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On generalized (k, r)-Pell and (k, r)-Pell–Lucas numbers

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Abstract: We introduce new kinds of k-Pell and k-Pell–Lucas numbers related to the distance between numbers by a recurrence relation and show their relation to the (k, r)-Pell and (k, r)-Pell–Lucas numbers. These sequences differ both according to the value of the natural number k and the value of a new parameter r in the definition of this distance. We give several properties of these sequences. In addition, we establish the generating functions, some important identities, as well as the sum of the terms of the generalized (k, r)-Pell and (k, r)-Pell–Lucas numbers. Furthermore, we indicate another way to obtain the generalized (k, r)-Pell and (k, r)-Pell–Lucas sequences from the generating function, in connection to graphs.
Keywords: Generalizations of Pell numbers, k-Pell numbers, r-distance Pell numbers, Graphs, Generating functions.
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1 Introduction

Number sequences have been a subject of study for many mathematicians since antiquity. Special focus has been on recurrent sequences such as the Fibonacci, Lucas, Pell, or Pell–Lucas sequences described below:

$$\begin{array}{rlll} \text{Fibonacci numbers} & (F_n)_{n\geq 0}: & F_0=0,\ F_1=1, & F_{n+2}=F_{n+1}+F_n; \\ \text{Lucas numbers} & (L_n)_{n\geq 0}: & L_0=2,\ L_1=1, & L_{n+2}=L_{n+1}+L_n; \\ \text{Pell numbers} & (P_n)_{n\geq 0}: & P_0=0,\ P_1=1, & P_{n+2}=2P_{n+1}+P_n; \\ \text{Pell–Lucas numbers} & (Q_n)_{n\geq 0}: & Q_0=2,\ Q_1=2, & Q_{n+2}=2Q_{n+1}+Q_n, \end{array}$$

In particular, Pell numbers are linked to approximations of $\sqrt{2}$ by rationals and Diophantine equations, and have the Pell–Lucas numbers as a natural companion. These sequences have also been extended and generalized for quaternions and octonions (see, e.g., [9]).

For a positive integer k and $D = k^2 + 4$, the k-Fibonacci and k-Lucas numbers are obtained as

$$F_{k,n} = \frac{1}{\sqrt{D}}\left(\left(\frac{k+\sqrt{D}}{2}\right)^n - \left(\frac{k-\sqrt{D}}{2}\right)^n\right), \quad L_{k,n} = \left(\frac{k+\sqrt{D}}{2}\right)^n + \left(\frac{k-\sqrt{D}}{2}\right)^n.$$

In particular, the "bronze" Fibonacci numbers $F_{3,n}$ indexed as A006190 in OEIS and starting with the terms

$$0, 1, 3, 10, 33, 109, 360, 1189, 3927, 12970, 42837, 141481, \ldots,$$

are related to the enumeration of fatty acids in [27]. Number sequences are generalized in different ways, see [1, 16, 17, 19, 21–24].

As a result of obtaining k-Fibonacci sequences, which are the more general versions of Fibonacci sequences, and by defining the distance of the sequences, it was possible to switch to r-distance (k, r)-Fibonacci sequences.

We can see some generalizations of Pell and Pell–Lucas numbers in [3–5, 7, 8, 12–15, 18, 25]. In [11], Falcon applied the definition of r-distance to the k-Fibonacci numbers in such a way that it generalized earlier results [6,29]. Moreover, Panwar et al. gave several identities for generalized (k, r)-Fibonacci numbers, see [26].

Now we recall the definition of k-Pell and k-Pell–Lucas sequence and its Binet Formula.

Definition 1.1 ([7]). *For $k \in \mathbf{N}$, the k-Pell sequence is defined as*

$$P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}, \textit{for} \quad n \geq 2$$

with $P_{k,0} = 0$ and $P_{k,1} = 1$.

Let us give the first few terms of k-Pell numbers as follows:

$$\begin{aligned} P_{k,0} &= 0 \\ P_{k,1} &= 1 \\ P_{k,2} &= 2 \\ P_{k,3} &= k + 4 \\ P_{k,4} &= 4k + 8 \\ P_{k,5} &= k^2 + 12k + 16 \\ P_{k,6} &= 6k^2 + 32k + 32 \\ P_{k,7} &= k^3 + 24k^2 + 80k + 64 \end{aligned}$$

For $k = 1$, the known Pell sequence is obtained. $P_1 = \{0, 1, 2, 5, 12, 29, 70, 169, \dots\}$. The n-th k-Pell sequence can be found by

$$P_{k,n} = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2},$$

where r_1 and r_2 are the roots of characteristic equation, $r^2 - 2r - k = 0$.

Definition 1.2 ([8]). *For $k \in \mathbf{N}$, the k-Pell–Lucas sequence is defined as*

$$Q_{k,n+1} = 2Q_{k,n} + kQ_{k,n-1}, \text{ for } n \geq 1$$

with $Q_{k,0} = 2$ and $Q_{k,1} = 2$.

Let us give the first few terms of k-Pell–Lucas numbers as follows:

$$\begin{aligned} Q_{k,0} &= 2 \\ Q_{k,1} &= 2 \\ Q_{k,2} &= 2k + 4 \\ Q_{k,3} &= 6k + 8 \\ Q_{k,4} &= 2k^2 + 16k + 16 \\ Q_{k,5} &= 10k^2 + 40k + 32 \end{aligned}$$

The n-th k-Pell–Lucas sequence can be found by

$$Q_{k,n} = {r_1}^n + {r_2}^n,$$

where r_1 and r_2 are the roots of characteristic equation, $r^2 - 2r - k = 0$.

We give several properties of these sequences with theorems. In addition, the generating functions, some important identities and the sum of the terms of the generalized (k, r)-Pell numbers are given. Also, we show another way of finding the generalized (k, r)-Pell sequence from the generating function with graphs.

Throughout this paper, we will provide numerical examples related to these sequences, and link them to the *Online Encyclopedia of Integer Sequences* (OEIS) [28]. This database currently has more than 350000 entries, and includes the Fibonacci, Lucas, Pell, and Pell–Lucas sequences as A000045, A000032, A000129, and A002203, respectively.

2 Generalized (k, r)-Pell numbers

Definition 2.1. *For the natural numbers $k \geq 1, n \geq 0$ and $r \geq 1$, the generalized (k, r)-Pell numbers $P_{k,n}(r)$ are defined by*

$$P_{k,n}(r) = 2P_{k,n-r}(r) + kP_{k,n-2}(r) \quad \text{for} \quad n \geq r \tag{1}$$

with $P_{k,n}(r) = 1$ for $n = 0, 1, 2, 3, \dots, r - 1$ and $P_{k,1}(1) = k$.

Let $P_k(r) = \{P_{k,n}(r) : n \in \mathbf{N}\}$. For some r, the following values are obtained.

$$\begin{aligned}
P_k(1) &= \{1, k, 3k, k^2 + 6k, 5k^2 + 12k, k^3 + 16k^2 + 24k, 7k^3 + 44k^2 + 48k, \ldots\} \\
P_k(2) &= \{1, 1, k + 2, k + 2, (k + 2)^2, (k + 2)^2, (k + 2)^3, (k + 2)^3, (k + 2)^4, (k + 2)^4, \ldots\} \\
P_k(3) &= \{1, 1, 1, k + 2, k + 2, k^2 + 2k + 2, (k + 2)2, k^3 + 2k^2 + 4k + 4, \ldots\} \\
P_k(4) &= \{1, 1, 1, 1, k + 2, k + 2, k^2 + 2k + 2, k^2 + 2k + 2, k^3 + 2k^2 + 4k + 4, \\
&\qquad k^3 + 2k^2 + 4k + 4, \ldots\} \\
P_k(5) &= \{1, 1, 1, 1, 1, k + 2, k + 2, k^2 + 2k + 2, k^2 + 2k + 2, k^3 + 2k^2 + 2k + 2, \\
&\qquad k^3 + 2k^2 + 4k + 4, \ldots\}
\end{aligned}$$

Now, let the generalized (k, r)-Pell numbers $P_{k,n}(r)$ particularize for $k = 1, 2, 3, \ldots$ For $k = 1$, we get the sequences in Table 1.

Table 1. $P_{1,n}(r)$ for some r

n	0	1	2	3	4	5	6	7	8
$P_{1,n}(1)$	1	1	3	7	17	41	99	239	577
$P_{1,n}(2)$	1	1	3	3	9	9	27	27	81
$P_{1,n}(3)$	1	1	1	3	3	5	9	11	15
$P_{1,n}(4)$	1	1	1	1	3	3	5	5	11
$P_{1,n}(5)$	1	1	1	1	1	3	3	5	5

In Table 1 for $P_{1,n}(1)$ if we take $1, 1+1, 1+1+3, 1+1+3+7, 1+1+3+7+17, \ldots$ respectively, we get the Pell numbers.

Table 2. $P_{2,n}(r)$ for some r

n	0	1	2	3	4	5	6	7
$P_{2,n}(1)$	1	2	6	16	44	120	328	896
$P_{2,n}(2)$	1	1	4	4	16	16	64	64
$P_{2,n}(3)$	1	1	1	4	4	10	16	28
$P_{2,n}(4)$	1	1	1	1	4	4	10	10
$P_{2,n}(5)$	1	1	1	1	1	4	4	10

For $k = 2$, we get the sequences in Table 2. For $k = 3$, we get the sequences in Table 3.

Table 3. $P_{3,n}(r)$ for some r

n	0	1	2	3	4	5	6	7
$P_{3,n}(1)$	1	3	9	27	81	243	729	2187
$P_{3,n}(2)$	1	1	5	5	25	25	125	125
$P_{3,n}(3)$	1	1	1	5	5	17	25	61
$P_{3,n}(4)$	1	1	1	1	5	5	17	17

It is worthy to be noted that only nine (k,r)-Pell sequences are referenced in the *On-Line Encyclopedia of Integer Sequences* [28] with the numbers given in Table 4. $P_{1,n}(2), P_{1,n}(4)$, $P_{2,n}(2), P_{2,n}(4)$ and $P_{3,n}(2), P_{3,n}(4)$ are double sequences.

Table 4. (k,r)-Pell sequences for $k = 1, 2, 3$

$P_{1,n}(1)$	A001333
$P_{1,n}(2)$	A108411
$P_{1,n}(3)$	A117433
$P_{1,n}(4)$	A146245
$P_{1,n}(5)$	A237714
$P_{2,n}(1)$	A002605
$P_{2,n}(2)$	A001333
$P_{3,n}(1)$	A000244
$P_{3,n}(2)$	A074872

If we write the sequence and show some numbers in bold

$$\begin{aligned}P_{1,n}(1) = \{&1, 1, 3, \mathbf{7}, 17, \mathbf{41}, 99, \mathbf{239}, 577, 1393, 3363, 8119, 19601, 47321, 114243,\\ &275807, 665857, 1607521, 3880899, \mathbf{9369319}, \ldots, 26102926097, \mathbf{63018038201},\\ &152139002499, \ldots\},\end{aligned}$$

we see Newman–Shanks–Williams prime (NSW prime) [20] which is a prime number written as

$$S_{2m+1} = \frac{(1+\sqrt{2})^{2m+1} + (1-\sqrt{2})^{2m+1}}{2}.$$

It is indexed as A088165 in the OEIS [28].

Theorem 2.1. *Let* $r \geq 2$. *Then,*

$$P_{k,r}(r) = k + 2$$

Proof. Just apply Equation 1. □

Theorem 2.2. *If* r *is even number, we have*

$$P_{k,2n}(2m) = P_{k,2n+1}(2m).$$

Proof. We prove the result by induction on n. For $n = 0$, from Definition 2.1, we get $P_{k,0}(2m) = 1$ and $P_{k,1}(2m) = 1$. Let us assume this formula up to $2n + 1$ is correct. Then

$$\begin{aligned}P_{k,2n+2}(2m) &= 2P_{k,2n+2-2m}(2m) + kP_{k,2n}(2m)\\ &= 2P_{k,2(n+1-m)}(2m) + kP_{k,2n}(2m)\\ P_{k,2n+3}(2m) &= 2P_{k,2n+3-2m}(2m) + kP_{k,2n+1}(2m)\\ &= 2P_{k,2(n+1-m)+1}(2m) + kP_{k,2n+1}(2m).\end{aligned}$$

So, we obtain the desired result because

$$P_{k,2n+1}(2m) = P_{k,2n}(2m) \longrightarrow P_{k,2(n+1-m)+1}(2m) = P_{k,2(n+1-m)}(2m).$$ □

Theorem 2.3. $\sum_{j=1}^{n} k^{\frac{n-j}{2}}\Big(P_{k,j}(r)+P_{k,j-1}(r)\Big) = \dfrac{P_{k,n+r}(r)+P_{k,n+r-1}(r)-2k^{(n+1)/2}}{2}.$

Proof. $P_{k,n+r}(r)+P_{k,n+r-1}(r)$

$$\begin{aligned}
&= 2P_{k,n}(r)+kP_{k,n+r-2}(r)+2P_{k,n-1}(r)+kP_{k,n+r-3}(r)\\
&= 2P_{k,n}(r)+2P_{k,n-1}(r)+k\Big(2P_{k,n-2}(r)+kP_{k,n+r-4}(r)\Big)\\
&\quad + k\Big(2P_{k,n-3}(r)+kP_{k,n+r-5}(r)\Big)\\
&= 2P_{k,n}(r)+2P_{k,n-1}(r)+2kP_{k,n-2}(r)+2kP_{k,n-3}(r)\\
&\quad + k^2\Big(2P_{k,n-4}(r)+kP_{k,n+r-6}(r)\Big)+k^2\Big(2P_{k,n-5}(r)+kP_{k,n+r-7}(r)\Big)\\
&= 2P_{k,n}(r)+2P_{k,n-1}(r)+2kP_{k,n-2}(r)+2kP_{k,n-3}(r)+2k^2P_{k,n-4}(r)\\
&\quad + 2k^2P_{k,n-5}(r)+k^3P_{k,n+r-6}(r)+k^3P_{k,n+r-7}(r)\\
&= 2\Big(P_{k,n}(r)+P_{k,n-1}(r)+kP_{k,n-2}(r)+kP_{k,n-3}(r)+k^2P_{k,n-4}(r)\\
&\qquad + k^2P_{k,n-5}(r)+k^3P_{k,n-6}(r)+k^3P_{k,n-7}(r)+\cdots+k^{(n-1)/2}P_{k,1}(r)\\
&\qquad + k^{(n-1)/2}P_{k,0}(r)+k^{(n+1)/2}P_{k,r-1}(r)+k^{(n+1)/2}P_{k,r-2}(r)\Big)\\
&= 2\sum_{j=1}^{n}\Big(P_{k,j}(r)k^{(n-j)/2}+P_{k,j-1}(r)k^{(n-j)/2}\Big)+2k^{(n+1)/2}\\
&= P_{k,n+r}(r)+P_{k,n+r-1}(r) = \sum_{j=1}^{n}k^{(n-j)/2}\Big(P_{k,j}(r)+P_{k,j-1}(r)\Big)\\
&= \frac{P_{k,n+r}(r)+P_{k,n+r-1}(r)-2k^{(n+1)/2}}{2}.
\end{aligned}$$

As is desired. □

Theorem 2.4.

$$\sum_{j=0}^{n} k^{n-j}P_{k,2j}(r) = \frac{P_{k,2n+r}(r)-k^{n+1}}{2},$$

$$\sum_{j=0}^{n} k^{n-j}P_{k,2j+1}(r) = \frac{P_{k,2n+r+1}(r)-k^{n+1}}{2}.$$

Proof. Let us prove the first formula. From Equation (1), we obtain

$$\begin{aligned}
P_{k,2n+r}(r) &= 2P_{k,2n}(r)+kP_{k,2n+r-2}(r)\\
&= 2P_{k,2n}(r)+k\Big(2P_{k,2n-2}(r)+kP_{k,2n+r-4}(r)\Big)\\
&= 2P_{k,2n}(r)+2kP_{k,2n-2}(r)+k^2\Big(2P_{k,2n-4}(r)+kP_{k,2n+r-6}(r)\Big)\\
&= 2P_{k,2n}(r)+2kP_{k,2n-2}(r)+2k^2P_{k,2n-4}(r)+\cdots+2k^nP_{k,0}(r)+k^{n+1}P_{k,r-2}(r)\\
&= 2\sum_{j=0}^{n}k^{n-j}P_{k,2j}(r)+k^{n+1} = P_{k,2n+r}(r)\\
&= \sum_{j=0}^{n}k^{n-j}P_{k,2j}(r) = \frac{P_{k,2n+r}(r)-k^{n+1}}{2}.
\end{aligned}$$

Similarly, let us prove the second formula:

$$
\begin{aligned}
P_{k,2n+r+1}(r) &= 2P_{k,2n+1}(r) + kP_{k,2n+r-1}(r) \\
&= 2P_{k,2n+1}(r) + k\Big(2P_{k,2n-1}(r) + kP_{k,2n+r-3}(r)\Big) \\
&= 2P_{k,2n+1}(r) + 2kP_{k,2n-1}(r) + k^2\Big(2P_{k,2n-3}(r) + kP_{k,2n+r-5}(r)\Big) \\
&= 2P_{k,2n+1}(r) + 2kP_{k,2n-1}(r) + 2k^2P_{k,2n-3}(r) + \cdots + 2k^nP_{k,1}(r) + k^{n+1}P_{k,r-1}(r) \\
&= 2\sum_{j=0}^{n} k^{n-j}P_{k,2j+1}(r) + k^{n+1} = P_{k,2n+r+1}(r) \\
&= \sum_{j=0}^{n} k^{n-j}P_{k,2j+1}(r) = \frac{P_{k,2n+r+1}(r) - k^{n+1}}{2}. \qquad \square
\end{aligned}
$$

Theorem 2.5. *The generating function $p_k(r,x)$ for the sequence $\{P_{k,n}(r)\}$ is given by*

$$p_k(r,x) = \frac{1+x}{1-kx^2-2x^r}.$$

Proof.

$$
\begin{aligned}
\sum_{n=0}^{\infty} P_{k,n}(r)x^n &= P_{k,0}(r) + P_{k,1}(r)x + \sum_{n=2}^{\infty}\Big(2P_{k,n-r}(r) + kP_{k,n-2}(r)\Big)x^n \\
&= 1 + x + 2x^r\sum_{n=2}^{\infty} P_{k,n-r}(r)x^{n-r} + kx^2\sum_{n=2}^{\infty} P_{k,n-2}(r)x^{n-2} \\
&= 1 + x + 2x^r\sum_{p=0}^{\infty} P_{k,p}(r)x^p + kx^2\sum_{s=0}^{\infty} P_{k,s}(r)x^s,
\end{aligned}
$$

so that

$$(1-kx^2-2x^r)\textstyle\sum_{n=0}^{\infty} P_{k,n}(r)x^n = 1+x,$$

as desired. $\square$

Finally, considering that for $r \geq 2$ it is $P_{k,0}(r) = P_{k,1}(r) = 1$, we find the following generating functions.

Now, we give the graphs of the generating functions of the different (k,r)-Pell numbers in Fig. 1, Fig. 2 and Fig. 3. For $r=1$, we obtain the graph of $p_k(1,x) = \frac{1+x}{1-kx^2-2x}$ like Fig. 1.

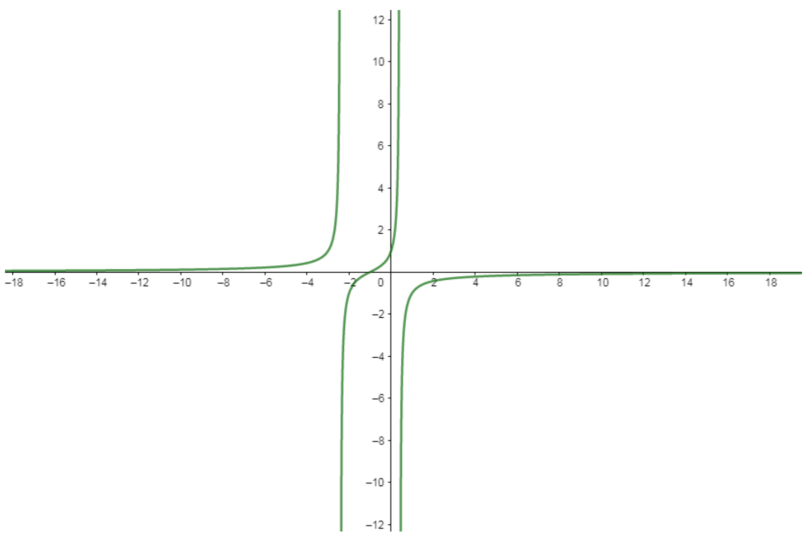

Figure 1. Generating function of $P_k(1)$

When r is an even number, we get Fig. 2 as r increases, the minimum and maximum values of this curve tend to $(-1^{+}, 0)$ and $(-1^{-}, 0)$ respectively.

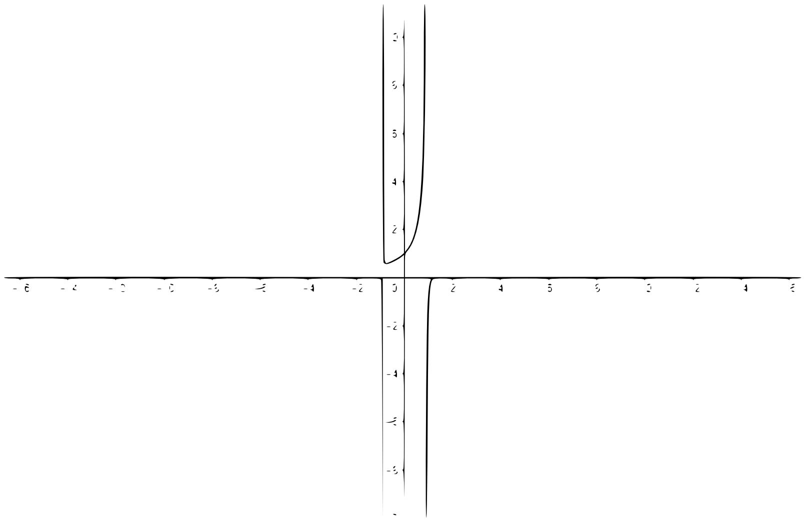

Figure 2. Generating function of $P_k(2m)$

If r is an odd number and $r \geq 1$, we have Fig. 3 as r increases, the minimum value of this curve tends to $(-1, 0)$.

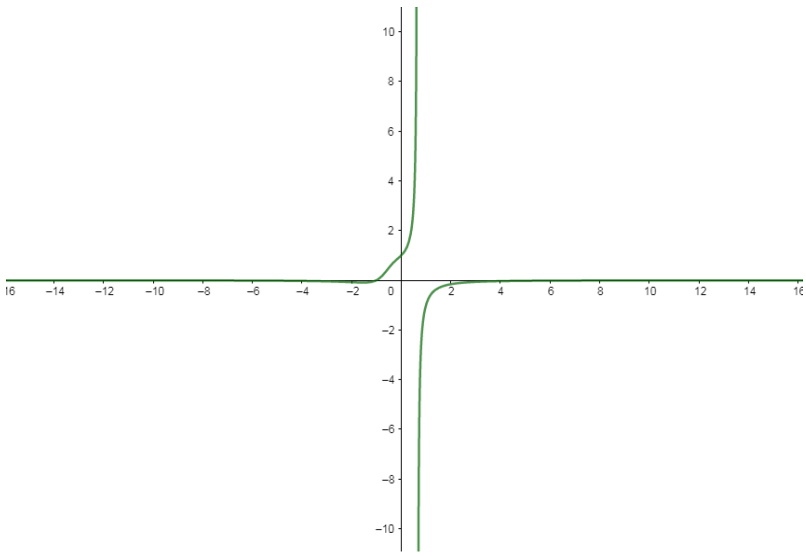

Figure 3. Generating function of $P_k(2m+1)$

3 Generalized (k, r)-Pell–Lucas numbers

Definition 3.1. *For the natural numbers $k \geq 1, n \geq 0$ and $r \geq 1$, the generalized (k, r)-Pell–Lucas numbers $Q_{k,n}(r)$ are defined by*

$$Q_{k,n}(r) = 2Q_{k,n-r}(r) + kQ_{k,n-2}(r) \quad \textit{for} \quad n \geq r \tag{2}$$

with $Q_{k,n}(r) = 2$ for $n = 0, 1, 2, 3, \ldots, r-1$ and $Q_{k,1}(1) = 2k$.

Let $Q_k(r) = \{Q_{k,n}(r) : n \in \mathbf{N}\}$. For some r, the following values are obtained.

$$Q_k(1) = \{2, 2k, 6k, 2k^2 + 12k, 10k^2 + 24k, 2k^3 + 32k^2 + 48k, 14k^3 + 88k^2 + 96k, \ldots\}$$
$$Q_k(2) = \{2, 2, 2k + 4, 2k + 4, 2(k + 2)^2, 2(k + 2)^2, 2(k + 2)^3, 2(k + 2)^3, 2(k + 2)^4, 2(k + 2)^4, \ldots\}$$
$$Q_k(3) = \{2, 2, 2, 2k + 4, 2k + 4, 2k^2 + 4k + 4, 2(k + 2)2, 2k^3 + 4k^2 + 8k + 8, \ldots\}$$

Now, let the generalized (k, r)-Pell–Lucas numbers $Q_{k,n}(r)$ particularize for $k = 1, 2, 3, \ldots$. For $k = 1$, we get the sequences in Table 5. For $Q_{1,n}(1)$ if we take $2, 2, 6, 14, 34, 82, \ldots$, respectively, we get the Pell–Lucas numbers in Table 5. For $k = 2$, we get the sequences in Table 6. For $k = 3$, we get the sequences in Table 7.

Table 5. $Q_{1,n}(r)$ for some r

n	0	1	2	3	4	5	6	7
$Q_{1,n}(1)$	2	2	6	14	34	82	198	478
$Q_{1,n}(2)$	2	2	6	6	18	18	54	54
$Q_{1,n}(3)$	2	2	2	6	6	10	18	22
$Q_{1,n}(4)$	2	2	2	2	6	6	10	10
$Q_{1,n}(5)$	2	2	2	2	2	6	6	10

Table 6. $Q_{2,n}(r)$ for some r

n	0	1	2	3	4	5	6	7
$Q_{2,n}(1)$	2	4	12	32	44	240	656	1792
$Q_{2,n}(2)$	2	2	8	8	32	32	128	128
$Q_{2,n}(3)$	2	2	2	8	8	20	32	56
$Q_{2,n}(4)$	2	2	2	2	8	8	20	20
$Q_{2,n}(5)$	2	2	2	2	2	8	8	20

Table 7. $Q_{3,n}(r)$ for some r

n	0	1	2	3	4	5	6	7
$Q_{3,n}(1)$	2	6	18	54	162	486	1458	4374
$Q_{3,n}(2)$	2	2	10	10	50	50	250	250
$Q_{3,n}(3)$	2	2	2	10	10	34	50	122
$Q_{3,n}(4)$	2	2	2	2	10	10	34	34

It is worth noting that only nine (k, r)-Pell–Lucas sequences are referenced in The On-Line Encyclopedia of Integer Sequences [28] with the numbers given in Table 8. $Q_{1,n}(2)$, $Q_{1,n}(4)$, $Q_{2,n}(2), Q_{2,n}(4)$ and $Q_{3,n}(2), Q_{3,n}(4)$ are double sequences.

Table 8. (k, r)-Pell–Lucas sequences for $k = 1, 2, 3$

$Q_{1,n}(1)$	A002203
$Q_{1,n}(2)$	A117855
$Q_{1,n}(4)$	A230096
$Q_{2,n}(1)$	A028860
$Q_{2,n}(2)$	A158302
$Q_{3,n}(1)$	A008776

Theorem 3.1. *Let* $r \geq 2$. *Then* $Q_{k,r}(r) = 2k + 4$.

Theorem 3.2. *If* r *is an even number, we have* $Q_{k,2n}(2m) = Q_{k,2n+1}(2m)$.

Theorem 3.3. $\sum_{j=1}^{n} k^{\frac{n-j}{2}} \left(Q_{k,j}(r) + Q_{k,j-1}(r) \right) = \frac{Q_{k,n+r}(r) + Q_{k,n+r-1}(r) - 4k^{(n+1)/2}}{2}$.

Theorem 3.4.

$$\sum_{j=0}^{n} k^{n-j} Q_{k,2j}(r) = \frac{Q_{k,2n+r}(r) - 2k^{n+1}}{2},$$

$$\sum_{j=0}^{n} k^{n-j} Q_{k,2j+1}(r) = \frac{Q_{k,2n+r+1}(r) - 2k^{n+1}}{2}.$$

Theorem 3.5. *The generating function* $q_k(r, x)$ *for* $Q_k(r) = \{Q_{k,n}(r)\}$ *is given by*

$$q_k(r, x) = \frac{2 + 2x}{1 - kx^2 - 2x^r}.$$

Now, we give the graphs of the generating functions of the different (k, r)-Pell–Lucas numbers in Fig. 4, Fig. 5 and Fig. 6. For $r = 1$, we obtain the graph of $q_k(1, x) = \frac{2 + 2x}{1 - kx^2 - 2x}$ like Fig. 4.

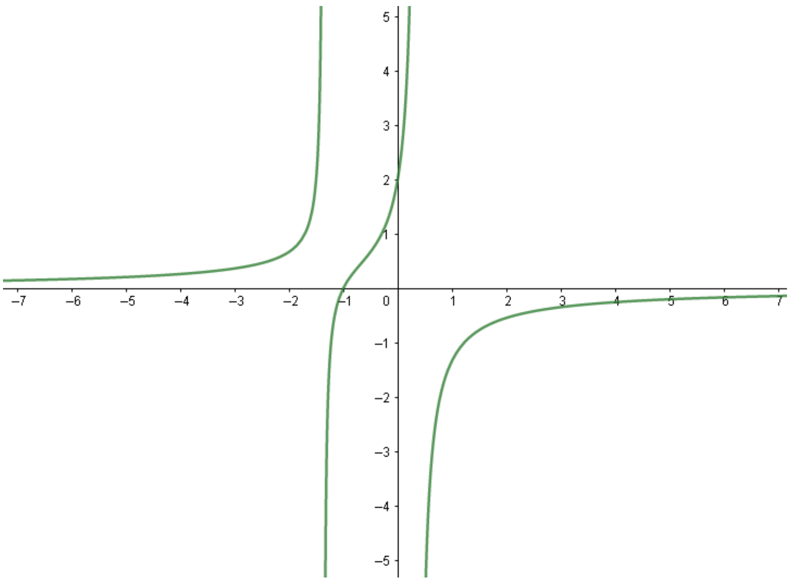

Figure 4. Generating function of $Q_k(1)$

When r is even number, we get Fig. 5 as r increases, the minimum and maximum values of this curve tend to $(-1^+, 0)$ and $(-1^-, 0)$, respectively.

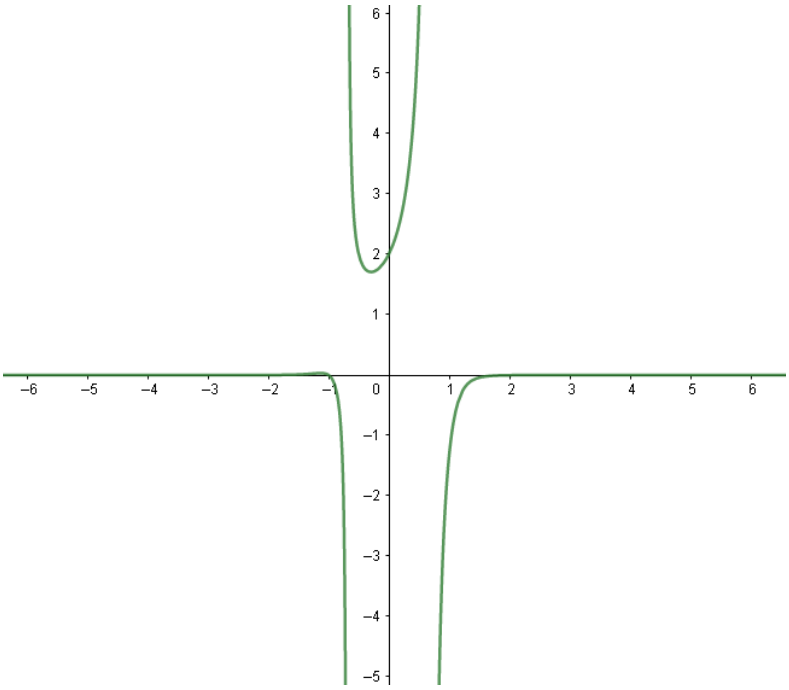

Figure 5. Generating function of $Q_k(2m)$

If r is odd number and $r \geq 1$, we have Fig. 6 as r increases, the minimum value of this curve tends to $(-1, 0)$.

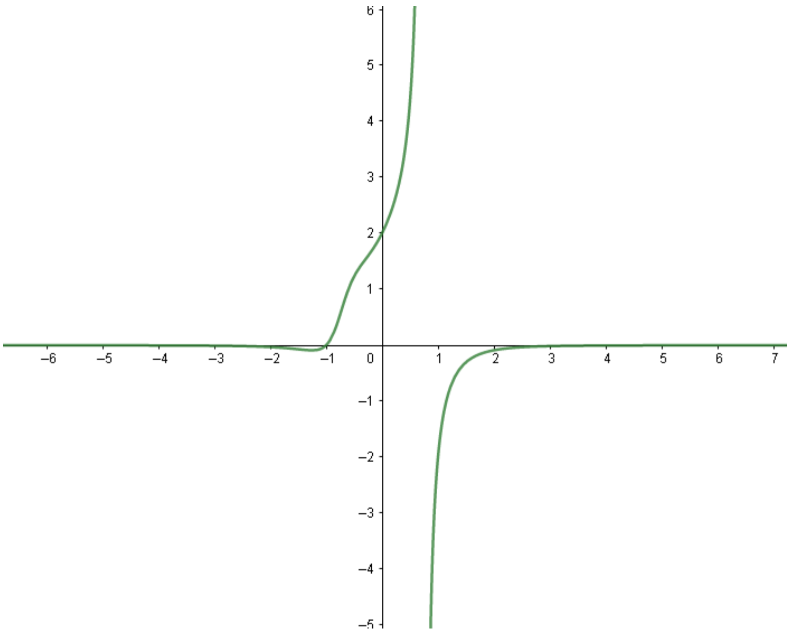

Figure 6. Generating function of $Q_k(2m+1)$

4 Conclusion

New generalized (k, r)-Pell and (k, r)-Pell–Lucas numbers have been introduced and studied. Several properties and theorems for these numbers were deduced. In addition, it was given the generating functions, some important identities and sum of the terms of the generalized (k, r)-Pell and (k, r)-Pell–Lucas.

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