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A study of the complexification process of the (s, t)-Perrin sequence

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Abstract: The present article deals with the study of the generalized (s, t)-Perrin sequence in its complex process. Thus, from the one-dimensional model of the generalized (s, t)-Perrin sequence, imaginary units are inserted, starting with the insertion of unit i, called two-dimensional relations. Altogether, we have the n-dimensional relationships of the generalized (s, t)-Perrin sequence.

Keywords: Generalization, Perrin sequence, Recurrences.
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1 Introduction

The Perrin sequence named after the Frenchman François Olivier Raoul Perrin (1841–1910). This sequence is linear and recurrent, having characteristics similar to the Padovan sequence P_n [4,5]. Thus, we have the recurrence of the Perrin sequence R_n, given by:

$$R_n = R_{n-2} + R_{n-3}, n \geqslant 3,$$

with $R_0 = 3, R_1 = 0, R_2 = 2$.

It is therefore noteworthy that the Perrin numbers differ from the Padovan numbers due to their initial values, given by: $P_0 = P_1 = P_2 = 1$.

Some studies are carried out around these sequences, highlighting the generalization of the coefficients of the Padovan sequence [3,6–8], called (s,t)-Padovan $P_n(s,t)$, where [2,7]:

$$P_n(s,t) = sP_{n-2}(s,t) + tP_{n-3}(s,t), n > 0,$$

assuming the initial values are $P_0(s,t) = 1, P_1(s,t) = 1, P_2(s,t) = s$ and $s \geqslant 0, t \neq 0$, $27t^2 - 4s^3 \neq 0$.

For the sequence (s,t)-Perrin, its recurrence is defined in the same way. Thus, one has:

$$R_n(s,t) = sR_{n-2}(s,t) + tR_{n-3}(s,t), n > 0,$$

assuming the initial values are $R_0(s,t) = 3, R_1(s,t) = 0, R_2(s,t) = 2$ and $s \geqslant 0, t \neq 0$, $27t^2 - 4s^3 \neq 0$.

Starting from Perrin's primitive sequence and its generalization (s,t)-Perrin, we have the complexification process, inserting the imaginary units studied by Vieira, Mangueira, Alves and Catarino [11].

From now on, the n-dimensional relations of the sequence (s,t)-Perrin will be studied, based on the work of Diskaya and Menken [1], in which it portrays the n-dimensional relations of the sequence of (p,q)-Fibonacci. With this, one can see the evolution of studies around these sequences, approaching the process of generalization and complexification of these numbers.

2 Two-dimensional recurrences of the (s,t)-Perrin sequence

In this section, we introduce the two-dimensional recurrences of the (s,t)-Perrin sequence based on the one-dimensional recurrence.

Definition 2.1. *For $n, m \in \mathbb{N}$ and $s \geqslant 0, t \neq 0, 27t^2 - 4s^3 \neq 0$, the two-dimensional of (s,t)-Perrin sequence $R_{s,t}^{n,m}$ is defined by the recurrences:*

$$R_{s,t}^{n+3,m} = sR_{s,t}^{n+1,m} + tR_{s,t}^{n,m},$$

$$R_{s,t}^{n,m+3} = sR_{s,t}^{n,m+1} + tR_{s,t}^{n,m},$$

with the initial values:

$$
\begin{aligned}
&R_{s,t}^{0,0} = 3, && R_{s,t}^{1,0} = 0, && R_{s,t}^{2,0} = 2,\\
&R_{s,t}^{0,1} = 3 + 2i && R_{s,t}^{1,1} = 2i, && R_{s,t}^{2,1} = 2s + 2si,\\
&R_{s,t}^{0,2} = 3s + 3ti, && R_{s,t}^{1,2} = 3ti, && R_{s,t}^{2,2} = 2s + 3sti,
\end{aligned}
$$

where $i^2 = -1$.

Property 2.1. *The following properties are valid:*

(a) $R_{s,t}^{n,0} = R_n(s,t)$;

(b) $R_{s,t}^{0,m} = 3P_m(s,t) + iR_{m+1}(s,t)$;

(c) $R_{s,t}^{n,1} = R_n(s,t) + i2P_n(s,t)$;

(d) $R_{s,t}^{1,m} = iR_{m+1}(s,t)$;

(e) $R_{s,t}^{n,m} = R_n(s,t)P_m(s,t) + iP_n(s,t)R_{m+1}(s,t)$.

Proof. (a) By the mathematical induction principle and by recurrence: $R_{s,t}^{n+3,m} = R_{s,t}^{n+1,m} + R_{s,t}^{n,m}$, $m = 0$, we can prove the first proposition.

For $n = 0$:

$$R_{s,t}^{0,0} = R_0(s,t) = 3.$$

Suppose that the desired equality is true for any $n \geqslant k, k \in \mathbb{N}$, we have to:

$$R_{s,t}^{k,0} = R_k(s,t).$$

Let us show that it is true for $k+1$.

$$
\begin{aligned}
R_{s,t}^{k+4,0} &= pR_{s,t}^{k+2,0} + qR_{s,t}^{k+1,0}\\
&= R_{k+2}(s,t) + R_{k+1}(s,t)\\
&= R_{k+4}(s,t).
\end{aligned}
$$

Thus proposition (a) is validated.

(b) By the mathematical induction principle and by recurrence: $R_{s,t}^{n,m+3} = R_{s,t}^{n,m+1} + R_{s,t}^{n,m}$, $n = 0$, we can prove the proposition.

For $m = 0$:

$$R_{s,t}^{0,0} = 3P_0(s,t) + iR_1(s,t) = 3.$$

Suppose that the desired equality is true for any $m \geqslant k, k \in \mathbb{N}$, we have to:

$$R_{s,t}^{0,k} = 3P_k(s,t) + iR_{k+1}(s,t).$$

Let us show that it is true for $k+1$.

$$
\begin{aligned}
R_{s,t}^{0,k+4} &= pR_{s,t}^{0,k+2} + qR_{s,t}^{0,k+1}\\
&= 3P_{k+2}(s,t) + iR_{k+3}(s,t) + 3P_{k+1}(s,t) + iR_{k+2}(s,t)\\
&= 3P_{k+4}(s,t) + iR_{k+5}(s,t).
\end{aligned}
$$

Thus proposition (b) is validated.

(c) By the mathematical induction principle and by recurrence: $R_{s,t}^{n+3,m} = R_{s,t}^{n+1,m} + R_{s,t}^{n,m}$, $m = 1$, we can prove the proposition.

For $n = 0$:

$$R_{s,t}^{0,1} = R_0(s,t) + i2P_0(s,t) = 3 + 2i.$$

Suppose that the desired equality is true for any $n \geqslant k, k \in \mathbb{N}$, we have:

$$R_{s,t}^{k,1} = R_k(s,t) + i2P_k(s,t).$$

Let us show that it is true for $k + 1$.

$$\begin{aligned} R_{s,t}^{k+4,1} &= pR_{s,t}^{k+2,1} + qR_{s,t}^{k+1,1} \\ &= R_{k+2}(s,t) + i2P_{k+2}(s,t) + R_{k+1}(s,t) + i2P_{k+1}(s,t) \\ &= R_{k+4}(s,t) + i2P_{k+4}(s,t) \end{aligned}$$

Thus proposition (c) is validated.

(d) The mathematical induction principle and by recurrence: $R_{s,t}^{n,m+3} = R_{s,t}^{n,m+1} + R_{s,t}^{n,m}$, $n = 1$, we can prove the proposition.

For $m = 0$:

$$R_{s,t}^{1,0} = iR_1(s,t) = 0.$$

Suppose that the desired equality is true for any $m \geqslant k, k \in \mathbb{N}$, we have to:

$$R_{s,t}^{1,k} = iR_{k+1}(s,t).$$

Let us show that it is true for $k + 1$.

$$\begin{aligned} R_{s,t}^{1,k+4} &= pR_{s,t}^{1,k+2} + qR_{s,t}^{1,k+1} \\ &= iR_{k+3}(s,t) + iR_{k+2}(s,t) \\ &= iR_{k+5}(s,t) \end{aligned}$$

Thus proposition (d) is validated.

(e) By the mathematical induction principle and by recurrences: $R_{s,t}^{n+3,m} = R_{s,t}^{n+1,m} + R_{s,t}^{n,m}$ and $R_{s,t}^{n,m+3} = R_{s,t}^{n,m+1} + R_{s,t}^{n,m}$, we can prove the proposition.

For $n = 0$:

$$\begin{aligned} R_{s,t}^{0,m} &= R_0(s,t)P_m(s,t) + iP_0(s,t)R_{m+1}(s,t) \\ &= 3P_m(s,t) + iP_{m+1}(s,t). \end{aligned}$$

For $m = 0$:

$$\begin{aligned} P_{s,t}^{n,0} &= R_n(s,t)P_0(s,t) + iP_n(s,t)R_1(s,t) \\ &= R_n(s,t). \end{aligned}$$

Suppose that the desired equality is true for any $n \geqslant k, m \geqslant k, k \in \mathbb{N}$. We have:

$$R_{s,t}^{k,m} = R_k(s,t)P_m(s,t) + iP_k(s,t)R_{m+1}(s,t)$$

and

$$R_{s,t}^{n,k} = R_n(s,t)P_k(s,t) + iP_n(s,t)R_{k+1}(s,t).$$

Let us show that it is true for $n = k+1$.

$$\begin{aligned} R_{s,t}^{k+4,m} &= pR_{s,t}^{k+2,m} + qR_{s,t}^{k+1,m} \\ &= R_{k+2}(s,t)P_m(s,t) + iP_{k+1}(s,t)R_{m+1}(s,t) \\ &\quad + R_{k+1}(s,t)P_m(s,t) + iP_{k+2}(s,t)R_{m+1}(s,t) \\ &= R_{k+4}(s,t)P_m(s,t) + iP_{k+4}(s,t)R_{m+1}(s,t). \end{aligned}$$

For $m = k+1$:

$$\begin{aligned} R_{s,t}^{n,k+4} &= pR_{s,t}^{n,k+2} + qR_{s,t}^{n,k+1} \\ &= R_n(s,t)P_{k+2}(s,t) + iP_n(s,t)R_{k+3}(s,t) \\ &\quad + R_n(s,t)P_{k+1}(s,t) + iP_n(s,t)R_{k+2}(s,t) \\ &= R_n(s,t)P_{k+4}(s,t) + iP_n(s,t)R_{k+5}(s,t). \end{aligned}$$

Thus proposition (e) is validated. □

3 Three-dimensional recurrences of the (s,t)-Perrin sequence

In this section, we introduce the three-dimensional recurrences of the (s,t)-Perrin sequence based on the one-dimensional recurrence.

Definition 3.1. *For $n, m, p \in \mathbb{N}$ and $s \geqslant 0, t \neq 0, 27t^2 - 4s^3 \neq 0$, the three-dimensional of (s,t)-Perrin sequence $R_{s,t}^{n,m,p}$ is defined by the recurrences:*

$$\begin{aligned} R_{s,t}^{n+3,m,p} &= sR_{s,t}^{n+1,m,p} + tR_{s,t}^{n,m,p}, \\ R_{s,t}^{n,m+3,p} &= sR_{s,t}^{n,m+1,p} + tR_{s,t}^{n,m,p}, \\ R_{s,t}^{n,m,p+3} &= sR_{s,t}^{n,m,p+1} + tR_{s,t}^{n,m,p}, \end{aligned}$$

with the initial values:

$$\begin{array}{lll} R_{s,t}^{0,0,0} = 3, & R_{s,t}^{1,0,0} = 0, & R_{s,t}^{2,0,0} = 2, \\ R_{s,t}^{0,0,1} = 3 + 2j, & R_{s,t}^{1,0,1} = 2j, & R_{s,t}^{2,0,1} = 2 + 2sj, \\ R_{s,t}^{0,0,2} = 3s + 3tj, & R_{s,t}^{1,0,2} = 3tj, & R_{s,t}^{2,0,2} = 2s + 3stj, \\ R_{s,t}^{0,1,0} = 3 + 2i, & R_{s,t}^{1,1,0} = 2i, & R_{s,t}^{2,1,0} = 2 + 2si, \\ R_{s,t}^{0,1,1} = 3 + 2i + 2j, & R_{s,t}^{1,1,1} = 2i + 2j, & R_{s,t}^{2,1,1} = 2 + 2si + 2sj, \\ R_{s,t}^{0,1,2} = 3s + 2si + 3tj, & R_{s,t}^{1,1,2} = 2si + 3tj, & R_{s,t}^{2,1,2} = 2s + 2s^2i + 3stj, \\ R_{s,t}^{0,2,0} = 3s + 3ti, & R_{s,t}^{1,2,0} = 3ti, & R_{s,t}^{2,2,0} = 2s + 3sti, \\ R_{s,t}^{0,2,1} = 3s + 3ti + 2sj, & R_{s,t}^{1,2,1} = 3ti + 2sj, & R_{s,t}^{2,2,1} = 2s + 3sti + 2s^2j, \\ R_{s,t}^{0,2,2} = 3s^2 + 3sti + 3stj, & R_{s,t}^{1,2,2} = 3sti + 3stj, & R_{s,t}^{2,2,2} = 2s^2 + 3s^2ti + 3s^2tj, \end{array}$$

where $i^2 = j^2 = -1$.

Property 3.1. *The following properties are valid:*

$$
\begin{aligned}
\text{(a) } R_{s,t}^{n,0,0} &= R_n(s,t);\\
\text{(b) } R_{s,t}^{n,0,1} &= R_n(s,t) + j2P_n(s,t);\\
\text{(c) } R_{s,t}^{n,1,0} &= R_n(s,t) + i2P_n(s,t);\\
\text{(d) } R_{s,t}^{n,1,1} &= R_n(s,t) + i2P_n(s,t) + j2P_n(s,t);\\
\text{(e) } R_{s,t}^{0,m,0} &= 3P_m(s,t) + iR_{m+1}(s,t);\\
\text{(f) } R_{s,t}^{0,m,1} &= 3P_m(s,t) + iR_{m+1}(s,t) + j2P_m(s,t);\\
\text{(g) } R_{s,t}^{1,m,0} &= iR_{m+1}(s,t);\\
\text{(h) } R_{s,t}^{1,m,1} &= iR_{m+1}(s,t) + j2P_m(s,t);\\
\text{(i) } R_{s,t}^{0,0,p} &= 3P_p(s,t) + jR_{p+1}(s,t);\\
\text{(j) } R_{s,t}^{0,1,p} &= 3P_p(s,t) + i2P_p(s,t) + jR_{p+1}(s,t);\\
\text{(k) } R_{s,t}^{1,0,p} &= jR_{p+1}(s,t);\\
\text{(l) } R_{s,t}^{1,1,p} &= i2P_p(s,t) + jR_{p+1}(s,t);\\
\text{(m) } R_{s,t}^{n,m,p} &= R_n(s,t)P_m(s,t)P_p(s,t) + iP_n(s,t)R_{m+1}(s,t)P_p(s,t)\\
&\quad + jP_n(s,t)P_m(s,t)R_{p+1}(s,t).
\end{aligned}
$$

Proof. The demonstrations are carried out in accordance with the Proposition 2.1. □

4 n-dimensional recurrences of the (s,t)-Perrin sequence

In this section, we introduce the n-dimensional recurrences of the (s,t)-Perrin sequence based on the one-dimensional, two-dimensional and three-dimensional recurrences.

Definition 4.1. *For $n_0, n_1, \ldots, n_{n-1} \in \mathbb{N}$ and $s^2 + 4t > 0$, the n-dimensional of (s,t)-Perrin sequence $R_{s,t}^{n_0,n_1,\ldots,n_{n-1}}$ is defined by the recurrences:*

$$
\begin{aligned}
R_{s,t}^{n_0+3,n_1,\ldots,n_{n-1}} &= sR_{s,t}^{n_0+1,n_1,\ldots,n_{n-1}} + tR_{s,t}^{n_0,n_1,\ldots,n_{n-1}}\\
R_{s,t}^{n_0,n_1+3,\ldots,n_{n-1}} &= sR_{s,t}^{n_0,n_1+1,\ldots,n_{n-1}} + tR_{s,t}^{n_0,n_1,\ldots,n_{n-1}}\\
&\vdots\\
R_{s,t}^{n_0,n_1,\ldots,n_{n-1}+3} &= sR_{s,t}^{n_0,n_1,\ldots,n_{n-1}+1} + tR_{s,t}^{n_0,n_1,\ldots,n_{n-1}},
\end{aligned}
$$

Theorem 4.1. *Numbers of the form $R_{s,t}^{n_0,n_1,\ldots,n_{n-1}}$, such that $n_1, n_2, n_3, \ldots, n_n \in \mathbb{N}$, are determined by:*

$$
\begin{aligned}
R_{s,t}^{(n_1,n_2,n_3,\ldots,n_n)} &= (R_{n_1}(s,t)P_{n_2}(s,t)P_{n_3}(s,t)\ldots P_{n_n}(s,t))\\
&\quad + (P_{n_1}(s,t)R_{n_2+1}(s,t)P_{n_3}(s,t)\ldots P_{n_n}(s,t))\mu_1\\
&\quad + \cdots + (P_{n_1}(s,t)P_{n_2}(s,t)P_{n_3}(s,t)\ldots R_{n_n+1}(s,t))\mu_n.
\end{aligned}
$$

Proof. Starting from the demonstrations carried out in the previous subsections, in which the theorems are valid:

$$\begin{aligned}
R_{s,t}^{n,m} &= R_n(s,t)P_m(s,t) + iP_n(s,t)R_{m+1}(s,t),\\
R_{s,t}^{n,m,p} &= R_n(s,t)P_m(s,t)P_p(s,t) + iP_n(s,t)R_{m+1}(s,t)P_p(s,t)\\
&\quad + jP_n(s,t)P_m(s,t)R_{p+1}(s,t).
\end{aligned}$$

Thus, through the inductive step, it can be verified that:

$$\begin{aligned}
Pe_{s,t}^{n_1,n_2} &= R_{n_1}(s,t)P_{n_2}(s,t) + P_{n_1}(s,t)R_{n_2+1}(s,t)\mu_1\\
Pe_{s,t}^{n_1,n_2,n_3} &= R_{n_1}(s,t)P_{n_2}(s,t)P_{n_3}(s,t)\\
&\quad + P_{n_1}(s,t)R_{n_2+1}(s,t)P_{n_3}(s,t)\mu_1\\
&\quad + P_{n_1}(s,t)P_{n_2}(s,t)R_{n_3+1}(s,t)\mu_2\\
Pe_{s,t}^{n_1,n_2,n_3,n_4} &= R_{n_1}(s,t)P_{n_2}(s,t)P_{n_3}(s,t)P_{n_4}(s,t)\\
&\quad + P_{n_1}(s,t)R_{n_2+1}(s,t)P_{n_3}(s,t)P_{n_4}(s,t)\mu_1\\
&\quad + P_{n_1}(s,t)P_{n_2}(s,t)R_{n_3+1}(s,t)P_{n_4}(s,t)\mu_2\\
&\quad + P_{n_1}(s,t)P_{n_2}(s,t)P_{n_3}(s,t)R_{n_4+1}(s,t)\mu_3\\
&\vdots\\
Pe_{s,t}^{n_1,n_2,n_3,\ldots,n_n} &= (R_{n_1}(s,t)P_{n_2}(s,t)P_{n_3}(s,t)\ldots P_{n_n}(s,t))\\
&\quad + (P_{n_1}(s,t)R_{n_2+1}(s,t)P_{n_3}(s,t)\ldots P_{n_n}(s,t))\mu_1\\
&\quad + (P_{n_1}(s,t)P_{n_2}(s,t)R_{n_3+1}(s,t)\ldots P_{n_n}(s,t))\mu_2\\
&\quad + \cdots + (P_{n_1}(s,t)P_{n_2}(s,t)P_{n_3}(s,t)\ldots R_{n_n+1}(s,t))\mu_n. \qquad \square
\end{aligned}$$

5 Conslusion

From the one-dimensional model of the generalized sequence (s,t)-Perrin, it was possible to obtain the two-dimensional, three-dimensional and n-dimensional relationships of these numbers. With that, it was possible to perceive the insertion of imaginary units in the scope of the numerical sequences, thus taking place the process of complexification of these.

In fact, one can see the existence of two forms of generalization, one in relation to the coefficients of the sequence recurrence formula, called s,t. And the second in relation to the insertions of imaginary units, called n-dimensional relations. For future work, we seek to apply this process of complexification of this generalized sequence in other areas.

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