Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2023, Volume 29, Number 1, 17–23
DOI: 10.7546/nntdm.2023.29.1.17-23

On distribution of the number of semisimple rings of order at most x in an arithmetic progression

Thorranin Thansri[1], Teerapat Srichan[2,*] and Pinthira Tangsupphathawat[3]

[1] Department of Mathematics, Faculty of Science
Kasetsart University, Bangkok 10900, Thailand
e-mail: fscitnt@ku.ac.th

[2] Department of Mathematics, Faculty of Science
Kasetsart University, Bangkok 10900, Thailand
e-mail: fscitrp@ku.ac.th

[3] Department of Mathematics, Faculty of Science and Technology
Phranakorn Rajabhat University, Bangkok 10220, Thailand
e-mail: t.pinthira@hotmail.com

* *Corresponding author*

Received: 20 September 2022 **Revised:** 26 November 2022
Accepted: 31 January 2023 **Online First:** 6 February 2023

Abstract: Let ℓ and q denote relatively prime positive integers. In this article, we derive the asymptotic formula for the summation

$$\sum_{\substack{n\leq x\\ n\equiv \ell \pmod q}} S(n),$$

where $S(n)$ denotes the number of non-isomorphic finite semisimple rings with n elements.
Keywords: Abelian group, Arithmetical progression, Asymptotic mean value, Counting function, Semisimple group.
2020 Mathematics Subject Classification: 11N45, 11N37.

1 Introduction and result

As usual, let ϕ be Euler's totient function, ζ the Riemann zeta-function and $L(s,\chi)$ Dirichlet L-function associated with χ the Dirichlet character modulo q. Let $a(n)$ denotes the number of non-isomorphic abelian groups of order n and $A(x)$ denote the number of distinct abelian groups of order $\leq x$. The study of the estimating the asymptotic formula for $A(x)$ has been initiated by Erdős and Szekeres [5]. Various authors contributed to the subject, see in [7, 9–11, 13–16]. Another arithmetical function which shares some properties of $a(n)$ is $S(n)$ which denotes the number of non-isomorphic finite semisimple rings with n elements. The problem of estimating the asymptotic result about $\sum_{n\leq x} S(n)$ has been studied by Knopfmacher in [8] and Duttlinger in [4].

It is well-known fact that each semisimple finite ring can be expressed as a direct sum of a finite number of simple finite rings, in a way that is unique up to permutation. The Dirichlet series associated with $S(n)$ may be represented as

$$\sum_{n=1}^{\infty} \frac{S(n)}{n^s} = \prod_{r=1}^{\infty}\prod_{m=1}^{\infty} \zeta(rm^2 s), \ \ (\Re(s) > 1).$$

Note that,

$$\sum_{n=1}^{\infty} \frac{S(n)}{n^s} = \prod_{r=1}^{\infty} \zeta(rs) \prod_{m\geq 2}^{\infty} \zeta(rm^2 s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \prod_{m\geq 2}^{\infty} \zeta(rm^2 s), \tag{1}$$

where the last infinite product is convergent in $\Re(s) > 1/4$. Then, one can obtain an asymptotic result about $\sum_{n\leq x} S(n)$ from $A(x)$. Calderón and Zárate [2] used the expression in (1) to derive an asymptotic formula for $\sum_{n\leq x} S(n)$ which improves the previously result in [4, 8] and its O-term has the same order of $A(x)$ in [9]. Calderón and Zárate proved that, for $x > 1$,

$$\sum_{n\leq x} S(n) = A_1 x + A_2 x^{1/2} + A_3 x^{1/3} + O(x^{97/381} (\log x)^{35}),$$

where

$$A_j = \prod_{\substack{v=1\\ v\neq j}}^{\infty} \prod_{r=1}^{\infty} \prod_{m=2}^{\infty} \zeta\Big(\frac{v}{j}\Big) \zeta\Big(\frac{rm^2}{j}\Big), \ j = 1, 2, 3.$$

The O-term was improved later by Calderón [1] to $O(x^{55/219} (\log x)^7)$.

In other direction, the distribution of the arithmetical function $a(n)$ over an arithmetical progression is also studied in 1953 by Richert [12]. Analogously it natural to ask whether the bound given by Richert for $\sum_{\substack{n\leq x\\ n\equiv \ell \pmod q}} a(n)$ function is also satisfied for $\sum_{\substack{n\leq x\\ n\equiv \ell \pmod q}} S(n)$.

In this article we shall study on the asymptotic mean values over arithmetical progressions of $S(n)$ by using the idea of Calderón and Zárate in [2]. We prove the following theorem:

Theorem 1.1. *Let ℓ and q denote relatively prime positive integers. Then*

$$\sum_{\substack{n\le x\\ n\equiv \ell\ (\bmod q)}} S(n) = g_1(\chi_0)\frac{x}{q} + (g_2(\chi_0)+h_1(\ell))\frac{x^{1/2}}{q}$$
$$+ (g_3(\chi_0)+h_2(\ell))\frac{x^{1/3}}{q} + O(q^{8/5}x^{3/10}(\log x)^{9/10}),$$

where χ_0 denotes the principal character modulo q, and the term in χ_1, χ_2 occur if and only if there exist characters $\chi_1\neq\chi_0$, $\chi_2\neq\chi_0$, modulo q such that $\chi_1^2=\chi_0$, $\chi_2^3=\chi_0$, and

$$g_\mu(\chi) = \prod_{\substack{v=1\\ v\neq\mu}}^{\infty}\prod_{r=1}^{\infty}\prod_{m=2}^{\infty} L\Big(\frac{v}{\mu},\chi^v\Big)L\Big(\frac{rm^2}{\mu},\chi^{rm^2}\Big),\quad \mu=1,2,3,$$

$$h_1(\ell) = \begin{cases}\chi_1(\ell)g_2(\chi_1), & \text{for } q\ge 3,\\ 0, & \text{for } q=2,\end{cases}$$

$$h_2(\ell) = \begin{cases}2\Re\big(\overline{\chi}_2(\ell)g_3(\chi_2)\big), & \text{for } 3\mid\phi(q),\\ 0, & \text{otherwise}.\end{cases}$$

Remark 1.2. *Since the rate of convergence of $L(s,\chi)$ to 1 as $\Re(s)\to\infty$, (see examples pp. 226 in [6]), all infinite products in this present paper converge. From proposition 1. in [3], our infinite products always converge uniformly in ever closed half-plane $\Re(s)>0$ when all terms are regular.*

Remark 1.3. *Since we use the properties of Dirichlet Characters, the restrition on the coprime of ℓ and q is necessary condition.*

To prove our result we will give the following lemma.

Lemma 1.1. *Let χ be any character modulo q. Let $S_4^*(n)$ be the arithmetical function such that for $\Re(s)>1/4$*

$$\sum_{n=1}^{\infty}\frac{S_4^*(n)}{n^s} = \prod_{r=1}^{\infty}\prod_{m=2}^{\infty}\zeta(rm^2 s). \tag{2}$$

Then, for $\sigma>1/4$, we have

$$\sum_{n\le x}\frac{S_4^*(n)\chi(n)}{n^\sigma} = \sum_{n=1}^{\infty}\frac{S_4^*(n)\chi(n)}{n^\sigma} + O(x^{1/4-\sigma}). \tag{3}$$

Proof. Given any character χ modulo q and $\Re(s)>1/4$, we write (2) as

$$\sum_{n=1}^{\infty}\frac{S_4^*(n)\chi(n)}{n^s} = \prod_{r=1}^{\infty}L(4rs,\chi^{4r})\prod_{m=3}^{\infty}L(rm^2 s,\chi^{rm^2})$$
$$= \Big(\sum_{n=1}^{\infty}\frac{a(n)\chi(n)}{n^{4s}}\Big)\Big(\sum_{m=1}^{\infty}\frac{S_9^*(m)\chi(m)}{m^s}\Big),$$

where $a(n)$ is the number of non-isomorphic abelian groups with n elements and the second series is convergent for $\Re(s)>1/9$. Thus, we have

$$
\begin{aligned}
\sum_{n\le x} S_4^*(n)\chi(n) &= \sum_{n^4 m\le x} a(n)\chi(n)S_9^*(m)\chi(m)\\
&= \sum_{m\le x} S_9^*(m)\chi(m) \sum_{n\le x^{1/4}m^{-1/4}} a(n)\chi(n)\\
&= O\Big(x^{1/4}\sum_{m\le x}\frac{S_9^*(m)}{m^{1/4}}\Big)\\
&= O(x^{1/4}).
\end{aligned}
$$

Now, for $\Re(s)=\sigma>1/4$,

$$
\sum_{n>x}\frac{S_4^*(n)\chi(n)}{n^\sigma} = O\Big(\int_x^\infty \frac{dt}{t^{\sigma+1-1/4}}\Big) = O(x^{1/4-\sigma}).
$$

Then, (3) follows. □

2 Proof

Proof of Theorem 1.1. Given any character χ modulo q, the Dirichlet series associated with $S(n)$ are

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{S(n)\chi(n)}{n^s} &= \prod_{r=1}^{\infty}\prod_{m=1}^{\infty} L(rm^2 s,\chi^{rm^2}),\ \ (\Re(s)>1)\\
&= \prod_{r=1}^{\infty} L(rs,\chi^r)\prod_{m\ge 2}^{\infty} L(rm^2 s,\chi^{rm^2}) = \Big(\sum_{n=1}^{\infty}\frac{a(n)\chi(n)}{n^s}\Big)\Big(\sum_{m=1}^{\infty}\frac{S_4^*(m)\chi(m)}{m^s}\Big),
\end{aligned}
$$

where the last series is convergent in $\Re(s)>1/4$. Thus, we have

$$
\sum_{n\le x} S(n)\chi(n) = \sum_{nm\le x} a(n)\chi(n)S_4^*(m)\chi(m) = \sum_{m\le x} S_4^*(m)\chi(m)\sum_{n\le x/m} a(n)\chi(n). \tag{4}
$$

Let χ_0 is the principal character modulo q. Let χ_1,χ_2 denote the non-principal character that having order 2 and 3, with respectively. Now, by equation (72-75) of Richert [12], we have

$$
\begin{aligned}
\sum_{n\le x} a(n)\chi_0(n) &= \frac{\phi(q)}{q}c_1(\chi_0)x + \frac{\phi(q)}{q}c_2(\chi_0)x^{1/2}\\
&\quad + \frac{\phi(q)}{q}c_3(\chi_0)x^{1/3} + O(\phi(q)q^{8/5}x^{3/10}(\log x)^{9/10}),
\end{aligned} \tag{5}
$$

$$
\sum_{n\le x} a(n)\chi_1(n) = \frac{\phi(q)}{q}c_2(\chi_1)x^{1/2} + O(\phi(q)q^{4/3}x^{3/10}(\log x)^{9/10}), \tag{6}
$$

$$
\sum_{n\le x} a(n)\chi_2(n) = \frac{\phi(q)}{q}c_3(\chi_2)x^{1/3} + O(\phi(q)q^{3/5}x^{3/10}(\log x)^{9/10}), \tag{7}
$$

and

$$
\sum_{n\le x} a(n)\chi(n) = O(\phi(q)q^{8/5}x^{3/10}(\log x)^{9/10}),\ \text{if } \chi^2\ne\chi_0,\ \chi^3\ne\chi_0. \tag{8}
$$

In view of (4) and (5), we have

$$\sum_{n\le x} S(n)\chi_0(n) = \sum_{m\le x} S_4^*(m)\chi_0(m)\Big(\frac{\phi(q)}{q}c_1(\chi_0)(\frac{x}{m}) + \frac{\phi(q)}{q}c_2(\chi_0)(\frac{x}{m})^{1/2} + \frac{\phi(q)}{q}c_3(\chi_0)(\frac{x}{m})^{1/3} + O(\phi(q)q^{8/5}(\frac{x}{m})^{3/10}(\log(\frac{x}{m}))^{9/10})\Big)$$

$$= \frac{\phi(q)}{q}c_1(\chi_0)x\sum_{m\le x}\frac{S_4^*(m)\chi_0(m)}{m} + \frac{\phi(q)}{q}c_2(\chi_0)x^{1/2}\sum_{m\le x}\frac{S_4^*(m)\chi_0(m)}{m^{1/2}} + \frac{\phi(q)}{q}c_3(\chi_0)x^{1/3}\sum_{m\le x}\frac{S_4^*(m)\chi_0(m)}{m^{1/3}} + O(\phi(q)q^{8/5}x^{3/10}(\log x)^{9/10})\sum_{m\le x}\frac{S_4^*(m)}{m^{3/10}}.$$

From Lemma 1.1, we have

$$\sum_{n\le x} S(n)\chi_0(n) = \frac{\phi(q)}{q}c_1(\chi_0)x\sum_{m=1}^{\infty}\frac{S_4^*(m)\chi_0(m)}{m} + \frac{\phi(q)}{q}c_2(\chi_0)x^{1/2}\sum_{m=1}^{\infty}\frac{S_4^*(m)\chi_0(m)}{m^{1/2}} + \frac{\phi(q)}{q}c_3(\chi_0)x^{1/3}\sum_{m=1}^{\infty}\frac{S_4^*(m)\chi_0(m)}{m^{1/3}} + O(\phi(q)q^{8/5}x^{3/10}(\log x)^{9/10}).$$

Similarly, from (4)-(8) and Lemma 1.1, we have

$$\sum_{n\le x} S(n)\chi_1(n) = \sum_{m\le x} S_4^*(m)\chi_1(m)\Big(\frac{\phi(q)}{q}c_2(\chi_1)(\frac{x}{m})^{1/2} + O(\phi(q)q^{4/3}(\frac{x}{m})^{3/10}(\log(\frac{x}{m}))^{9/10})\Big)$$

$$= \frac{\phi(q)}{q}c_2(\chi_1)x^{1/2}\sum_{m\le x}\frac{S_4^*(m)\chi_1(m)}{m^{1/2}} + O(\phi(q)q^{4/3}x^{3/10}(\log x)^{9/10})\sum_{m\le x}\frac{S_4^*(m)}{m^{3/10}}$$

$$= \frac{\phi(q)}{q}c_2(\chi_1)x^{1/2}\sum_{m=1}^{\infty}\frac{S_4^*(m)\chi_1(m)}{m^{1/2}} + O(\phi(q)q^{4/3}x^{3/10}(\log x)^{9/10}),$$

$$\sum_{n\le x} S(n)\chi_2(n) = \sum_{m\le x} S_4^*(m)\chi_2(m)\Big(\frac{\phi(q)}{q}c_3(\chi_2)(\frac{x}{m})^{1/3} + O(\phi(q)q^{3/5}(\frac{x}{m})^{3/10}(\log(\frac{x}{m}))^{9/10})\Big)$$

$$= \frac{\phi(q)}{q}c_3(\chi_2)x^{1/3}\sum_{m\le x}\frac{S_4^*(m)\chi_2(m)}{m^{1/3}} + O(\phi(q)q^{3/5}x^{3/10}(\log x)^{9/10})\sum_{m\le x}\frac{S_4^*(m)}{m^{3/10}}$$

$$= \frac{\phi(q)}{q}c_3(\chi_2)x^{1/3}\sum_{m=1}^{\infty}\frac{S_4^*(m)\chi_2(m)}{m^{1/3}} + O(\phi(q)q^{3/5}x^{3/10}(\log x)^{9/10}),$$

and for $\chi \ne \chi_1, \chi_2$,

$$\sum_{n\le x} S(n)\chi(n) = O(\phi(q)q^{3/5}x^{3/10}(\log x)^{9/10}|\sum_{m\le x}\frac{S_4^*(m)}{m^{3/10}}|)$$

$$= O(\phi(q)q^{3/5}x^{3/10}(\log x)^{9/10}).$$

The theorem then follows from the equation

$$\sum_{\substack{n\le x\\ n\equiv \ell \pmod q}} S(n) = \frac{1}{\phi(q)}\sum_{\chi \pmod q}\bar{\chi}(\ell)\sum_{n\le x} S(n)\chi(n). \qquad \square$$

Acknowledgements

This work was financially supported by Office of the Permanent Secretary, Ministry of Higher Education, Science, Research and Innovation, Grant No. RGNS 63-40.

References

[1] Calderón, C. (2003). Asymptotic estimates on finite abelian groups. *Publications de l'Institut Mathématique*, 74(88), 57–70.

[2] Calderón, C., & Zárate, M. J. (1992). The number of semisimple rings of order at most x. *Extracta Mathematicae*, 7(2-3), 144–147.

[3] Connell, I. G. (1964). A number theory problem concerning finite groups and rings. *Canadian Mathematical Bulletin*, 7, 23–34.

[4] Duttlinger, J. (1974). Eine Bemerkung zu einer asymptotischen Formel von Herrn Knopfmacher. *Journal für die reine und angewandte Mathematik*, 266, 104–106.

[5] Erdős, P., & Szekeres, G. (1934). Über die Anzahl der Abelschen Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem. *Acta Scientiarum Mathematicarum (Szeged)*, 7, 95–102.

[6] Ivić, A. (1985). *The Theory of the Riemann Zeta Function*. Wiley, New York.

[7] Kendall, D. G., & Rankin, R. A. (1947). On the number of abelian groups of a given order. *The Quarterly Journal of Mathematics*, 18(1), 197–208.

[8] Knopfmacher, J. (1972). Arithmetical properties of finite rings and algebras, and analytic number theory. I. *Journal für die reine und angewandte Mathematik*, 252, 16–43.

[9] Kolesnik, G. (1981). On the number of Abelian groups of a given order. *Journal für die reine und angewandte Mathematik*, 329, 164–175.

[10] Liu, H. Q. (1991). On the number of abelian groups of a given order. *Acta Arithmetica*, 59, 261–277.

[11] Richert, H.-E. (1952). Über die Anzahl Abelscher Gruppen gegebener Ordnung. I. *Mathematische Zeitschrift*, 56, 21–32.

[12] Richert, H.-E. (1953). Über die Anzahl Abelscher Gruppen gegebener Ordnung. II. *Mathematische Zeitschrift*, 58(1), 71–84 .

[13] Sargos, P., & Wu, J. (2000). Multiple exponential sums with monomials and their applications in number theory. *Acta Mathematica Hungarica*, 87(4), 333–354.

[14] Schmidt, P. G. (1968). Zur Anzahl Abelscher Gruppen gegebener Ordnung. *Journal für die reine und angewandte Mathematik*, 229, 34–42.

[15] Schmidt, P. G. (1968). Zur Anzahl Abelscher Gruppen gegebener Ordnung II. *Acta Arithmetica*, 13, 405–417.

[16] Schwarz, W. (1966). Über die Anzahl Abelscher Gruppen gegebener Ordnung I. *Mathematische Zeitschrift*, 92, 314–320.