Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2023, Volume 29, Number 1, 30–39
DOI: 10.7546/nntdm.2023.29.1.30-39

Two generalizations of Liouville λ function

André Pierro de Camargo

Federal University of the ABC region, Brazil
e-mail: andrecamargo.math@gmail.com

Received: 6 September 2022 **Revised:** 15 December 2022
Accepted: 10 February 2023 **Online First:** 13 February 2023

Abstract: We study the properties of two classes of functions λ_k and $\tilde{\lambda}_k$ that generalize the Liouville λ function, including some equivalencies between the Riemann hypothesis and some assertions about the asymptotic behavior of the summatory functions of λ_k and $\tilde{\lambda}_k$. Similar results are obtained for the generalization of the Möbius function considered by Tanaka.
Keywords: Liouville function, Möbius function, Prime Number Theorem, Riemann Hypothesis.
2020 Mathematics Subject Classification: 11N56, 11N99.

1 Introduction

The Möbius μ and the Liouville λ functions are important and closely related arithmetic functions connected with the distribution of the prime numbers

$$\mu(n) = \begin{cases} (-1)^k, & \text{if } n \text{ is the product of } k \text{ distinct prime numbers,} \\ 0, & \text{otherwise,} \end{cases}$$

$$\lambda(n) = (-1)^k, \quad k \text{ is the number of prime factors of } n \text{ counted with multiplicity.}$$

In fact, the Prime Number Theorem and the Riemann Hypothesis are equivalent [5], respectively, to

$$M(x) = o(x) \quad \text{or} \quad L(x) = o(x) \tag{1}$$

and

$$M(x) \;=\; O\left(x^{1/2+\epsilon}\right)\;\forall\,\epsilon > 0 \quad \text{or} \quad L(x) \;=\; O\left(x^{1/2+\epsilon}\right)\;\;\forall\,\epsilon > 0, \tag{2}$$

where M and L are the summatory functions of μ and λ, that is,

$$M(x) \;=\; \sum_{j \le x} \mu(j) \quad \text{and} \quad L(x) \;=\; \sum_{j \le x} \lambda(j), \quad x \ge 1$$

(M is known as the Mertens function in the Literature). The equivalences in (1) and (2) can be obtained, for example, via the classical relations [5]

$$M(x) \;=\; \sum_{j \le \sqrt{x}} \mu(j) L\left(\frac{x}{j^2}\right), \quad L(x) \;=\; \sum_{j \le \sqrt{x}} M\left(\frac{x}{j^2}\right) \tag{3}$$

between M and L.

Generalizations of the Möbius function were previously presented by Apostol and Tanaka: Apostol's generalization μ_k of $\mu = \mu_1$, $k \ge 1$, is defined in [2] by:

$$\mu_k(j) \;=\; \begin{cases} 0, & \text{if } q_{k+1}(j) \;=\; 0, \\ (-1)^r, & \text{if } j \;=\; (p_1 p_2 \dots p_r)^k\, m, \; p_i \text{ prime}; \; q_k(m) \ne 0, \\ 1, & \text{otherwise,} \end{cases}$$

where q_ℓ is the characteristic function of ℓ-free integers:

$$q_\ell(n) \;=\; \begin{cases} 1, & \text{if } n \text{ is } \ell\text{-free}, \\ 0, & \text{otherwise} \end{cases}$$

(a positive integer n is ℓ-free if n is not divisible by the ℓ-th power of any prime number).

Tanaka's generalization $\tilde{\mu}_k$ of $\mu = \tilde{\mu}_2$ is defined in [9] by:

$$\tilde{\mu}_k(j) \;=\; q_k(j)\lambda(j), \; k \ge 2, \; j \ge 1. \tag{4}$$

In our previous paper [3], we rediscovered the curious relation [7]

$$n = \sum_{j=1}^{n} |\mu(j)| \left\lfloor \sqrt{n/j} \right\rfloor, \;\; n \ge 1, \tag{5}$$

which can be obtained by the properties of the Liouville function λ and the fact the λ is the Dirichlet inverse of $|\mu| = q_2$. In an attempt of generalizing (5) for q_k, $k \ge 3$, we crossed with two classes of functions which generalizes the Liouville function. The first generalization $\lambda_k, k \ge 2$, of $\lambda = \lambda_2$ is the Dirichlet inverse of q_k. Explicitly (see Section 3.1):

$$\lambda_k(n) \;=\; \mu(D_{free}(k,n)), \; n \ge 1, \tag{6}$$

where $D_{free}(k,n)$ is the k–free part of n, that is

$$\frac{n}{D_{free}(k,n)} \tag{7}$$

is the largest k-th integral power that divides n.

The definition of λ_k is motivated by the generalizations of the Möbius function mentioned above, which extends the relation between μ and q_2:

$$q_k \;=\; |\mu_{k-1}| \;=\; |\tilde{\mu}_k|,\; k \geq 2. \tag{8}$$

The second generalization of λ is defined by

$$\tilde{\lambda}_k(n) \;=\; \lambda(D_{free}(k,n)),\; n \geq 1, \tag{9}$$

and is obtained by replacing μ by λ in (6). Our interest here in $\tilde{\lambda}_k$ is only for k odd because

$$\tilde{\lambda}_{2\ell} \;=\; \lambda \,\forall\, \ell \geq 1. \tag{10}$$

In this note we present some properties of λ_k and $\tilde{\lambda}_k$. We generalize (3) and obtain analogues of (1) and (2) for $\lambda_k, \tilde{\lambda}_k$ and for Tanakas's generalization $\tilde{\mu}_k$ of $\mu = \tilde{\mu}_2$ and we also give an asymptotic formula for the summatory function of $|\lambda_k|$.

Remark 1. *After we finished the paper, we found that this subject was partially explored in the unpublished paper [4]. However, our results and our methodology are significantly distinct from that of [4], as the reader will conclude by himself/herself.*

2 Preliminaries

Lemma 2.1. *If $\beta = \beta_s$ is a completely multiplicative function (occasionally depending on a complex parameter s) and g is defined by*

$$g(x) \;=\; \sum_{j \leq x} \beta(j),$$

for every $k \geq 2$ and $x \geq 1$,

$$\sum_{j \leq x} q_k(j)\beta(j) \;=\; \sum_{j \leq \sqrt[k]{x}} \mu(j)\beta(j)^k g\left(\frac{x}{j^k}\right). \tag{11}$$

Proof. For $j \leq \sqrt[k]{x}$, we have

$$\sum_{i \leq \frac{x}{j^k}} \beta\left(j^k i\right) \;=\; \beta\left(j^k\right) g\left(\frac{x}{j^k}\right).$$

Hence,

$$\begin{aligned}
\sum_{j \leq \sqrt[k]{x}} \mu(j)\beta\left(j^k\right) g\left(\tfrac{x}{j^k}\right) \;&=\; \sum_{j \leq \sqrt[k]{x}} \mu(j) \sum_{i \leq \frac{x}{j^k}} \beta\left(j^k i\right) \\
&=\; \sum_{n \leq x} \sum_{ij^k \,=\, n} \mu(j)\beta\left(j^k i\right). \\
&=\; \sum_{n \leq x} \beta(n) \sum_{j^k | n} \mu(j).
\end{aligned}$$

The proof is completed by using that [8]

$$\sum_{j^k | n} \mu(j) \;=\; q_k(n).$$

□

We will also use the following generalization of Theorem 2.22 of [1].

Lemma 2.2. *For $k \geq 1$, if α has a Dirichlet inverse α^{-1} and f and g are arithmetic functions related by*

$$g(x) \;=\; \sum_{j \leq \sqrt[k]{x}} \alpha(j) f\left(x/j^k\right), x \geq 1,$$

then

$$f(x) \;=\; \sum_{j \leq \sqrt[k]{x}} \alpha^{-1}(j) g\left(x/j^k\right), x \geq 1.$$

Proof. In fact,

$$\sum_{j \leq \sqrt[k]{x}} \alpha^{-1}(j) g\left(x/j^k\right) = \sum_{j \leq \sqrt[k]{x}} \alpha^{-1}(j) \left(\sum_{i \leq \sqrt[k]{\frac{x}{j^k}}} \alpha(i) f\left([x/j^k]/i^k\right) \right)$$

$$= \sum_{n \leq \sqrt[k]{x}} \sum_{i\,j = n} \alpha^{-1}(j)\alpha(i) f\left(x/n^k\right) \;=\; f(x). \qquad \square$$

Theorem 2.3. *Let g be an arithmetic function. For $k \geq 2$, let $\tilde{g}$ and $\tilde{G}$ be defined by*

$$\tilde{g}(n) \;=\; g(D_{free}(k, n)),\; n \geq 1, \quad \tilde{G}(x) \;=\; \sum_{j \leq x} \tilde{g}(j),\; x \geq 1,$$

with D_{free} defined by (7)*. For $x \geq 1$,*

$$\sum_{j \leq x} q_k(j) g(j) \;=\; \sum_{j \leq \sqrt[k]{x}} \mu(j) \tilde{G}\left(\frac{x}{j^k}\right). \tag{12}$$

Proof. For $j \leq \sqrt[k]{x}$, we have

$$\sum_{i \leq \frac{x}{j^k}} \tilde{g}\left(j^k i\right) \;=\; \sum_{i \leq \frac{x}{j^k}} \tilde{g}(i) \;=\; \tilde{G}\left(\frac{x}{j^k}\right). \tag{13}$$

Hence,

$$\begin{aligned} \sum_{j \leq \sqrt[k]{x}} \mu(j) \tilde{G}\left(\tfrac{x}{j^k}\right) \;&=\; \sum_{j \leq \sqrt[k]{x}} \mu(j) \left(\sum_{i \leq \frac{x}{j^k}} \tilde{g}\left(j^k i\right) \right) \\ &=\; \sum_{n \leq x} \sum_{ij^k \,=\, n} \mu(j) \tilde{g}\left(j^k i\right). \end{aligned} \tag{14}$$

Write each $n \leq x$ as $n = u^k v$, with v k-free. The summatory above extends over all divisors j of u and i is uniquely determined by $i = (u/j)^k v$, that is

$$\begin{aligned} \sum_{ij^k \,=\, n} \mu(j) \tilde{g}\left(j^k i\right) \;&=\; \sum_{j|u} \mu(j) \tilde{g}\left(j^k (u/j)^k v\right) \;=\; \sum_{j|u} \mu(j) \tilde{g}\left(v\right) \\ &\overset{(7)}{=} \begin{cases} \tilde{g}(n), & u = 1, \\ 0, & u > 1. \end{cases} \end{aligned} \tag{15}$$

Recalling that $\tilde{g} = g$ over k-free integers, by (14) and (15),

$$\sum_{j \leq \sqrt[k]{x}} \mu(j) \tilde{G}\left(\tfrac{x}{j^k}\right) \;=\; \sum_{n \leq x} q_k(n) g(n). \tag{16}$$

This completes the proof. $\square$

3 Some properties of λ_k and $\tilde{\lambda}_k$

As we can see by (6) and (9), λ_k and $\tilde{\lambda}_k$ are multiplicative for $k \geq 3$, but are completely multiplicative only for $k = 2$.

The following well known properties of λ, [7] and [1], p. 38, are extended to λ_k.

Lemma 3.1. *For $k \geq 2$ and $x \geq 1$, we have*

$$\sum_{d|n} \lambda_k(d) = \begin{cases} 1, & \text{if } n = m^k \text{ for some } m \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases} \tag{17}$$

$$\sum_{j \leq x} \lambda_k(j) \left\lfloor \frac{x}{j} \right\rfloor = \lfloor \sqrt[k]{x} \rfloor. \tag{18}$$

Proof. Write $n = u^k v$ with v k–free. Note that, for each divisor d of n, there are unique u', v', b such that $u = bu'$

$$d = b^k v', \quad v' | (u')^k v \quad \text{and} \quad v' \text{ is } k-\text{free}.$$

Hence,

$$\sum_{d|n} \lambda_k(d) = \sum_{u'|u} \sum_{v'|(u')^k v} q_k(v')\lambda_k(v') \overset{(6)}{=} \sum_{u'|u} \sum_{v'|(u')^k v} \mu(v')$$
$$= \begin{cases} 1, & v = 1 \\ 0, & v > 1. \end{cases}$$

This proves (17).

Relation (18) is obtained by summing up in n:

$$\sum_{j \leq n+1} \lambda_k(j) \lfloor (n+1)/j \rfloor - \sum_{j \leq n} \lambda_k(j) \lfloor n/j \rfloor$$
$$= \sum_{j \leq n+1} \lambda_k(j) \left(\lfloor (n+1)/j \rfloor - \lfloor n/j \rfloor \right)$$
$$= \sum_{d \mid n+1} \lambda_k(d) \overset{(17)}{=} \begin{cases} 1, & \text{if } (n+1) = m^k \text{ for some } m \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

□

Recalling that λ_k is the Dirichlet inverse of q_k, we obtain the following generalization of (5) by (18) and Lemma 2.2:

Corollary 1. *For $n \geq 1$ and $k \geq 2$,*

$$n = \sum_{j=1}^{n} q_k(j) \left\lfloor \sqrt[k]{n/j} \right\rfloor. \tag{19}$$

The Dirichlet convolution $1 * \tilde{\lambda}_k$ has not a much simpler form (see next section). Consequently, there are no simple analogues of (17), (18) and (19) for $\tilde{\lambda}_k$. However, note that

$$f(j) \in \{-1, 1\} \; \forall j \geq 1 \tag{20}$$

holds for $f = \tilde{\lambda}_k$, but not for $f = \lambda_k$ for $k \geq 3$. In addition, recall that

$$\mu(j) = \lambda(j)$$

whenever $\mu(j) \neq 0$. This relation extends to $\tilde{\lambda}_k$ and Tanaka's generalization $\tilde{\mu}_k$ (4) of the Möbius function:

$$\tilde{\mu}_k(j) = \tilde{\lambda}_k(j) \tag{21}$$

whenever $\tilde{\mu}_k(j) \neq 0$. Nevertheless, (21) holds for λ_k in the place of $\tilde{\lambda}_k$ only when $\tilde{\mu}_k$ and λ_k are both non-vanishing.

3.1 Dirichlet series associated to λ_k and $\tilde{\lambda}_k$

The Dirichlet series associated to q_k is [8]

$$\sum_{j=1}^{\infty} \frac{q_k(j)}{j^s} \;=\; \frac{\zeta(s)}{\zeta(ks)},\ \Re e(s) > 1$$

(ζ is the Riemann zeta function). Because λ_k is the Dirichlet inverse of q_k, we have

$$\sum_{j=1}^{\infty} \frac{\lambda_k(j)}{j^s} \;=\; \frac{\zeta(ks)}{\zeta(s)}. \tag{22}$$

Consequently,

$$\lambda_k(n) \;=\; \sum_{j^k i \,=\, n} \mu(i) \;=\; \sum_{j^k | n} \mu\left(\frac{n}{j^k}\right).$$

Clearly, $\mu\left(\frac{n}{j^k}\right) \;=\; 0$ if $\frac{n}{j^k}$ is not k-free and we obtain (6). Once we now that $|\lambda_k(j)| \leq 1 \ \forall j \geq 1$, we have that the series in the left-hand side of (22) converges absolutely for $\Re e(s) > 1$.

The Dirichlet series associated to $\tilde{\lambda}_k$, k odd, can be obtained as follows. First, note that, taking the limit $x \to \infty$ in (11), for $\beta_s(j) := \frac{\lambda(j)}{j^s}$ and $\Re e(s) > 1$,

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{q_k(j)\lambda(j)}{j^s} \quad &= \quad \left(\sum_{j=1}^{\infty} \mu(j)\frac{\lambda(j)^k}{j^{ks}}\right)\left(\sum_{j=1}^{\infty} \frac{\lambda(j)}{j^s}\right) \\
&\overset{k \text{ is odd}}{=} \quad \left(\sum_{j=1}^{\infty} \frac{|\mu(j)|}{j^{ks}}\right)\left(\sum_{j=1}^{\infty} \frac{\lambda(j)}{j^s}\right) \\
&= \quad \frac{\zeta(ks)}{\zeta(2ks)}\frac{\zeta(2s)}{\zeta(s)}
\end{aligned} \tag{23}$$

(relation (23) was reported in [9]. In fact, the left-hand side of (23) is the Dirichlet series associated to $\tilde{\mu}_k$).

By (23) and the fact that $\mathbb{N} \;=\; \bigcup_{r \geq 1} V_r$ is the disjoint union of the sets

$$V_r \;=\; \{r^k j : j \;\in\; \mathbb{N},\ q_k(j) \neq 0\},\ r \geq 1,$$

we get

$$
\begin{aligned}
\sum_{j=1}^{\infty} \frac{\tilde{\lambda}_k(j)}{j^s} &= \sum_{r\geq 1} \sum_{n\,\in\,V_r} \frac{\tilde{\lambda}_k(n)}{n^s} \\
&= \sum_{r\geq 1} \sum_{j=1}^{\infty} \frac{q_k(j)\lambda(j)}{(r^k j)^s} \\
&= \frac{\zeta(ks)^2}{\zeta(2ks)} \frac{\zeta(2s)}{\zeta(s)}.
\end{aligned} \tag{24}
$$

Therefore, we have

Lemma 3.2. *For* $\Re e(s) > 1$*,*

$$
\sum_{j=1}^{\infty} \frac{\tilde{\lambda}_k(j)}{j^s} \overset{(10)}{=} \begin{cases} \frac{\zeta(ks)^2}{\zeta(2ks)} \frac{\zeta(2s)}{\zeta(s)}, & k \text{ odd}, \\ \frac{\zeta(2s)}{\zeta(s)}, & k \text{ even}. \end{cases} \tag{25}
$$

4 On the summatory functions of λ_k, $\tilde{\lambda}_k$ and $\tilde{\mu}_k$

For $k \geq 2$, let L_k and $\tilde{L}_k$ be the summatory functions of λ_k and $\tilde{\lambda}_k$, that is,

$$
L_k(x) = \sum_{j\leq x} \lambda_k(j), \quad \tilde{L}_k(x) = \sum_{j\leq x} \tilde{\lambda}_k(j) \quad x \geq 1.
$$

The explicit expressions for the Dirichlet series associated to λ_k and $\tilde{\lambda}_k$ obtained in the previous section can be used to analyze the asymptotic behavior of L_k and $\tilde{L}_k$ via standard analytical methods. Below we present an alternative analysis which explores the relations between L_k and $\tilde{L}_k$ and M and L.

We generalize (3) for L_k as follows.

Corollary 2. *For $k \geq 2$ and $x \geq 1$,*

$$
M(x) = \sum_{j\leq \sqrt[k]{x}} \mu(j) L_k\left(\frac{x}{j^k}\right), \quad L_k(x) = \sum_{j\leq \sqrt[k]{x}} M\left(\frac{x}{j^k}\right). \tag{26}
$$

Proof. The first relation in (26) is obtained by Theorem 2.3 for $g = \mu$. The second relation in (26) is obtained by the previous one and Lemma 2.2. □

Put

$$
\tilde{M}_k(x) = \sum_{j\leq x} \tilde{\mu}_k(j), \quad x \geq 1,
$$

with $\tilde{\mu}_k$ defined by (4). Theorem 2.3 and Lemma 2.2 give the next Corollary 3.

Corollary 3. *For $k \geq 1$ and $x \geq 1$,*

$$
\tilde{M}_k(x) = \sum_{j\leq \sqrt[k]{x}} \mu(j) \tilde{L}_k\left(\frac{x}{j^k}\right), \quad \tilde{L}_k(x) = \sum_{j\leq \sqrt[k]{x}} \tilde{M}_k\left(\frac{x}{j^k}\right). \tag{27}
$$

In addition, Lemma 2.1 and Lemma 2.2 yields the following Corollary 4.

Corollary 4. *For $k \geq 2$ and $x \geq 1$,*

$$\tilde{M}_k(x) = \sum_{j \leq \sqrt[k]{x}} \mu(j)^{k+1} L\left(\frac{x}{j^k}\right), \quad L(x) = \sum_{j \leq \sqrt[k]{x}} \tilde{M}_k\left(\frac{x}{j^k}\right), \quad (k \textit{ even}) \tag{28}$$

and

$$L(x) = \sum_{j \leq \sqrt[k]{x}} \lambda(j) \tilde{M}_k\left(\frac{x}{j^k}\right), \quad (k \textit{ odd}). \tag{29}$$

Using Corollaries 2, 3 and 4, one can readily extend the equivalencies in (1) and (2) to $L_k, \tilde{L}_k$ and $\tilde{M}_k$:

Corollary 5. *For $k \geq 2, k' \geq 2$ and $k'' \geq 2$, the following are equivalent:*

$$L_k(x) = o(x), \quad \tilde{L}_{k'}(x) = o(x), \quad \tilde{M}_{k''}(x) = o(x). \tag{30}$$

Corollary 6. *For $k \geq 2, k' \geq 2$ and $k'' \geq 2$, the following are equivalent:*

$$\begin{aligned} L_k(x) &= O\left(x^{1/2+\epsilon}\right) \ \forall\, \epsilon > 0, \quad \tilde{L}_{k'}(x) = O\left(x^{1/2+\epsilon}\right) \ \forall\, \epsilon > 0, \\ \tilde{M}_{k''}(x) &= O\left(x^{1/2+\epsilon}\right) \ \forall\, \epsilon > 0. \end{aligned} \tag{31}$$

Remark 2. *Tanaka remarked that $\tilde{M}_k(x) = O\left(x^{1/2+\epsilon}\right) \ \forall\, \epsilon > 0$ under the Riemann hypothesis [9].*

5 On the summatory functions of $|\lambda_k|$ and $|\tilde{\lambda}_k|$

Let $|L|_k$ and $|\tilde{L}|_k$ be defined by

$$|L|_k(x) = \sum_{j \leq x} |\lambda_k(j)| \quad \text{and} \quad |\tilde{L}|_k(x) = \sum_{j \leq x} |\tilde{\lambda}_k(j)|, \quad x \geq 1.$$

By (20), we get

$$|\tilde{L}|_k(x) = \lfloor x \rfloor, \ k \geq 1, \ x \geq 1. \tag{32}$$

For $|L|_k$, we have the following theorem.

Theorem 5.1. *For $k \geq 2$,*

$$|L|_k(x) = \frac{\zeta(k)}{\zeta(2)} x + O\left(\sqrt{x}\right). \tag{33}$$

Proof. We only need to prove the case $k \geq 3$. Applying Theorem 2.3 for $g = |\mu|$, we get

$$Q(x) = \sum_{j \leq \sqrt[k]{x}} \mu(j) |L|_k\left(\frac{x}{j^k}\right), \quad Q(x) := \sum_{j \leq x} |\mu(j)| \tag{34}$$

(the case $k = 1$ of (34) is well-known [6]: $Q(x) = \sum_{j \leq \sqrt{x}} \mu(j) \left\lfloor \frac{x}{j^2} \right\rfloor$). Hence, by Lemma 2.2,

$$|L|_k(x) = \sum_{j \leq \sqrt[k]{x}} Q\left(\frac{x}{j^k}\right), \quad x \geq 1.$$

Using that [6]

$$Q(x) \;=\; \frac{1}{\zeta(2)}x + O(\sqrt{x}),$$

and recalling that $k \geq 3$, we obtain

$$|L|_k(x) \;=\; \frac{1}{\zeta(2)}\left(\sum_{j\leq \sqrt[k]{x}} \frac{1}{j^k}\right) x \;+\; O(\sqrt{x}) \;=\; \frac{\zeta(k)}{\zeta(2)}x \;+\; O(\sqrt{x}). \qquad \square$$

For $k \geq 2$, let $\lambda_k^+, \tilde{\lambda}_k^+, \lambda_k^-$ and $\tilde{\lambda}_k^-$ be defined by

$$\lambda_k^{\pm}(j) \;=\frac{|\lambda_k(j)| \pm \lambda_k(j)}{2}, \quad \tilde{\lambda}_k^{\pm}(j) \;= \frac{|\tilde{\lambda}_k(j)| \pm \tilde{\lambda}_k(j)}{2},\; j \geq 1.$$

The functions $\lambda^+ \;:=\; \lambda_2^+ = \tilde{\lambda}_2^+$ and $\lambda^- \;:=\; \lambda_2^- \;=\; \tilde{\lambda}_2^-$ are the characteristic functions of the sets of the numbers that have an even and an odd number of prime factors, respectively. The version $L(x) = o(x)$ of the Prime Number Theorem states that these two sets of integers have asymptotically the same density:

$$\sum_{j\leq x} \lambda^+(j) \sim \frac{1}{2}x \;\text{ and }\; \sum_{j\leq x} \lambda^-(j) \sim \frac{1}{2}x. \qquad (35)$$

Using (32) and (33) and the versions $L_k(x) \;=\; o(x)$ and $\tilde{L}_k(x) \;=\; o(x)$ of the Prime Number Theorem given in Corollary 5, we can generalize (35).

Corollary 7. *For* $k \geq 2$*,*

$$\sum_{j\leq x} \lambda_k^{\pm}(j) \sim \frac{\zeta(k)}{2\zeta(2)}x \;\textit{ and }\; \sum_{j\leq x} \tilde{\lambda}_k^{\pm}(j) \sim \frac{1}{2}x. \qquad (36)$$

6 Summary

The next Table 1 summarizes the properties of the Liouville function which we extended to λ_k or / and $\tilde{\lambda}_k$. It shows that λ_k and $\tilde{\lambda}_k$ inherit different properties of λ, but also have several similar properties. They also have different interactions with Tanakas's generalization of the Möbius function: (3) can be extended to $\tilde{\lambda}_k$ and $\tilde{\mu}_k$ with the same index k, while

$$|\tilde{\mu}_k| \overset{(8)}{=} \lambda_k^{-1}, k \geq 2.$$

Hence, we may say that there is no favorite among λ_k and $\tilde{\lambda}_k$.

Just to finish, let us mention that the generalization $\mu_k, k \geq 1$, of the Möbius function cited at the introductory section was introduced by Apostol [2] ten years earlier than Tanaka's work, but μ_k has a distinct behavior than $\mu_1 = \mu$ in general. For instance, the summatory function of μ_k satisfies

$$\sum_{j\leq x} \mu_k(j) \;=\; o(x)$$

only for $k = 1$.

Table 1. Some properties of λ and their extensions to λ_k and $\tilde{\lambda}_k$.

holds for $\boldsymbol{\lambda}$	**has a simple analogue for λ_k**	**has a simple analogue for $\tilde{\lambda}_k$**
$\sum_{d\mid n} \lambda(d) = \begin{cases} 1, & \text{if } n = m^2 \text{ for some } m \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$	✓(see (17))	
$\sum_{j \leq x} \lambda(j) \left\lfloor \frac{x}{j} \right\rfloor = \lfloor \sqrt{x} \rfloor$	✓(see (18))	
$n = \sum_{j=1}^{n} \lvert\mu(j)\rvert \left\lfloor \sqrt{n/j} \right\rfloor, \ n \geq 1$	✓(see (19))	
$\lambda(j) \in \{-1, 1\} \ \forall j \geq 1$		✓(see (20))
$\mu(j) = \lambda(j)$ for $\mu(j) \neq 0$		✓(see (21))
$M(x) = \sum_{j \leq \sqrt{x}} \mu(j) L\left(\frac{x}{j^2}\right), \quad L(x) = \sum_{j \leq \sqrt{x}} M\left(\frac{x}{j^2}\right)$	✓(see (26))	✓(see (27))
$L(x) = o(x)$	✓(see (30))	✓(see (30))
$L(x) = O\left(x^{1/2+\epsilon}\right) \ \forall\, \epsilon > 0$ is equivalent to the Riemann Hypothesis	✓(see (31))	✓(see (31))
$\sum_{j \leq x} \lambda^{\pm}(j) \sim \frac{1}{2} x$	✓(see (36))	✓(see (36))

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