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# The integrality of the Genocchi numbers obtained through a new identity and other results

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**Abstract:** In this note, we investigate some properties of the integer sequence of general term $a_n := \sum_{k=0}^{n-1} k!(n-k-1)!$ ($\forall n \geq 1$) to derive a new identity of the Genocchi numbers $G_n$ ($n \in \mathbb{N}$), which immediately shows that $G_n \in \mathbb{Z}$ for any $n \in \mathbb{N}$. In another direction, we obtain nontrivial lower bounds for the 2-adic valuations of the rational numbers $\sum_{k=1}^{n} \frac{2^k}{k}$.
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## 1 Introduction and Notation

Throughout this note, we let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ denote the set of non-negative integers. For $x \in \mathbb{R}$, we let $\lfloor x \rfloor$ denote the integer part of $x$. Let $s(n, k)$ and $S(n, k)$ (with $n, k \in \mathbb{N}_0$, $n \geq k$) respectively denote the Stirling numbers of the first and second kinds, which can be defined as the integer coefficients appearing in the polynomial identities:

$$\begin{aligned} X(X-1)\cdots(X-n+1) &= \sum_{k=0}^{n} s(n,k) X^k, \\ X^n &= \sum_{k=0}^{n} S(n,k) X(X-1)\cdots(X-k+1) \end{aligned} \qquad \text{(for every } n \in \mathbb{N}_0\text{)}.$$

This immediately implies the orthogonality relations (see, e.g., [1,8]):

$$\sum_{k\leq i\leq n} s(n,i)S(i,k) = \sum_{k\leq i\leq n} S(n,i)s(i,k) = \delta_{nk} \qquad \text{(for every } n,k\in\mathbb{N}_0, n\geq k), \tag{1.1}$$

where $\delta_{nk}$ is the Kronecker delta. Among the many formulas related to the Stirling numbers, we mention the following result (see, e.g., [1,6,8]):

$$\frac{\log^k(1+x)}{k!} = \sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!} \qquad \text{(for every } k\in\mathbb{N}_0), \tag{1.2}$$

which is needed later on. We let in addition $B_n$ and $G_n$ ($n \in \mathbb{N}_0$) respectively denote the Bernoulli and the Genocchi numbers, which can be defined by their respective exponential generating functions (see, e.g., [1,8]):

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n\frac{x^n}{n!} \quad \text{and} \quad \frac{2x}{e^x+1} = \sum_{n=0}^{\infty} G_n\frac{x^n}{n!}. \tag{1.3}$$

The famous Genocchi theorem [7] states that the $G_n$'s are all integers. There are at least two beautiful proofs of the Genocchi theorem: the first one uses the formula $G_n = 2(1-2^n)B_n$ (see, e.g., [1]) together with the Fermat little theorem and the von Staudt-Clausen theorem, while the second one uses the remarkable Seidel formula [13]:

$$\sum_{k=0}^{n}\binom{n}{k}G_{n+k} = 0 \qquad \text{(for every } n\in\mathbb{N}_0).$$

In this note, we give a new proof of the integrality of the $G_n$'s by expressing them in terms of the Stirling numbers of the second kind. The starting point of this research is the study of the integer sequence $(a_n)_{n\in\mathbb{N}_0}$, defined by:

$$a_0 = 0 \ \text{ and } \ a_n := \sum_{k=0}^{n-1} k!(n-k-1)! \quad \text{(for every } n\in\mathbb{N}). \tag{1.4}$$

This sequence is closely related to the sum of the inverses of binomial coefficients, which is studied by several authors (see [10,12,16–18,20,21]). It must be noted that both Stirling numbers, Genocchi numbers, and the numbers $a_n$ ($n \in \mathbb{N}_0$) have combinatorial interpretations (see, e.g., [1, 15] for the Stirling numbers, [4, 19] for the Genocchi numbers, and the sequence $A003149$ of [14] for the $a_n$'s).

Next, the least common multiple of given positive integers $u_1, u_2, \ldots, u_n$ ($n \in \mathbb{N}$) is denoted by $\mathrm{lcm}(u_1, u_2, \ldots, u_n)$ or by $\mathrm{lcm}\{u_1, u_2, \ldots, u_n\}$ if this is more convenient. For a given prime number $p$ and a given positive integer $n$, we let $\vartheta_p(n)$ and $s_p(n)$ respectively denote the usual $p$-adic valuation of $n$ (that is the greatest $e \in \mathbb{N}_0$ satisfying $p^e \mid n$) and the sum of base-$p$ digits of $n$. The function $\vartheta_p$ ($p$ a prime) is naturally extended to $\mathbb{Q}^*$ by defining $\vartheta_p(\pm\frac{a}{b}) = \vartheta_p(a) - \vartheta_p(b)$, for any positive integers $a$ and $b$. With this extension, the function $\vartheta_p$ ($p$ a prime) satisfies several elementary properties; among them, we cite:

$$
\begin{aligned}
\vartheta_p(rs) &= \vartheta_p(r) + \vartheta_p(s) && \text{(for every } r, s \in \mathbb{Q}^*\text{)},\\
\vartheta_p\left(\frac{r}{s}\right) &= \vartheta_p(r) - \vartheta_p(s) && \text{(for every } r, s \in \mathbb{Q}^*\text{)},\\
\vartheta_p\left(\mathrm{lcm}(1, 2, \dots, n)\right) &= \left\lfloor \frac{\log n}{\log p} \right\rfloor && \text{(for every } n \in \mathbb{N}\text{)},\\
\vartheta_p(r) &\geq 0 && \text{(for every } r \in \mathbb{Z}^*\text{)}.
\end{aligned} \tag{1.5}
$$

Furthermore, a well-known formula of Legendre (see, e.g., [9, Theorem 2.6.4, page 77]) states that for any prime number $p$ and any positive integer $n$, we have

$$\vartheta_p(n!) = \frac{n - s_p(n)}{p - 1}. \tag{1.6}$$

In another direction, by leaning on Legendre's formula (1.6), an identity due to Rockett [12], and another identity due to the author [5], we obtain nontrivial lower bounds for the $2$-adic valuations of the rational numbers $\sum_{k=1}^{n} \frac{2^k}{k}$ $(n \in \mathbb{N})$.

# 2 The results and the proofs

Our main result is the following:

**Theorem 2.1.** *For all positive integer $n$, we have*

$$G_n = \sum_{1 \leq \ell \leq k \leq n} (-1)^{k-1} (\ell - 1)! (k - \ell)! S(n, k). \tag{2.1}$$

*In particular, $G_n$ is an integer for any $n \in \mathbb{N}$.*

To prove this theorem, we need some intermediary results. The first one (Proposotion 2.2 below) can be immediately derived from the following identity of Rockett [12]:

$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k} \qquad \text{(for every } n \in \mathbb{N}_0\text{)}. \tag{2.2}$$

But for convenience, we prefer reproduce its proof here.

**Proposition 2.2.** *For all positive integer $n$, we have*

$$a_n = \frac{n!}{2^n} \sum_{k=1}^{n} \frac{2^k}{k}. \tag{2.3}$$

*Proof.* We begin by establishing a recurrent formula for the sequence $(a_n)_n$. For any integer $n \geq 2$, we have:

$$
\begin{aligned}
a_n &:= \sum_{k=0}^{n-1} k!(n-k-1)! \\
&= \sum_{k=0}^{n-2} k!(n-k-1)! + (n-1)! \\
&= \sum_{k=0}^{n-2} k!(n-k-2)!(n-k-1) + (n-1)! \\
&= n\sum_{k=0}^{n-2} k!(n-k-2)! - \sum_{k=0}^{n-2}(k+1)!(n-k-2)! + (n-1)!.
\end{aligned}
$$

But since

$$\sum_{k=0}^{n-2} k!(n-k-2)! = a_{n-1}$$

and

$$
\begin{aligned}
\sum_{k=0}^{n-2}(k+1)!(n-k-2)! &= \sum_{\ell=1}^{n-1} \ell!(n-\ell-1)! \qquad \text{(by putting } \ell = k+1\text{)} \\
&= \sum_{\ell=0}^{n-1} \ell!(n-\ell-1)! - (n-1)! \\
&= a_n - (n-1)!,
\end{aligned}
$$

it follows that:

$$a_n = na_{n-1} - a_n + 2 \cdot (n-1)!\ .$$

Hence

$$a_n = \frac{n}{2}a_{n-1} + (n-1)!. \tag{2.4}$$

Further, we remark that Formula (2.4) also holds for $n = 1$. Now, according to Formula (2.4), we have for any positive integer $k$:

$$\frac{2^k}{k!}a_k - \frac{2^{k-1}}{(k-1)!}a_{k-1} = \frac{2^k}{k}.$$

Then by summing both sides of the last equality from $k = 1$ to $n$, we obtain (because the sum on the left is telescopic and $a_0 = 0$) that:

$$\frac{2^n}{n!}a_n = \sum_{k=1}^{n} \frac{2^k}{k},$$

which gives the required formula. The proof is achieved. □

**Corollary 2.3.** *The exponential generating function of the sequence $(a_n)_n$ is given by:*

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \frac{-2\log(1-x)}{2-x}. \tag{2.5}$$

*Proof.* Using Formula (2.3) of Proposition 2.2, we have

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \sum_{n=1}^{\infty} \left( \frac{1}{2^n} \sum_{k=1}^{n} \frac{2^k}{k} \right) x^n = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{1}{2^n} \frac{2^k}{k} x^n = \sum_{k=1}^{\infty} \frac{2^k}{k} \left( \sum_{n=k}^{\infty} \left( \frac{x}{2} \right)^n \right).$$

But since $\sum_{n=k}^{\infty} \left(\frac{x}{2}\right)^n = \left(\frac{x}{2}\right)^k \frac{1}{1-\frac{x}{2}} = \frac{x^k}{2^k} \cdot \frac{2}{2-x}$, we get

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \frac{2}{2-x} \sum_{k=1}^{\infty} \frac{x^k}{k} = \frac{2}{2-x} \left( -\log(1-x) \right),$$

as required. This achieves the proof. □

Next, from Corollary 2.3 and Formula (1.2), we derive the following corollary:

**Corollary 2.4.** *For any non-negative integer $n$, we have*

$$a_n = (-1)^{n-1} \sum_{k=0}^{n} G_k s(n,k). \tag{2.6}$$

*Proof.* Let us consider the following three functions (which are analytic on the neighborhood of zero):

$$f(x) := \frac{-2\log(1-x)}{2-x}, \quad g(x) := \frac{2x}{e^x+1}, \quad \text{and } h(x) := \log(1-x).$$

We easily check that $f = -g \circ h$. Since in addition $h(0) = 0$ then the power series expansion of $f$ about the origin can be obtained by substituting $h$ in the power series expansion of $g$ about the origin (which is given by (1.3)) and multiplying by $(-1)$. Doing so, we get

$$f(x) = -\sum_{k=0}^{\infty} G_k \frac{(h(x))^k}{k!} = -\sum_{k=0}^{\infty} G_k \frac{\log^k(1-x)}{k!}. \tag{2.7}$$

Further, by substituting in (1.2) $x$ by $(-x)$, we have for any $k \in \mathbb{N}_0$:

$$\frac{\log^k(1-x)}{k!} = \sum_{n=k}^{\infty} (-1)^n s(n,k) \frac{x^n}{n!}.$$

So, by inserting this last expression into (2.7), we get

$$\begin{aligned} f(x) &= -\sum_{k=0}^{\infty} G_k \sum_{n=k}^{\infty} (-1)^n s(n,k) \frac{x^n}{n!} \\ &= -\sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n G_k s(n,k) \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[ (-1)^{n-1} \sum_{k=0}^{n} G_k s(n,k) \right] \frac{x^n}{n!}. \end{aligned}$$

Comparing this last formula with Formula (2.5) of Corollary 2.3, we conclude that:

$$a_n = (-1)^{n-1} \sum_{k=0}^{n} G_k s(n,k) \qquad (\text{for every } n \in \mathbb{N}_0),$$

as required. □

We finally derive our main result from Corollary 2.4 above by applying the well-known inversion formula recalled in the following proposition.

**Proposition 1.** *Let* $(u_n)_{n\in\mathbb{N}_0}$ *and* $(v_n)_{n\in\mathbb{N}_0}$ *be two real sequences. Then the two following identities* $(I)$ *and* $(II)$ *are equivalent:*

$$u_n = \sum_{k=0}^{n} v_k s(n,k) \qquad (\textit{for every } n \in \mathbb{N}_0), \tag{$I$}$$

$$v_n = \sum_{k=0}^{n} u_k S(n,k) \qquad (\textit{for every } n \in \mathbb{N}_0). \tag{$II$}$$

*Proof.* Use the orthogonality relations (1.1) (see, e.g., [1] or [11] for the details). □

*Proof of Theorem 2.1.* It suffices to apply Proposition 1 for $u_n = (-1)^{n-1}a_n$ and $v_n = G_n$, for every $n \in \mathbb{N}_0$. In view of (2.6), Identity $(I)$ holds; so $(II)$ also, that is

$$G_n = \sum_{k=0}^{n} (-1)^{k-1} a_k S(n,k) \qquad (\text{for every } n \in \mathbb{N}_0).$$

Finally, by substituting in this last equality $a_k$ by its expression given by (1.4), we get for any $n \in \mathbb{N}$:

$$\begin{aligned} G_n &= \sum_{k=1}^{n} (-1)^{k-1} \left( \sum_{i=0}^{k-1} i!(k-i-1)! \right) S(n,k) \\ &= \sum_{k=1}^{n} (-1)^{k-1} \left( \sum_{\ell=1}^{k} (\ell-1)!(k-\ell)! \right) S(n,k) \qquad (\text{by putting } \ell = i+1) \\ &= \sum_{1\le \ell \le k \le n} (-1)^{k-1} (\ell-1)!(k-\ell)! S(n,k), \end{aligned}$$

as required. □

*Remark* 2.5. In the relatively recent literature, there are several ways to explain the integrality of the Genocchi numbers. For example, it is shown (see [3]) that the Genocchi numbers are (up to a sign) the values of the Gandhi polynomials (lying in $\mathbb{Z}[X]$) at 1. On the other hand, the combinatorial interpretation of the Genocchi numbers discovered by Dumont (see, e.g., [4, 19]) immediately explains the integrality of the $G_n$'s.

Now, we turn to present another type of results providing nontrivial lower bounds for the 2-adic valuations of the rational numbers $\sum_{k=1}^{n} \frac{2^k}{k}$ ($n \in \mathbb{N}$).

**Theorem 2.6.** *For any positive integer* $n$*, we have*

$$\vartheta_2 \left( \sum_{k=1}^{n} \frac{2^k}{k} \right) \ge s_2(n) \tag{2.8}$$

*and* (*more strongly*)*:*

$$\vartheta_2 \left( \sum_{k=1}^{n} \frac{2^k}{k} \right) \ge n - \left\lfloor \frac{\log n}{\log 2} \right\rfloor. \tag{2.9}$$

*Proof.* Let $n$ be a fixed positive integer. Since $a_n \in \mathbb{Z}$ then we have $\vartheta_2(a_n) \geq 0$. But, by using Formula (2.3) of Proposition 2.2 together with the properties of (1.5), this is equivalent to:

$$\vartheta_2(n!) - n + \vartheta_2\left(\sum_{k=1}^{n} \frac{2^k}{k}\right) \geq 0.$$

Then, using the Legendre formula (1.6) for the prime number $p = 2$, which says that $\vartheta_2(n!) = \frac{n - s_2(n)}{2-1} = n - s_2(n)$, we get

$$\vartheta_2\left(\sum_{k=1}^{n} \frac{2^k}{k}\right) \geq s_2(n),$$

confirming (2.8). To establish the stronger lower bound (2.9), we use the Rockett formula (2.2) together with the identity:

$$\operatorname{lcm}\left\{\binom{m}{0}, \binom{m}{1}, \dots, \binom{m}{m}\right\} = \frac{\operatorname{lcm}(1, 2, \dots, m, m+1)}{m+1} \qquad \text{(for every } m \in \mathbb{N}_0\text{)}, \quad (2.10)$$

established by the author in [5]. According to (2.2) and (2.10), we have that

$$\sum_{k=1}^{n} \frac{2^k}{k} = \frac{2^n}{n} \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}} \text{ and } 1 = \frac{n}{\operatorname{lcm}(1, 2, \dots, n)} \cdot \operatorname{lcm}\left\{\binom{n-1}{0}, \binom{n-1}{1}, \dots, \binom{n-1}{n-1}\right\}.$$

By multiplying side by side these last equalities, we get

$$\sum_{k=1}^{n} \frac{2^k}{k} = \frac{2^n}{\operatorname{lcm}(1, 2, \dots, n)} \cdot \operatorname{lcm}\left\{\binom{n-1}{0}, \binom{n-1}{1}, \dots, \binom{n-1}{n-1}\right\} \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}}.$$

But since the rational number $\operatorname{lcm}\{\binom{n-1}{0}, \binom{n-1}{1}, \dots, \binom{n-1}{n-1}\} \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}}$ is obviously a positive integer, then it has a nonnegative 2-adic valuation; so it follows (according to the properties of the functions $\vartheta_p$ given in (1.5)) that:

$$\vartheta_2\left(\sum_{k=1}^{n} \frac{2^k}{k}\right) \geq \vartheta_2\left(\frac{2^n}{\operatorname{lcm}(1, 2, \dots, n)}\right) = n - \left\lfloor \frac{\log n}{\log 2} \right\rfloor,$$

confirming (2.9) and completes the proof. □

*Remark* 2.7. Very recently, Dubickas [2] has shown that the lower bound (2.9) of Theorem 2.6 is essentially optimal (it is attained if $n$ has the form $2^k - 1$, $k \in \mathbb{N}$).

# 3 Two open problems

**Open problem 1.** Find a generalization of Theorem 2.6 to other prime numbers $p$ other than $p = 2$. Notice that the generalization that might immediately come to mind:

$$\vartheta_p\left(\sum_{k=1}^{n} \frac{p^k}{k}\right) \geq s_p(n)$$

is false for $p > 2$ (take for example $n = 2$).

**Open problem 2.** Since every term in Formula (2.1) of Theorem 2.1 has an easily understood combinatorial meaning (see [1] for the factorials and the Stirling numbers and [4, 19] for the Genocchi numbers), it is natural to ask whether there exists a combinatorial proof of that formula (permitting us to understand it intuitively). Note that one of the classic references providing an enormous amount of combinatorial proofs is the Stanley book [15].

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