Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2022, Volume 28, Number 4, 758–764
DOI: 10.7546/nntdm.2022.28.4.758-764

On combined 3-Fibonacci sequences

Krassimir T. Atanassov[1,2], Lilija C. Atanassova[3] and Anthony G. Shannon[4]

[1] Department of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences
"Acad. G. Bonchev" Str. Bl. 105, Sofia 1113, Bulgaria
e-mail: krat@bas.bg

[2] Intelligent Systems Laboratory, "Prof. Asen Zlatarov" University
1 "Prof. Yakimov" Blvd., Burgas 8010, Bulgaria

[3] Institute of Information and Communication Technologies,
Bulgarian Academy of Sciences
"Acad. G. Bonchev" Str., Bl. 2, Sofia 1113, Bulgaria
e-mail: l.c.atanassova@gmail.com

[4] Warrane College, The University of New South Wales
Kensington, NSW 2033, Australia
email: tshannon38@gmail.com

Received: 20 July 2022 **Revised:** 27 October 2022
Accepted: 14 November 2022 **Online First:** 16 November 2022

Abstract: The term 'combined' sequence includes any of the 'coupled', 'intercalated' and 'pulsated' sequences. In this paper, $k = 3$, so new combined 3-Fibonacci sequences, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, are introduced and the explicit formulae for their general terms are developed. That is, there are three such sequences, each with a linear recurrence relation which contains terms from the other two. In effect, each such recurrence relation is second order, with two initial terms which specify the subsequent delineation of the terms of the sequences. The initial terms are, respectively, $\langle \alpha_0, \alpha_1 \rangle = \langle 2a, 2d \rangle$, $\langle \beta_0, \beta_1 \rangle = \langle b, e \rangle$ and $\langle \gamma_0, \gamma_1 \rangle = \langle 2c, 2f \rangle$ in turn. These result in neat inter-relationships among the three sequences, which can lead to intriguing connections with known sequences, and to a surprisingly simple graphical representation of the whole process. The references include a comprehensive cover of the pertinent literature on these aspects of recursive sequences particularly during the last seventy years.

A secondary goal of the paper is to put the disarray of this part of number theory into some semblance of order with a selection of representative references. This gives rise to a 'combobulated sequence', so-called because it restores partial order to a disarray of many papers into three classes, which are fuzzy in both their membership and non-membership because of their diverse and non-systematic derivations.
Keywords: Fibonacci sequence, 3-Fibonacci sequence, Combobulated sequence.
2020 Mathematics Subject Classification: 11B39.

1 Introduction

The first extension of the Fibonacci sequence in the direction of increasing of the number of the sequences was described in [2, 10] (see, also [11]). There, two parallel Fibonacci sequences were introduced. Later, these research efforts were extended to three parallel sequences, to two-Tribonacci sequences, etc. In general, this research can be separated into three classes of a "Combobulated Sequence", so-called because it restores partial order to a disarray of many papers into three classes. Some of the sequences in different classes have common properties because they were originally developed by diverse authors with different aims:

- Coupled sequences – see [2–17, 22, 34–39];
- Diatomic sequences – see [1, 18–21, 29–31, 41, 42];
- Equivalence classes – see [23–28, 32, 33, 40, 43].

In [12–14, 19], seven different combined 3-Fibonacci sequences were introduced.

Here, we continue this direction of research, introducing new 3-Fibonacci sequences that are different from the previous ones. So, the series of extensions of the nature of the Fibonacci sequence (see, e.g., [11]) is continued.

2 Main results

Let everywhere below, a, b, c, d, e, f be arbitrary real numbers. The new 3-Fibonacci sequences have the form:

$$\alpha_0 = 2a,\ \beta_0 = b,\ \gamma_0 = 2c,\ \alpha_1 = 2d,\ \beta_1 = e,\ \gamma_1 = 2f$$

and for each natural number $n \geq 1$:

$$\begin{aligned}\alpha_{n+1} &= \beta_n + \gamma_{n-1},\\ \beta_{n+1} &= \frac{\alpha_n + \gamma_n}{2} + \beta_{n-1},\\ \gamma_{n+1} &= \beta_n + \alpha_{n-1}.\end{aligned}$$

The first values of sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are given in the following Table 1.

Table 1. First values of sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$

n	α_n	β_n	γ_n
0	$2a$	b	$2c$
1	$2d$	e	$2f$
2	$2c+e$	$b+d+f$	$2a+e$
3	$b+d+3f$	$a+c+2e$	$b+3d+f$
4	$3a+c+3e$	$2b+3d+3f$	$a+3c+3e$
5	$3b+6d+4f$	$3a+3c+5e$	$3b+4d+6f$
6	$4a+6c+8e$	$5b+8d+8f$	$6a+4c+8e$
7	$8b+12d+14f$	$8a+8c+13e$	$8b+14d+12f$
8	$14a+21c+21e$	$13b+21d+21f$	$12a+14c+21e$
9	$21b+35d+33f$	$21a+21c+34e$	$21b+33d+35f$
10	$33a+35c+55e$	$34b+55d+55f$	$35a+33c+55e$
	$\vdots$	$\vdots$	$\vdots$

Graphically, we can represent the scheme of their construction as shown below (Figure 1).

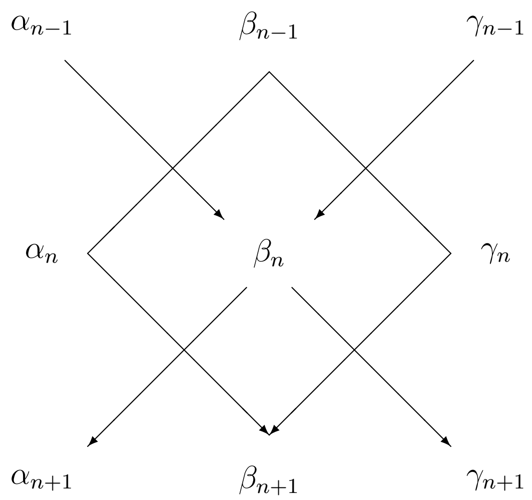

Figure 1. Scheme of construction of the sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$

Let $\{F_n\}_{n=0}^{\infty}$ be the standard Fibonacci sequence, where $F_0 = 0, F_1 = 1$, and

$$F_{n+2} = F_{n+1} + F_n$$

for each natural number $n \geq 0$.

Theorem 1. *For each natural number $n \geq 1$:*

$$\begin{aligned}
\alpha_{4n} &= (F_{4n-1}+1)a + (F_{4n-1}-1)c + F_{4n}e,\\
\beta_{4n} &= F_{4n-1}b + F_{4n}d + F_{4n}f,\\
\gamma_{4n} &= (F_{4n-1}-1)a + (F_{4n-1}+1)c + F_{4n}e,
\end{aligned}$$

$$\begin{aligned}
\alpha_{4n+1} &= F_{4n}b + (F_{4n+1}+1)d + (F_{4n+1}-1)f,\\
\beta_{4n+1} &= F_{4n}a + F_{4n}c + F_{4n+1}e,\\
\gamma_{4n+1} &= F_{4n}b + (F_{4n+1}-1)d + (F_{4n+1}+1)f,
\end{aligned}$$

$$\begin{aligned}
\alpha_{4n+2} &= (F_{4n+1}-1)a + (F_{4n+1}+1)c + F_{4n+2}e,\\
\beta_{4n+2} &= F_{4n+1}b + F_{4n+2}d + F_{4n+2}f,\\
\gamma_{4n+2} &= (F_{4n+1}+1)a + (F_{4n+1}-1)c + F_{4n+2}e,
\end{aligned}$$

$$\begin{aligned}
\alpha_{4n+3} &= F_{4n+2}b + (F_{4n+3}-1)d + (F_{4n+3}+1)f,\\
\beta_{4n+3} &= F_{4n+2}a + F_{4n+2}c + F_{4n+3}e,\\
\gamma_{4n+3} &= F_{4n+2}b + (F_{4n+3}+1)d + (F_{4n+3}-1)f.
\end{aligned}$$

Proof. We can prove the Theorem, for example, by induction. For $n = 1$, the validity of the Theorem is checked directly from the above table. Let us assume that the Theorem is valid for some natural number $n \geq 1$. Then:

$$\begin{aligned}
\alpha_{4n+4} &= \beta_{4n+3} + \gamma_{4n+2}\\
&= F_{4n+2}a + F_{4n+2}c + F_{4n+3}e + (F_{4n+1}+1)a + (F_{4n+1}-1)c + F_{4n+2}e\\
&= (F_{4n+3}+1)a + (F_{4n+3}-1)c + F_{4n+4}e.
\end{aligned}$$

$$\begin{aligned}
\beta_{4n+4} &= \frac{\alpha_{4n+3}+\gamma_{4n+3}}{2} + \beta_{4n+2}\\
&= \frac{F_{4n+2}b + (F_{4n+3}-1)d + (F_{4n+3}+1)f + F_{4n+2}b + (F_{4n+3}+1)d + (F_{4n+3}-1)f}{2}\\
&\quad + F_{4n+1}b + F_{4n+2}d + F_{4n+2}f\\
&= \frac{2F_{4n+2}b + (2(F_{4n+3}-1+1)d + (2F_{4n+3}+1-1)f}{2} + F_{4n+1}b + F_{4n+2}d + F_{4n+2}f\\
&= F_{4n+2}b + F_{4n+3}d + F_{4n+3}f + F_{4n+1}b + F_{4n+2}d + F_{4n+2}f\\
&= F_{4n+3}b + F_{4n+4}d + F_{4n+4}f.
\end{aligned}$$

$$\begin{aligned}
\gamma_{4n+4} &= \beta_{4n+3} + \alpha_{4n+2}\\
&= F_{4n+2}a + F_{4n+2}c + F_{4n+3}e + (F_{4n+1}-1)a + (F_{4n+1}+1)c + F_{4n+2}e\\
&= (F_{4n+3}-1)a + (F_{4n+3}+1)c + F_{4n+4}e.
\end{aligned}$$

The rest formulas are checked by analogy. □

3 Conclusion

In this paper, new combined 3-Fibonacci sequences of a new type have been introduced and explicit formulas for their elements have been developed algebraically and demonstrated graphically. Some readers may be tempted to relate these sequences to better-known standard sequences. In their future research, the authors plan to modify the standard forms of the 2- and 3-Fibonacci sequences.

References

[1] Ando, S., & Hayashi, M. (1997). Counting the number of equivalence classes of (m, F) sequences and their generalizations. *The Fibonacci Quarterly*, 35(1), 3–8.

[2] Atanassov, K. T. (1986). On a second new generalization of the Fibonacci sequence. *The Fibonacci Quarterly*, 24(4), 362–365.

[3] Atanassov, K. T. (1995). Remark on a new direction for a generalization of the Fibonacci sequence. *The Fibonacci Quarterly*, 33(3), 249–250.

[4] Atanassov, K. T. (2013). Pulsating Fibonacci sequence. *Notes on Number Theory and Discrete Mathematics*, 19(3), 12–14.

[5] Atanassov, K. T. (2013). Pulsated Fibonacci sequence. Part 2. *Notes on Number Theory and Discrete Mathematics*, 19(4), (2013), 33–36.

[6] Atanassov, K. T. (2014). n–pulsated Fibonacci sequence. *Notes on Number Theory and Discrete Mathematics*, 20(1), 32–35.

[7] Atanassov, K. T. (2018). On two new combined 3-Fibonacci sequences. *Notes on Number Theory and Discrete Mathematics*, 24(2), 90–93.

[8] Atanassov, K. T. (2018). On two new combined 3-Fibonacci sequences. Part 2. *Notes on Number Theory and Discrete Mathematics*, 24(3), 111–114.

[9] Atanassov, K. T. (2022). On two new combined 3-Fibonacci sequences. Part 3. *Notes on Number Theory and Discrete Mathematics*, 28(1), 143–146.

[10] Atanassov, K. T., Atanassova, L. C., & Sasselov, D. D. (1985). A new perspective to the generalization of the Fibonacci sequence. *The Fibonacci Quarterly*, 23(1), 21–28.

[11] Atanassov, K. T., Atanassova, V. K., Shannon, A. G., & Turner, J. C. (2002). *New Visual Perspectives on Fibonacci Numbers*. World Scientific, New Jersey.

[12] Atanassov, K. T., Deford, D. R., & Shannon, A. G. (2015). Pulsated Fibonacci recurrences. In: *Anderson, P. G., Ballot, C., & Webb, W. (Eds). Proceedings of the Sixteenth International Conference on Fibonacci Numbers and Their Applications*. *The Fibonacci Quarterly*, 52(5), 22–27.

[13] Atanassov, K. T., Hlebarova, J., & Mihov, S. (1992). Recurrent formulas of the generalized Fibonacci and Tribonacci sequences. *The Fibonacci Quarterly*, 30(1), 77–79.

[14] Atanassov, K. T., & Shannon, A. G. (2016). Combined 3-Fibonacci sequences from a new type. *Notes on Number Theory and Discrete Mathematics*, 22(3), 5–8.

[15] Atanassov, K. T., & Shannon, A. G. (2020). On intercalated Fibonacci sequences. *Notes on Number Theory and Discrete Mathematics*, 26(3), 218–223.

[16] Atanassov, K. T., Shannon, A. G., & Turner, J. C. (1995). The generation of trees from coupled third order recurrence relations. In: *Shtrakov, S., & Mirchev, I. (Eds.) Discrete Mathematics and Applications*. Neofit Rilski University, Blagoevgrad, 46–56.

[17] Badshah, V., & Khan, I. (2009). New generalization of the Fibonacci sequence in case of 4-order recurrence equations. *International Journal of Theoretical & Applied Sciences*, 1(2), 93–96.

[18] Bicknell-Johnson, M. (2003). Stern diatomic array applied to Fibonacci representations. *The Fibonacci Quarterly*, 41(2), 169–180.

[19] Coons, M. (2018). Proof of Northshield's conjecture concerning an analogue of Stern's sequence for $\mathbb{Z}[\sqrt{2}]$. *Australasian Journal of Combinatorics*, 71(1), 113–120.

[20] Coons, M., & Shallit, J. (2011). Pattern sequence approach to Stern's sequence. *Discrete Mathematics*, 311, 2630–2633.

[21] Frankl, P., & Wilson, R. M. (1981). Intersection theory with geometric consequences. *Combinatorica*, 1(4), 357–368.

[22] Georgiev, P. G., & Atanassov, K. T. (1996). On one generalization of the Fibonacci sequence. Part IV: Multiplicity roots of the characteristic equation. *Notes on Number Theory and Discrete Mathematics*, 2(4), 3–7.

[23] Gould, H. W. (1965). Non-Fibonacci numbers. *The Fibonacci Quarterly*, 3(3), 177–183.

[24] Gould, H. W. (1968). Up-down permutations. *S.I.A.M. Review*, 10, 225–226.

[25] Horadam, A. F. (1966). Generalization of two theorems of K. Subba Rao. *Bulletin of the Calcutta Mathematical Society*, 58(1), 23–29.

[26] Kiss, P. (1982). On common terms of linear recurrences. *Acta Mathematica Academiae Scientiarum Hungarica*, 40(1/2), 119–123.

[27] Krishna, H. V. (1973). A note on number quartets. *The Mathematics Student*, 40(1), 1.

[28] Laxton, R. R. (1974). On a problem of M. Ward. *The Fibonacci Quarterly*, 12(1), 41–44.

[29] Lee, J.-Z., & Lee, J.-S. (1987). Some properties of the generalization of the Fibonacci sequence. *The Fibonacci Quarterly*, 25(2), 111–117.

[30] Lehmer, D. H. (1929). On Stern's diatomic sequence. *American Mathematical Monthly*, 36(1), 59–67.

[31] Northshield. S. (2015). Three analogues of Stern's diatomic sequence. In: Anderson, P. G., Ballot, C., & Webb, W. (Eds.). *Proceedings of the Sixteenth International Conference on Fibonacci Numbers and Their Applications*, *The Fibonacci Quarterly*, 52(5), 168–186.

[32] Rao, K. S. (1954). Some properties of the Fibonacci numbers–I. *Bulletin of the Calcutta Mathematical Society*, 46, 253–257.

[33] Rao, K. S. (1959). Some properties of the Fibonacci numbers–II. *The Mathematics Student*, 27(1), 19–23.

[34] Sedgewick, R., & Flajot, F. (2013). *An Introduction to the Analysis of Algorithms*. Addison-Wesley, Boston, MA, Ch.2.

[35] Shannon, A. G. (1983). Intersections of second order linear recursive sequences. *The Fibonacci Quarterly*, 21(1), 6–12.

[36] Shannon, A. G., Turner, J. C., & Atanassov, K. T. (1991). A generalized tableau associated with coloured convolution trees. *Discrete Mathematics*, 92, 329–340.

[37] Spickerman, W., & Creech, R. (1997). The $(2, T)$ generalized Fibonacci sequences. *The Fibonacci Quarterly*, 35(4), 358–360.

[38] Spickerman, W., Creech, R., & Joyner, R. (1992). On the $(2, F)$ generalizations of the Fibonacci sequence. *The Fibonacci Quarterly*, 30(4), 310–314.

[39] Spickerman, W., Creech, R., & Joyner, R. (1993). On the structure of the set of difference systems defining the $(3, F)$ generalized Fibonacci sequence. *The Fibonacci Quarterly*, 31(4), 333–337.

[40] Spickerman, W., Creech, R., & Joyner, R. (1995). On the $(3, F)$ generalizations of the Fibonacci sequence. *The Fibonacci Quarterly*, 33(1), 9–12.

[41] Stein, M. A. (1858). Über eine zahlentheoretische Funktion. *Journal für die reine und angewandte Mathematik*, 55, 193–220.

[42] Stein, S. K. (1962). The intersection of Fibonacci sequences. *Michigan Mathematics Journal*, 9(4), 399–402.

[43] Ward, M. (1955). The laws of apparition and repetition of primes in a cubic recurrence. *Transactions of the American Mathematical Society*, 79(1), 72–90.