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Combinatorial proofs of identities for the generalized Leonardo numbers

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Abstract: In this paper, we provide combinatorial proofs of several prior identities satisfied by the recently introduced generalized Leonardo numbers, denoted by $\mathcal{L}_{k,n}$, as well as derive some new formulas. To do so, we interpret $\mathcal{L}_{k,n}$ as the enumerator of two classes of linear colored tilings of length n. A comparable treatment is also given for the incomplete generalized Leonardo numbers. Finally, a (p, q)-generalization of $\mathcal{L}_{k,n}$ is obtained by considering the joint distribution of a pair of statistics on one of the aforementioned classes of colored tilings.

Keywords: Leonardo number, Fibonacci number, Linear tiling, Combinatorial proof.
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1 Introduction

The sequence $\{\mathcal{L}_{k,n}\}_{n\geq 0}$ of *generalized Leonardo numbers* was introduced by Kuhapatanakul and Chobsorn in [8] and is defined recursively by

$$\mathcal{L}_{k,n} = \mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2} + k, \qquad n \geq 2, \tag{1.1}$$

with initial values $\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1$. The parameter k is a fixed positive integer, which can also be taken to be an indeterminate. Members of the sequence $\mathcal{L}_{k,n}$ in the case $k = 1$ correspond to what are known as the *Leonardo numbers* [1, 3, 4], denoted by Le_n, which were first studied by Edsger Dijkstra in conjunction with his smoothsort algorithm [5] (see also [6]). For other extensions of Le_n, we refer the reader to [7, 9–11, 13]. In [8], several identities are proven for $\mathcal{L}_{k,n}$ by algebraic methods and an incomplete version of the sequence is introduced. Here, we wish to provide a combinatorial framework for these and other identities satisfied by $\mathcal{L}_{k,n}$.

Let F_n denote the Fibonacci number defined recursively by $F_n = F_{n-1} + F_{n-2}$ if $n \geq 2$, with $F_0 = 0$ and $F_1 = 1$. Let $f_n = F_{n+1}$ and we express our results in terms of f_n, as it is often more convenient notionally when studying combinatorial properties of the Fibonacci sequence. Further, the f_n notation is consistent with that used by Benjamin and Quinn in their text [2]. To shorten notation further, we will denote $\mathcal{L}_{k,n}$ here by $a_n = a_{k,n}$, since the k parameter is constant in the identities featured. Note that a_n reduces to f_n for all $n \geq 0$ when $k = 0$, and hence a_n provides a common generalization of the Fibonacci and Leonardo number sequences.

In the next section, we find two combinatorial interpretations for a_n in terms of certain colored linear tilings and another in terms of marked binary sequences (see the proof of Identity 2.6 below). We make use of these interpretations and extend arguments given in [2] in providing combinatorial proofs of the identities for a_n from [8], which were shown by various algebraic methods such as induction and matrix representations of sequences. A similar treatment is also provided for the incomplete generalized Leonardo numbers, denoted here by $a_n^{(\ell)}$. Further, we apply our combinatorial model in finding new identities satisfied by a_n and $a_n^{(\ell)}$. In the third section, we introduce a bivariate polynomial generalization of a_n, denoted by $a_n(p,q)$, which is obtained by introducing two parameters on a set of tilings enumerated by a_n and is related to previously studied q-analogues of the Fibonacci numbers. We find an explicit summation formula for $a_n(p,q)$, which when restricted accordingly yields a polynomial generalization of $a_n^{(\ell)}$.

2 Combinatorial proofs of identities

We first find a combinatorial interpretation of the sequence a_n in terms of tilings. Recall that a (linear) tiling is a covering of the members of $[n] = \{1, \ldots, n\}$, written in a row, by a sequence of squares and dominos, where a domino covers two consecutive numbers and a square covers a single number. Here, a square or domino is considered indistinguishable from other pieces of the same kind and is denoted by s or d, respectively. See [2] for a discussion of various types of tilings. Let $\mathcal{L}_n$ denote the set of linear tilings of length n. Considering whether the last piece within a tiling of length n is an s or d implies $|\mathcal{L}_n| = f_n$ for all $n \geq 0$. Note that a member of $\mathcal{L}_n$ containing exactly i dominos must contain $n - 2i$ squares, and hence there are $\binom{n-i}{i}$ such members of $\mathcal{L}_n$ for $0 \leq i \leq \lfloor n/2 \rfloor$.

In order to provide combinatorial proofs of identities involving a_n, we extend the linear tiling structure by introducing a third kind of tile.

Definition 2.1. *A k-tile is a rectangular piece coming in one of k colors which*

- *must occur as the first piece in a tiling, if it occurs at all,*
- *has arbitrary length greater than or equal two.*

A k-tile of length ℓ will be denoted by k_ℓ for all $\ell \geq 2$.

We define a new set of tilings as follows.

Definition 2.2. *Let $\mathcal{K}_n$ denote the set of linear tilings using squares, dominos and k-tiles where the combined length of all the pieces is equal n.*

We will represent the members of $\mathcal{K}_n$ using sequences in s, d and k_ℓ. For example, we have

$$\mathcal{K}_4 = \{s^4, s^2d, sds, ds^2, d^2, k_2s^2, k_2d, k_3s, k_4\}.$$

Since each k_ℓ piece comes in one of k colors, we have $|\mathcal{K}_4| = 4k + 5 = a_4$. Considering whether the final piece of $\lambda \in \mathcal{K}_n$ where $n \geq 2$ is an s or d or if it equals k_n (in which case, λ consists of a single k-tile of length n), we get $|\mathcal{K}_n| = |\mathcal{K}_{n-1}| + |\mathcal{K}_{n-2}| + k$. Since k-tiles must have length greater than one, we have $|\mathcal{K}_n| = 1 = a_n$ for $n = 0, 1$, and hence we have shown the following.

Proposition 2.1. *If $n \geq 0$, then $a_n = |\mathcal{K}_n|$.*

We will make use of this interpretation of a_n in providing combinatorial explanations of several identities. We first prove the alternative homogeneous third-order linear recurrence satisfied by a_n.

Identity 2.1. *We have*

$$a_n = 2a_{n-1} - a_{n-3}, \qquad n \geq 3, \tag{2.1}$$

with $a_0 = a_1 = 1$ and $a_2 = k + 2$.

Proof. The initial conditions follow easily from the definitions. To show (2.1), suppose $n \geq 3$ and first note that there are a_{n-1} members of $\mathcal{K}_n$ that end in s. Let S denote the subset of $\mathcal{K}_{n-1}$ consisting of those tilings that do not end in d. By subtraction, we have $|S| = a_{n-1} - a_{n-3}$. If $\lambda \in S$ ends in s, then let λ' be obtained from λ by replacing the final s with a d. Otherwise, $n \geq 3$ implies $\lambda = k_{n-1}$ is also possible, where k_{n-1} comes in one of k colors. In this case, we let $\lambda' = k_n$, keeping the color the same. Then the mapping $\lambda \mapsto \lambda'$ is a bijection from S to the subset of $\mathcal{K}_n$ whose members do not end in s, and hence they number $a_{n-1} - a_{n-3}$, which completes the proof. $\square$

Identity 2.2. *If $n \geq 0$, then*

$$a_n = (k+1)f_n - k. \tag{2.2}$$

Proof. Suppose $\lambda \in \mathcal{K}_n$, where $n \geq 2$, starts with k_ℓ for some $2 \leq \ell \leq n$. Consider the following further cases: **(i)** $\lambda = k_\ell d\lambda'$, **(ii)** $\lambda = k_\ell s\lambda'$, or **(iii)** $\lambda = k_n$, where λ' is a (possibly empty) linear tiling in (i) and (ii). We convert such λ into members of $\mathcal{L}_n$ as follows: if (i) holds, let $f(\lambda) = s^\ell d\lambda'$; if (ii), let

$$f(\lambda) = \begin{cases} d^{\ell/2}s\lambda', & \text{if } \ell \text{ is even}, \\ sd^{(\ell-1)/2}s\lambda', & \text{if } \ell \text{ is odd}; \end{cases}$$

if (iii), let

$$f(\lambda) = \begin{cases} d^{n/2}, & \text{if } n \text{ is even}, \\ sd^{(n-1)/2}, & \text{if } n \text{ is odd}. \end{cases}$$

If λ does not start with k_ℓ for some ℓ, i.e., if $\lambda \in \mathcal{L}_n$, then let $f(\lambda) = \lambda$. Note that if λ starts with k_ℓ, then $f(\lambda)$ must contain at least one d. One may verify that f maps $\mathcal{K}_n$ onto $\mathcal{L}_n$ such that every member of $\mathcal{L}_n - \{s^n\}$ has exactly $k+1$ pre-images in $\mathcal{K}_n$, with the tiling s^n having a single pre-image. Thus, we get $|\mathcal{K}_n| = (k+1)(|\mathcal{L}_n| - 1) + 1$, i.e., $a_n = (k+1)f_n - k$. □

Identity 2.3. *For $n \geq 0$, we have*

$$\sum_{i=0}^{n} a_i = a_{n+2} - k(n+1) - 1, \tag{2.3}$$

$$\sum_{i=0}^{n} a_{2i} = a_{2n+1} - kn, \tag{2.4}$$

$$\sum_{i=0}^{n} a_{2i+1} = a_{2n+2} - k(n+1) - 1. \tag{2.5}$$

Proof. To show (2.3), consider the largest i such that a d covers the numbers $i+1, i+2$ within $\lambda \in \mathcal{K}_{n+2}$ for some $0 \leq i \leq n$. Such λ may be expressed as $\lambda = \lambda' d s^{n-i}$, where $\lambda' \in \mathcal{K}_i$, and hence there are a_i possible λ. Summing over i gives all $\lambda \in \mathcal{K}_{n+2}$ containing at least one d. Otherwise, $\lambda = k_\ell s^{n+2-\ell}$ for some $2 \leq \ell \leq n+2$ or $\lambda = s^{n+2}$, for which there are $k(n+1)+1$ possibilities. Combining the previous cases gives $a_{n+2} = \sum_{i=0}^{n} a_i + k(n+1) + 1$, as desired. To show (2.4), consider the number $2i+1$ covered by the rightmost square within $\lambda \in \mathcal{K}_{2n+1}$, where $0 \leq i \leq n$. Then $\lambda = \lambda' s d^{n-i}$, where $\lambda' \in \mathcal{K}_{2i}$, and considering all i gives $\sum_{i=0}^{n} a_{2i}$ possibilities. Otherwise, $\lambda = k_{2i+1} d^{n-i}$ for some $1 \leq i \leq n$, which gives kn further members of $\mathcal{K}_{2n+1}$ and completes the proof of (2.4). A similar argument applies to (2.5), upon observing that members of $\mathcal{K}_{2n+2}$ not containing a square are either of the form $\lambda = k_{2i+2} d^{n-i}$ for some $0 \leq i \leq n$ or equal to d^{n+1}, for which there are $k(n+1)+1$ possibilities. □

Remark. *Note that* (2.5) *slightly corrects the third formula from [8, Theorem 3], replacing the subtracted quantity* $k(n+2)$ *on the right-hand side with* $k(n+1)+1$.

The next identity was shown in [8] using the matrix representation of the generalized Leonardo numbers.

Identity 2.4. *If $m, n \geq 1$, then*

$$a_m a_{n-1} + a_{m-1} a_n = a_{m+1} a_{n+1} - (k+1) a_{m+n} - k. \tag{2.6}$$

Proof. The identity is clear if m or n equals 1, so assume $m, n \geq 2$. We prove the more suggestive form

$$(k+1) a_{m+n} + a_m a_{n-1} + a_{m-1} a_n + k = a_{m+1} a_{n+1}, \tag{2.7}$$

and make use of the second combinatorial interpretation of a_n as the enumerator of members of $\mathcal{C}_n$. To do so, let $A = [k+1] \times \mathcal{C}_{m+n}$, $B = \mathcal{C}_m \times \mathcal{C}_{n-1}$, $C = \mathcal{C}_{m-1} \times \mathcal{C}_n$ and $D = [k]$. Let $\mathcal{S} = A \cup B \cup C \cup D$ and $\mathcal{T} = \mathcal{C}_{m+1} \times \mathcal{C}_{n+1}$. Note that the left and right sides of (2.7) give $|\mathcal{S}|$ and $|\mathcal{T}|$, respectively, and thus it suffices to show $|\mathcal{S}| = |\mathcal{T}|$.

We define $f : \mathcal{S} \to \mathcal{T}$ as follows. Let $(i, \lambda) \in A$. First suppose λ is of the form $\lambda = \lambda' d \lambda''$, where $|\lambda'| = m-1$ and $|\lambda''| = n-1$ (here, and elsewhere, $|\rho|$ denotes the length of a tiling ρ).

Then let $f(i,\lambda) = (\lambda' d, \text{rev}(d\lambda'')) \in \mathcal{T}$, where $\text{rev}(\rho)$ denotes the tiling obtained by reversing the order of the tiles within ρ. Here, it is understood that the color assigned to the first piece of λ is transferred to the first piece of $\lambda' d$, with the first piece in the tiling $\text{rev}(d\lambda'')$ assigned the i-th possible color. On the other hand, if $\lambda = \lambda'\lambda''$, where $|\lambda'| = m$ and $|\lambda''| = n$, with neither λ' nor λ'' all squares, then let $f(i,\lambda) = (\lambda' s, \lambda'' s)$, where the colors assigned to the respective first pieces of $\lambda' s$ and $\lambda'' s$ are stipulated as before.

If $(\rho', \rho'') \in B$, with ρ'' not all squares, then let $f(\rho', \rho'') = (\rho' s, \rho'' d)$, where $\rho' s$ and $\rho'' d$ receive the colors assigned to ρ' and ρ''. Note that if $\rho' = s^m \in \mathcal{C}_m$ in the first coordinate, then $\rho' s$ is also not to be assigned a color. If $(\rho', \rho'') \in C$, with ρ' not all squares, then let $f(\rho', \rho'') = (\rho' d, \rho'' s)$, where the colors are assigned as in the preceding case. This completes the definition of the mapping f. One may verify that f is one-to-one, where defined. Further, the mapping f is not defined for the following members of $\mathcal{S}$: **(i)** $(i,\lambda) \in A$, where $\lambda = \lambda'\lambda''$, with $|\lambda'| = m$ and $|\lambda''| = n$ and at least one of λ', λ'' consisting of all squares, **(ii)** $(\lambda', \lambda'') \in B$ with $\lambda'' = s^{n-1}$, **(iii)** $(\lambda', \lambda'') \in C$ with $\lambda' = s^{m-1}$, or **(iv)** $i \in D$. Note that the elements of $\mathcal{S}$ in (i)–(iv) have combined cardinality

$$(k+1)(a_m + a_n - 1) + a_m + a_n + k = (k+2)(a_m + a_n) - 1 := p.$$

Finally, the elements of $\mathcal{T}$ of the following forms are seen not to belong to the range of f: **(a)** $(s^{m-1}d, \lambda'' s)$, where $|\lambda''| = n$, **(b)** $(\lambda' s, s^{n-1}d)$, where $|\lambda'| = m$, or **(c)** $(\lambda' s, \lambda'' s)$, where at least one of λ', λ'' consists only of squares. The members of $\mathcal{T}$ satisfying (a), (b) and (c) number $(k+1)a_n$, $(k+1)a_m$ and $a_m + a_n - 1$, respectively. Thus, there are exactly p members of $\mathcal{T}$ in all that fail to belong to the range of f. Since f is one-to-one and not defined on p members of the domain $\mathcal{S}$, it follows that $|\mathcal{S}| = |\mathcal{T}|$, as desired. $\square$

The Fibonacci number formulas

$$\sum_{i=0}^{n} f_i^2 = f_n f_{n+1}, \qquad n \geq 0,$$

and

$$\sum_{i=0}^{n-2} f_i 2^{n-2-i} = 2^n - f_{n+1}, \qquad n \geq 0,$$

involving the sum of squares and the convolution with the sequence 2^n appear respectively as Identities 9 and 10 in [2]. Generalizing the arguments from [2] given for these identities yields the following new relations involving a_n via combinatorial arguments.

Identity 2.5. *If* $n \geq 1$, *then*

$$\sum_{i=0}^{n} a_i^2 = a_n a_{n+1} - k(a_{n+2} - k(n+1) - 2). \tag{2.8}$$

Proof. Let $A = \mathcal{K}_n \times \mathcal{K}_{n+1}$, and we enumerate members $(\alpha, \beta) \in A$ as follows. Clearly, we may assume $n \geq 2$ as the result is clear for $n = 0, 1$. First, let $\mathcal{L} \subseteq A$ consist of those pairs (α, β) such that α and β can be decomposed as either **(i)** $\alpha = \alpha' s d^j$ and $\beta = \beta' d^{j+1}$, or **(ii)** $\alpha = \alpha' d^j$ and

$\beta = \beta' s d^j$ for some $j \geq 0$. Note that $|\alpha'| = |\beta'|$ in either case and hence $(\alpha', \beta') \in \mathcal{K}_i \times \mathcal{K}_i$ for some $0 \leq i \leq n$, where i and n are of opposite parity in (i), and of the same parity in (ii). Thus, combining all of the possibilities from (i) and (ii) implies the subset $\mathcal{L}$ has cardinality $\sum_{i=0}^{n} a_i^2$.

Now suppose $(\alpha, \beta) \in A - \mathcal{L}$. Then it is seen that at least one of α, β must begin with a k-tile. First assume α starts with k_ℓ for some $2 \leq \ell \leq n$. Assume further that β starts with k_j for some $2 \leq j < \ell$ or does not contain a k-piece. Note that $(\alpha, \beta) \in A - \mathcal{L}$ implies α cannot contain a square. To see this, suppose, to the contrary, that $\alpha = \rho s d^m$ for some $m \geq 0$, where $\rho \in \mathcal{K}_{n-1-2m}$. Then in order to avoid (α, β) belonging to $\mathcal{L}$, we must have $\beta = \gamma d^m$, where $\gamma \in \mathcal{K}_{n+1-2m}$. If the last piece of γ is a square or domino, then $(\alpha, \beta) \in \mathcal{L}$, contrary to assumption. Otherwise, we have $\gamma = k_{n+1-2m}$, but then β would start with a k-piece that is of strictly greater length than the k-piece in α, which contradicts the assumption on β. Thus, we have $\alpha = k_\ell d^j$ for some $2 \leq \ell \leq n$ and $j \geq 0$. Then β must be of the form $\beta = \beta' d^{j+1}$, where $\beta' \in \mathcal{K}_{\ell-1}$. On the other hand, if β starts with k_ℓ and α either starts with k_j for some $2 \leq j < \ell$ or does not contain a k-piece, then by similar reasoning, we must have $\alpha = \alpha' d^j$ and $\beta = k_\ell d^j$, where $2 \leq \ell \leq n+1$ and $\alpha' \in \mathcal{K}_{\ell-1}$. Note that $\beta' \in \mathcal{K}_{\ell-1}$ in the first case is such that $n - |\beta'|$ is odd, whereas $n - |\alpha'|$ is even for $\alpha' \in \mathcal{K}_{\ell-1}$ in the second.

Let $B = \cup_{\ell=1}^{n} \mathcal{K}_\ell$. Then it is seen from the preceding that the cardinality of $A - \mathcal{L}$ is equal to $k|B|$. By formula (2.3) above, which was explained bijectively, we have $|B| = a_{n+2} - k(n+1) - 2$. Therefore, we get

$$|A| = |\mathcal{L}| + k|B| = \sum_{i=0}^{n} a_i^2 + k(a_{n+2} - k(n+1) - 2),$$

which implies (2.8). □

Identity 2.6. *If* $n \geq 1$, *then*

$$\sum_{i=0}^{n-2} a_i 2^{n-2-i} = 2^n + k2^{n-1} - a_{n+1}. \tag{2.9}$$

Proof. We show equivalently

$$a_n + a_{n-1} + \sum_{i=0}^{n-2} a_i 2^{n-2-i} = 2^n + k(2^{n-1} - 1), \qquad n \geq 1. \tag{2.10}$$

Let $\mathcal{A}_n$ denote the set of all binary sequences of length n in which an initial sequence of 1's of length at least two may be marked in one of k ways. Note that the initial marked sequence of 1's need not be maximal. For example, if $n = 3$, we have

$$\mathcal{A}_3 = \{abc : a, b, c \in \{0, 1\}\} \cup \{\underline{11}0, \underline{11}1, \underline{111}\},$$

where the marked sequence is underlined in the second set, and hence $|\mathcal{A}_3| = 3k + 8$. Let $\mathcal{A}'_3$ denote the subset of $\mathcal{A}_n$ whose members end in 1 and not containing two consecutive zeros. For example, $\mathcal{A}'_3 = \{011, 101, 111\} \cup \{\underline{11}1, \underline{111}\}$, and thus $|\mathcal{A}'_3| = 2k + 3 = a_3$. In general, we have $|\mathcal{A}'_n| = a_n$, upon converting 01's to d's, changing any remaining 1's (not part of a marked sequence) to s's and replacing an initial marked sequence of 1's of length ℓ for some $\ell \geq 2$ with k_ℓ.

Let $\lambda = \lambda_1 \cdots \lambda_n \in \mathcal{A}_n$. Suppose first that λ contains two consecutive zeros. Let i be the smallest r such that $\lambda_{r+1} = \lambda_{r+2} = 0$, where $0 \leq i \leq n-2$. Then $\lambda_1 \cdots \lambda_i \in \mathcal{A}_i'$, as $\lambda_i = 1$ if $i > 0$, by the minimality of i, with no restrictions on the remaining letters of λ. This gives $a_i 2^{n-2-i}$ possible λ and considering all $0 \leq i \leq n-2$ yields $\sum_{i=0}^{n-2} a_i 2^{n-2-i}$ members of $\mathcal{A}_n$ that contain two consecutive zeros. Otherwise, λ must either belong to $\mathcal{A}_n'$ or be of the form $\lambda = \lambda' 0$, where $\lambda' \in \mathcal{A}_{n-1}'$. This yields $a_n + a_{n-1}$ additional members of $\mathcal{A}_n$, and hence we have that the left side of (2.10) gives $|\mathcal{A}_n|$.

To complete the proof of (2.10), we must then show $|\mathcal{A}_n| = 2^n + k(2^{n-1} - 1)$. Clearly, there are 2^n members of $\mathcal{A}_n$ not containing a marked sequence. Otherwise, consider an arbitrary binary sequence ρ of length $n-1$, not all 1's. Let $\rho = 1^m 0 \rho'$, where $0 \leq m \leq n-2$ and thus ρ' is of length $n-2-m$. Let $g(\rho) = \underline{1 \cdots 1}\rho'$ be the member of $\mathcal{A}_n$, where the underlined sequence of 1's is understood to have length $m+2$ and the marking is not specified. Then g is seen to be a 1-to-k mapping onto the subset of $\mathcal{A}_n$ whose members contain an initial marked sequence of 1's, and hence there are $k(2^{n-1} - 1)$ such members of $\mathcal{A}_n$. Combining this with the previous case gives the stated formula for $|\mathcal{A}_n|$ and completes the proof. $\square$

The next result is a new relation for a_n which is an extension of the Cassini identity for Fibonacci numbers. It can be explained by extending a combinatorial procedure known as tail swapping.

Identity 2.7. *We have*

$$a_n^2 - a_{n-1}a_{n+1} = (-1)^n (k+1)^2 + k(k+1) f_{n-3}, \qquad n \geq 1, \tag{2.11}$$

where $f_{-2} = 1$ and $f_{-1} = 0$.

Proof. We treat only the even case, as the modifications required to handle the odd case will be apparent. Let $n = 2m$, where we may assume $m \geq 2$ (as the $n = 2$ case is easily verified). We make use of the tail swapping involution described in [2, Chapter 1] (see also [15]) to define a one-to-one mapping f between the sets $\mathcal{A} = \mathcal{K}_{2m} \times \mathcal{K}_{2m}$ and $\mathcal{B} = \mathcal{K}_{2m-1} \times \mathcal{K}_{2m+1}$. Note that f is defined only on members of $\mathcal{A}$ that contain at least one fault line when a pair of tilings is arranged in a staggered formation offset by one position; further, one must be mindful not to swap tails if it results in one of the tilings containing a k-tile that is not at the beginning. Let $\mathcal{A}' \subseteq \mathcal{A}$ consist of those tiling pairs for which f is not defined and let $\mathcal{B}' = \mathcal{B} - \text{range}(f)$. It then suffices to show

$$|\mathcal{A}'| - |\mathcal{B}'| = (k+1)^2 + k(k+1) f_{2m-3}.$$

We first determine $|\mathcal{A}'|$. Note that $\mathcal{A}'$ consists of those $(\lambda, \lambda') \in \mathcal{A}$ having one of the following forms: **(i)** $\lambda = \lambda' = d^m$, **(ii)** $\lambda = k_{2i} d^{m-i}$ and $\lambda' = \alpha d^{m-i}$, or **(iii)** $\lambda = \beta d^{m-i+1}$ and $\lambda' = k_{2i} d^{m-i}$, where $\alpha \in \mathcal{K}_{2i}$, $\beta \in \mathcal{K}_{2i-2}$ and $1 \leq i \leq m$ (in both (ii) and (iii)). Note that f fails to be defined for members of $\mathcal{A}'$ in (ii) where $\alpha = s\alpha'$ for some $\alpha \in \mathcal{L}_{2i-1}$ since swapping tails would result in $f(\lambda, \lambda')$ being equal $(\alpha' d^{m-i}, s k_{2i} d^{m-i})$ in that case, which is not permitted since the tiling in the second coordinate contains a k-tile that is not at the beginning. Further, f is not defined on the remaining tilings in (ii), and also (i) and (iii), since there exist no fault lines when λ and λ' are arranged in a staggered formation in each of these cases. Note that the cardinalities of the members of $\mathcal{A}'$ satisfying (ii) or (iii) are given by $k\sum_{i=1}^{m} a_{2i}$ and $k\sum_{i=1}^{m} a_{2i-2}$, respectively. Thus, we have

$$\begin{aligned}|\mathcal{A}'| &= 1-k-ka_{2m}+2k\sum_{i=0}^{m} a_{2i} = 1-k-ka_{2m}+2k(a_{2m+1}-mk)\\ &= 1-k-(2m-1)k^2+k(a_{2m+1}+a_{2m-1}),\end{aligned}$$

where we have made use of (2.4) and (1.1).

We now find $|\mathcal{B}'|$. Note that $\mathcal{B}'$ consists of those $(\lambda,\lambda')\in\mathcal{B}$ such that either **(I)** $\lambda = k_{2i-1}d^{m-i}$ and $\lambda' = \alpha d^{m-i+1}$ with $2\le i\le m$, or **(II)** $\lambda=\alpha d^{m-i}$ and $\lambda' = k_{2i+1}d^{m-i}$ with $1\le i\le m$, where $\alpha\in\mathcal{K}_{2i-1}$ in both cases. Note that (I) yields $k\sum_{i=2}^{m} a_{2i-1}$ possibilities, whereas (II) gives $k\sum_{i=1}^{m} a_{2i-1}$, and hence

$$|\mathcal{B}'| = 2k\sum_{i=1}^{m} a_{2i-1}-k = 2k(a_{2m}-mk-1)-k = 2ka_{2m}-2mk^2-3k,$$

by (2.5). Therefore, we have

$$\begin{aligned}|\mathcal{A}'|-|\mathcal{B}'| &= 1-k-(2m-1)k^2+k(a_{2m+1}+a_{2m-1})-(2ka_{2m}-2mk^2-3k)\\ &= (k+1)^2+k(a_{2m+1}+a_{2m-1}-2a_{2m}) = (k+1)^2+k(a_{2m-3}+k)\\ &= (k+1)^2+k(k+1)f_{2m-3},\end{aligned}$$

as desired, where we have made use of (1.1) and (2.2). □

2.1 Identities for incomplete Leonardo numbers

The *incomplete* generalized Leonardo numbers are defined as follows.

Definition 2.3. [8] *If $k\ge 1$, then the incomplete generalized Leonardo numbers $a_n^{(\ell)} = a_{k,n}^{(\ell)}$ are given by*

$$a_n^{(\ell)} = (k+1)\sum_{i=0}^{\ell}\binom{n-i}{i}-k, \qquad 0\le \ell\le \lfloor n/2\rfloor.$$

Note that $a_n^{(0)}=1$ and $a_n^{(\lfloor n/2\rfloor)} = a_n$ for all $n\ge 0$, the latter holding by Identity 2.2 since $f_n=\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n-i}{i}$. We now develop a combinatorial interpretation for $a_n^{(\ell)}$ in terms of tilings. Let $\mathcal{C}_n$ denote the set consisting of tilings λ of length n such that the first piece of λ is assigned one of $k+1$ colors, provided λ is not the all squares tiling, in which case the first piece of λ is not assigned a color. Then $|\mathcal{C}_n| = (k+1)(f_n-1)+1=(k+1)f_n-k$, and the proof of Identity 2.2 demonstrates combinatorially the fact $|\mathcal{C}_n|=|\mathcal{K}_n|$.

Let $\mathcal{C}_n^{(\ell)}$ for $0\le\ell\le\lfloor n/2\rfloor$ denote the subset of $\mathcal{C}_n$ containing at most ℓ dominos. Then $|\mathcal{C}_n^{(\ell)}| = 1+(k+1)\sum_{i=1}^{\ell}\binom{n-i}{i} = a_n^{(\ell)}$, since the all-squares tiling belongs to $\mathcal{C}_n^{(\ell)}$ for each ℓ. Note that $\mathcal{C}_n^{(\lfloor n/2\rfloor)} = \mathcal{C}_n$ for all $n\ge 0$.

We now provide combinatorial proofs of the following two recurrences for $a_n^{(\ell)}$ from [8].

Identity 2.8. *If $n \geq 0$, then*

$$a_{n+2}^{(\ell)} = a_{n+1}^{(\ell)} + a_n^{(\ell)} + k - (k+1)\binom{n-\ell}{\ell}, \qquad 0 \leq \ell \leq \lfloor n/2 \rfloor. \tag{2.12}$$

Proof. If $\ell = 0$, then (2.12) is clear, so assume $\ell \geq 1$. We append squares and dominos respectively to the members of $\mathcal{C}_{n+1}^{(\ell)}$ and $\mathcal{C}_n^{(\ell)}$ to obtain members of $\mathcal{C}_{n+2}^{(\ell)}$ ending in s or d, which yields $a_{n+1}^{(\ell)} + a_n^{(\ell)}$ possibilities. Note that this counts the $k+1$ members of $\mathcal{C}_{n+2}^{(\ell)}$ of the form $s^n d$ only once, as s^n is counted just once by $a_n^{(\ell)}$. Hence, we must add k to account for these missed members of $\mathcal{C}_{n+2}^{(\ell)}$. Finally, we subtract $(k+1)\binom{n-\ell}{\ell}$ from the total, since appending d to any one of the $(k+1)\binom{n-\ell}{\ell}$ members of $\mathcal{C}_n^{(\ell)}$ containing exactly ℓ dominos is disallowed as the resulting tiling would contain too many dominos for membership in $\mathcal{C}_{n+2}^{(\ell)}$. $\square$

Identity 2.9. *If $n, t, \ell \geq 0$ with $\ell \leq \frac{n-t}{2}$, then*

$$a_{n+2t}^{(\ell+t)} = \sum_{i=0}^{t} \binom{t}{i} a_{n+i}^{(\ell+i)} + (2^t - 1)k. \tag{2.13}$$

Proof. If $t = 0$, then the formula is obvious, so assume $t \geq 1$. We enumerate $\lambda \in \mathcal{C}_{n+2t}^{(\ell+t)}$ according to the number i of squares among the final t pieces of λ. Note that such λ can be decomposed as $\lambda = \lambda'\alpha$, where $\lambda' \in \mathcal{C}_{n+i}^{(\ell+i)}$ and α is an arbitrary permutation of the multiset $\{s^i d^{t-i}\}$. Hence, there are $\binom{t}{i} a_{n+i}^{(t+i)}$ possibilities for each $0 \leq i \leq t$, and considering all i accounts for the summation in (2.13). Note that the summation misses most of the members of $\mathcal{C}_{n+2t}^{(\ell+t)}$ of the form $\rho = s^{n+i}\alpha$, where $0 \leq i \leq t-1$ and α is as before. Indeed, for each α fixed, it misses exactly k of the $k+1$ members $\rho \in \mathcal{C}_{n+2t}^{(\ell+t)}$ of the stated form. Note that there are $2^t - 1$ possible α among these ρ, since the $i = t$ case is to be excluded as s^{n+2t} is counted only once by $a_{n+2t}^{(\ell+t)}$. Hence, there are $(2^t - 1)k$ members of $\mathcal{C}_{n+2t}^{(\ell+t)}$ missed by the summation. Thus, the second term on the right side of (2.13) corrects for this undercount, which completes the proof. $\square$

Remarks. *Note that the $t = 1$ case of (2.13) gives*

$$a_{n+2}^{(\ell+1)} = a_{n+1}^{(\ell+1)} + a_n^{(\ell)} + k,$$

which occurs as [8, Theorem 5]. A different generalization of this identity is given by

$$a_{n+m}^{(\ell+1)} = a_n^{(\ell+1)} + \sum_{i=0}^{m-1} a_{n+i-1}^{(\ell)} + km, \tag{2.14}$$

where $m \geq 1$ and $n \geq 2\ell + 2$. Note that (2.14) reduces to [4, Proposition 4] when $k = 1$. A combinatorial proof of (2.14) can be had by considering if $\lambda \in \mathcal{C}_{n+m}^{(\ell+1)}$ is of the form $\lambda = \lambda' s^m$ or $\lambda = \lambda' d s^{m-1-i}$ for some $0 \leq i \leq m-1$ and further whether or not λ' in the second case contains a domino.

By a combinatorial argument, one can obtain the following new formula relating a_n and $a_n^{(\ell)}$.

Identity 2.10. *If $n \geq 0$, then*

$$\sum_{\ell=0}^{\lfloor n/2 \rfloor} a_n^{(\ell)} = (\lfloor n/2 \rfloor + 1)a_n - (k+1)\sum_{i=0}^{n-2} f_i f_{n-2-i}. \tag{2.15}$$

Proof. Suppose $\lambda \in \mathcal{C}_n$ contains exactly r dominos, where we assume initially $1 \leq r \leq \lfloor n/2 \rfloor$, and is assigned the j-th color for some $j \in [k+1]$. Then λ is counted $p - r$ times by the left-hand side of (2.15), where $p = \lfloor n/2 \rfloor + 1$, since λ is counted once by each $a_n^{(\ell)}$ term for $\ell \geq r$. For the right side of (2.15), first note that $(k+1)\sum_{i=0}^{n-2} f_i f_{n-2-i}$ enumerates all "marked" members of $\mathcal{C}_n - \{1^n\}$ wherein one of the dominos is distinguished. To realize this, note that the $(k+1)f_i f_{n-2-i}$ term accounts for marked members of $\mathcal{C}_n - \{1^n\}$ wherein the marked domino covers the numbers $i+1, i+2$ for some $0 \leq i \leq n-2$. Thus, λ containing $r \geq 1$ dominos implies it is counted exactly $p - r$ times by the difference $pa_n - (k+1)\sum_{i=0}^{n-2} f_i f_{n-2-i}$. Further, if λ contains no dominos (i.e., $\lambda = s^n$) and is thus assigned no color, then λ is seen to be counted p times by both sides of (2.15). Since the two sides of (2.15) agree for all $\lambda \in \mathcal{C}_n$ as to the number of times λ is counted, the proof is complete. $\square$

Remark. *Let $L_n = f_n + f_{n-2}$ denote the n-th Lucas number. Then the right side of* (2.15) *can be simplified by applying the formula $\sum_{i=0}^{n-2} f_i f_{n-2-i} = \frac{nL_n - f_{n-1}}{5}$; see [2, Identity 58], where a combinatorial proof was given.*

3 A polynomial generalization of a_n

In this section, we briefly introduce a polynomial generalization of a_n by considering the joint distribution of a pair of statistics on $\mathcal{K}_n$. By a *longer* piece within $\lambda \in \mathcal{K}_n$, we mean a domino or a k-tile k_ℓ for some $\ell \geq 2$. Let $\mu(\lambda)$ denote the number of longer pieces of λ and $\sigma(\lambda)$ be the sum of the numbers covered by the rightmost sections of the longer pieces. For example, if $n = 15$ and $\lambda = k_4 s d^2 s^3 d s \in \mathcal{K}_{15}$, then $\mu(\lambda) = 4$ and $\sigma(\lambda) = 4 + 7 + 9 + 14 = 34$. Define the joint distribution $a_n(p, q)$ by

$$a_n(p,q) = \sum_{\lambda \in \mathcal{K}_n} p^{\mu(\lambda)} q^{\sigma(\lambda)}, \qquad n \geq 1,$$

with $a_0(p,q) = 1$. For example, if $n \geq 3$, then we have $\mathcal{K}_3 = \{ds, sd, s^3, k_2 s, k_3\}$ and we get $a_3(p,q) = pq^2 + pq^3 + 1 + kpq^2 + kpq^3$. Note that $a_n(1,1) = a_n$ for all n.

Considering whether a member of $\mathcal{K}_n$ ends in s or d or consists of a single tile k_n implies the recurrence

$$a_n(p,q) = a_{n-1}(p,q) + pq^n a_{n-2}(p,q) + kpq^n, \qquad n \geq 2, \tag{3.1}$$

with $a_0(p,q) = a_1(p,q) = 1$. Note that $a_n(p,q) = F_n^{(2)}(q, pq)$ when $k = 0$, where $F_n^{(r)}(q,t)$ denotes the generalized r-Fibonacci polynomial studied in [12].

Let $\binom{n}{k}_q$ denote the q-binomial coefficient. Using (3.1), one can find an explicit formula for $a_n(p,q)$.

Theorem 3.1. *If $n \geq 0$, then*

$$a_n(p,q) = (k+1)\sum_{\ell=0}^{\lfloor n/2 \rfloor} p^\ell q^{\ell(\ell+1)} \binom{n-\ell}{\ell}_q - k. \tag{3.2}$$

Proof. Let $F(x) = \sum_{n \geq 0} a_n(p,q) x^n$. Then rewriting recurrence (3.1) in terms of generating functions leads to the functional equation

$$F(x) = \frac{1 - qx + kpq^2x^2}{(1-x)(1-qx)} + \frac{pq^2x^2}{1-x}F(qx). \tag{3.3}$$

Iterating (3.3) an infinite number of times gives

$$F(x) = \sum_{\ell \geq 0} \frac{p^\ell q^{\ell(\ell+1)} x^{2\ell}(1 - q^{\ell+1}x + kpq^{2\ell+2}x^2)}{\prod_{i=0}^{\ell+1}(1-q^ix)},$$

which can be rewritten as

$$\begin{aligned} F(x) &= \sum_{\ell \geq 0} \frac{p^\ell q^{\ell(\ell+1)} x^{2\ell}}{\prod_{i=0}^{\ell}(1-q^ix)} + k \sum_{\ell \geq 0} \frac{p^{\ell+1} q^{(\ell+1)(\ell+2)} x^{2\ell+2}}{\prod_{i=0}^{\ell+1}(1-q^ix)} \\ &= (k+1) \sum_{\ell \geq 0} \frac{p^\ell q^{\ell(\ell+1)} x^{2\ell}}{\prod_{i=0}^{\ell}(1-q^ix)} - \frac{k}{1-x}. \end{aligned} \tag{3.4}$$

Recall the well-known expansion (see, e.g., [14])

$$\frac{x^\ell}{(1-x)(1-qx)\cdots(1-q^\ell x)} = \sum_{n \geq \ell} \binom{n}{k}_q x^n, \qquad \ell \geq 0,$$

which implies

$$\sum_{\ell \geq 0} \frac{p^\ell q^{\ell(\ell+1)} x^{2\ell}}{\prod_{i=0}^{\ell}(1-q^ix)} = \sum_{\ell \geq 0} p^\ell q^{\ell(\ell+1)} x^\ell \sum_{n \geq \ell} \binom{n}{k}_q x^n = \sum_{n \geq 0} \sum_{\ell=0}^{\lfloor n/2 \rfloor} p^\ell q^{\ell(\ell+1)} \binom{n-\ell}{\ell}_q x^n.$$

Extracting the coefficient of x^n in (3.4) now gives (3.2). $\square$

Remarks. *Taking $p = q = 1$ in* (3.4) *yields the Leonardo number generating function formula*

$$\sum_{n \geq 0} a_n x^n = \frac{1 - x + kx^2}{(1-x)(1-x-x^2)}.$$

Let $(x;q)_n := (1-x)(1-qx)\cdots(1-q^{n-1}x)$ denote the q-Pochhammer symbol. Then the generating function $F(x)$ above is given in terms of q-series by $F(x) = (k+1)g(x) - k(1-x)^{-1}$, where

$$g(x) = \frac{1}{1-x} \sum_{\ell \geq 0} \frac{q^{\ell(\ell+1)}}{(qx;q)_\ell}(px^2)^\ell,$$

and one might consider studying further properties of g. When $p = q = 1$ in (3.2)*, one gets the formula for a_n in terms of the $\ell = \lfloor n/2 \rfloor$ case of Definition 2.3. By considering the partial sums of the formula in Theorem 3.1, one obtains a (p,q)-analogue of the incomplete generalized Leonardo numbers, denoted by $a_n^{(\ell)}(p,q)$. Note that extending the proofs of* (2.12) *and* (2.14) *above shows that $a_n^{(\ell)}(p,q)$ satisfies the recurrences*

$$a_{n+2}^{(\ell)}(p,q) = a_{n+1}^{(\ell)}(p,q) + pq^{n+2}a_n^{(\ell)}(p,q) + kpq^{n+2} - (k+1)p^{\ell+1}q^{\ell(\ell+1)+n+2}\binom{n-\ell}{\ell}_q$$

and

$$a_{n+m}^{(\ell+1)}(p,q) = a_n^{(\ell+1)}(p,q) + p\sum_{i=0}^{m-1} q^{n+i+1}\left(a_{n+i-1}^{(\ell)}(p,q) + k\right).$$

The $a_n(p,q)$ satisfy the following further recurrences.

Theorem 3.2. *If $n \geq 3$, then*

$$a_n(p,q) = (1+pq^n)a_{n-1}(p,q) - p^2q^{2n-1}a_{n-3}(p,q) + kpq^n(1-pq^{n-1}), \tag{3.5}$$

with $a_0(p,q) = a_1(p,q) = 1$ and $a_2(p,q) = 1 + (k+1)pq^2$.

Theorem 3.3. *For $n \geq 0$, we have*

$$a_{n+2}(p,q) = 1 + kpq^2(n+1)_q + pq^2 \sum_{i=0}^{n} q^i a_i(p,q), \tag{3.6}$$

$$a_{2n+1}(p,q) = a_{2n}(p,q) + \sum_{i=0}^{n-1} p^{n-i} q^{(n-i)(n+i+2)}(a_{2i}(p,q)+k), \tag{3.7}$$

$$a_{2n}(p,q) = a_{2n-1}(p,q) + (k+1)p^n q^{n(n+1)} + \sum_{i=1}^{n-1} p^{n-i} q^{(n-i)(n+i+1)}(a_{2i-1}(p,q)+k). \tag{3.8}$$

Theorems 3.2 and 3.3 may be shown by generalizing the combinatorial arguments of the comparable results above by employing a (p,q)-weighting on the members of $\mathcal{K}_n$, the details of which we leave to the interested reader. Furthermore, additional combinatorial identities for $a_n(p,q)$ may be obtained in this way, which we leave for the reader to explore.

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