i+j = C, \\ &I_o k^{e_1(n)} + J_o l^{e_2(n)} < i k^{e_1(n)} + j l^{e_2(n)} \text{ and } \tilde{a}_{e_1(n),e_2(n)}(I_o, J_o) \neq 0 \}. \end{aligned} \tag{40}$$

By (38) and (39), we have

$$\begin{aligned} &\tilde{Q}_{e_1(n),e_2(n)}\Big(\frac{1}{b^{k^{e_1(n)}}}, \frac{1}{b^{l^{e_2(n)}}}\Big) f_{e_1(n)}\Big(\frac{1}{b^{k^{e_1(n)}}}\Big) g_{e_2(n)}\Big(\frac{1}{b^{l^{e_2(n)}}}\Big) - \tilde{P}_{e_1(n),e_2(n)}\Big(\frac{1}{b^{k^{e_1(n)}}}, \frac{1}{b^{l^{e_2(n)}}}\Big) \\ &= \sum_{i+j \geq C} \tilde{a}_{e_1(n),e_2(n)}(i,j) \frac{1}{b^{i k^{e_1(n)} + j l^{e_2(n)}}}. \end{aligned} \tag{41}$$

By Lemma 2.2 with $\frac{1}{2} < \gamma < 1$, (8), (12), (39), (40) and $|\tilde{a}_{e_1(n),e_2(n)}(I_o, J_o)| \geq 1$, for any sufficiently large integer n in N_8, we get

$$\begin{aligned} &\Big| \sum_{i+j \geq C} \tilde{a}_{e_1(n),e_2(n)}(i,j) \frac{1}{b^{i k^{e_1(n)} + j l^{e_2(n)}}} \Big| \\ &\geq \Big| \sum_{i+j = C} \tilde{a}_{e_1(n),e_2(n)}(i,j) \frac{1}{b^{i k^{e_1(n)} + j l^{e_2(n)}}} \Big| - \Big| \sum_{i+j \geq C+1} \tilde{a}_{e_1(n),e_2(n)}(i,j) \frac{1}{b^{i k^{e_1(n)} + j l^{e_2(n)}}} \Big| \\ &\geq \Big| \sum_{i+j = C} \tilde{a}_{e_1(n),e_2(n)}(i,j) \frac{1}{b^{i k^{e_1(n)} + j l^{e_2(n)}}} \Big| - \sum_{j=0}^{\infty} \frac{c_{r,1} A^r (A^3)^{j+1}}{b^{(C+1+j) l^{e_2(n)}}} \\ &\geq \frac{1}{b^{I_o k^{e_1(n)} + J_o l^{e_2(n)}}} - \frac{c_{r,1} C A^r (A^3)^{C+1}}{b^{I_o k^{e_1(n)} + J_o l^{e_2(n)} + k^{e_1(n)} \gamma^{e_1(n)}}} - \frac{c_{r,1} C A^r (A^3)^{C+1}}{b^{I_o k^{e_1(n)} + J_o l^{e_2(n)} + l^{e_2(n)} \gamma^{e_2(n)}}} - \sum_{j=0}^{\infty} \frac{c_{r,1} A^r (A^3)^{j+1}}{b^{(C+1+j) l^{e_2(n)}}} \\ &\geq \frac{1}{b^{I_o k^{e_1(n)} + J_o l^{e_2(n)}}} \Big(1 - \frac{c_{r,1} C A^r (A^3)^{C+1}}{b^{k^{e_1(n)} \gamma^{e_1(n)}}} - \frac{c_{r,1} C A^r (A^3)^{C+1}}{b^{l^{e_2(n)} \gamma^{e_2(n)}}}\Big) - \sum_{j=0}^{\infty} \frac{c_{r,1} A^r (A^3)^{j+1}}{b^{(C+1+j) l^{e_2(n)}}} \\ &\geq \frac{1}{2 b^{I_o k^{e_1(n)} + J_o l^{e_2(n)}}} - \sum_{j=0}^{\infty} \frac{c_{r,1} A^r (A^3)^{j+1}}{b^{(C+1+j) l^{e_2(n)}}} \\ &\geq \frac{1}{2 b^{C(1+\frac{1}{n}) l^{e_2(n)}}} - \sum_{j=0}^{\infty} \frac{c_{r,1} A^r (A^3)^{j+1}}{b^{(C+1+j) l^{e_2(n)}}} > 0. \end{aligned} \tag{42}$$

By Theorem 1.1, we only prove that $f_0(\frac{1}{b})g_0(\frac{1}{b})$ is irrational. We assume that $f_0(\frac{1}{b})g_0(\frac{1}{b}) = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ with $q \neq 0$. By (41), (42), (24) and $(1 + \sum_{i=1}^{k-1} \frac{\tau(i,y)}{b^{ik^y}})(1 + \sum_{j=1}^{l-1} \frac{\delta(j,y)}{b^{jl^y}}) \neq 0$ (for all $y \geq 0$), we have

$$\begin{aligned} 0 &< |Q_{e_1(n),e_2(n)}(\frac{1}{b^{k^{e_1(n)}}}, \frac{1}{b^{l^{e_2(n)}}})f_0(\frac{1}{b})g_0(\frac{1}{b}) - p_n| \\ &\leq \frac{b^2}{(b-1)^2} b^{k^{e_1(n)}} b^{l^{e_2(n)}} \sum_{j \geq [\sqrt{2}r]} \frac{c_{r,1}(A^3)^{j+1}}{b^{jl^{e_2(n)}}}. \end{aligned} \tag{43}$$

By the definitions of $Q_{e_1(n),e_2(n)}(\frac{1}{b^{k^{e_1(n)}}}, \frac{1}{b^{l^{e_2(n)}}})$ and p_n, we have

$$M_n := b^{(r+2)k^{e_1(n)}} q(Q_{e_1(n),e_2(n)}(\frac{1}{b^{k^{e_1(n)}}}, \frac{1}{b^{l^{e_2(n)}}})f_0(\frac{1}{b})g_0(\frac{1}{b}) - p_n) \in \mathbb{Z}. \tag{44}$$

On the other hand, by (43), $[\sqrt{2}r] - r - 6 > 0$ and (12), for any sufficiently large integer n in N_8, we get

$$0 < |M_n| < 1.$$

This contradicts (44). Therefore, $f_0(\frac{1}{b})g_0(\frac{1}{b})$ is irrational. □

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