Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2022, Volume 28, Number 4, 744–748
DOI: 10.7546/nntdm.2022.28.4.744-748

Explicit formulas for sums related to Dirichlet L-functions

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Received: 11 May 2022 **Revised:** 20 August 2022
Accepted: 10 November 2022 **Online First:** 11 November 2022

Abstract: Let $p \geq 3$ be a prime number and let m, n and l be integers with $\gcd(l, p) = 1$. Let χ be a Dirichlet character modulo p and $L(s, \chi)$ be the Dirichlet L-function corresponding to χ. Explicit formulas for:

$$\frac{2}{p-1} \sum_{\substack{\chi \bmod p \\ \chi(-1)=+1}} \chi(l) L(m, \chi) L(n, \overline{\chi}) \text{ and } \frac{2}{p-1} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(l) L(m, \chi) L(n, \overline{\chi})$$

are given in this paper by using the properties of character sums and Bernoulli polynomials.

Keywords: Character sum, Dirichlet L-function, Bernoulli number, Generalized Bernoulli number.
2020 Mathematics Subject Classification: 11M06, 11B68.

1 Introduction and main results

Let $k \geq 3$ and l be integers with $\gcd(l, k) = 1$. Let χ be a Dirichlet character modulo k and let $L(s, \chi)$ be the Dirichlet L-function corresponding to χ. Set

$$M(k, l, m, n) := \frac{2}{\varphi(k)} \sum_{\substack{\chi \bmod k \\ \chi(-1)=(-1)^m=(-1)^n}} \chi(l) L(m, \chi) L(n, \overline{\chi}),$$

where φ is the totient's Euler function.

Many mathematicians have been interested in evaluating and giving the mean values of $M(k,l,m,n)$ for different values of k, l, m, and n (see, e.g., [3, 4, 5, 6, 7]. Walum [7] for a prime p showed that

$$M(p,1,1,1) = \frac{\pi^2}{6}\frac{(p-1)(p-2)}{p^2}.$$

Next, Liu [3] for a prime $p > 3$, studied $M(p,l,1,n)$ and $M(p,l,2,n)$ with $l = 1,2,3,4$ and gave interesting results, for example he proved [3, Corollary 1.1] that

$$M(p,1,2,2) = \frac{\pi^4}{90}\frac{(p^2-1)(p^2+11)}{p^4}. \tag{1}$$

Also, in the aforementioned corollary, he provided an explicit formula for $M(p,3,2,2)$, which contained a misprint that was later corrected by Louboutin [6].

In this paper, for a prime $p \geq 3$ we give formulas for $M(p,l,m,n)$ where m and n have the same parity, by using the properties of character sums and Bernoulli polynomials $B_m(x)$ $(m = 0,1,2,\ldots)$ which are the coefficients in the power series expansion (see, e.g., [2]):

$$\frac{ze^{xz}}{e^z-1} = \sum_{m=0}^{\infty} B_m(x)\frac{z^m}{m!}, \quad |z| < 2\pi.$$

Our main formulas are the following.

Theorem 1.1. *Let χ be a Dirichlet character modulo a prime $p \geq 3$. Let m, n and l be integers with* $\gcd(p,l) = 1$.

1. *If* $m \equiv n \equiv 0 \pmod 2$*, then*

$$\begin{aligned} M(p,l,m,n) = (-1)^{\frac{m+n}{2}}\frac{(2\pi)^{m+n}}{4\cdot p\cdot m!\cdot n!}\Bigg\{&\Bigg(2\sum_{\substack{1\leq a,b\leq p-1\\ a\equiv bl \mod p}} B_m\left(\frac{a}{p}\right)B_n\left(\frac{b}{p}\right)\Bigg)\\ &+\frac{2p}{p-1}\left(1-\frac{1}{p^m}\right)\left(1-\frac{1}{p^n}\right)B_m(0)B_m(0)\\ &-\frac{2}{p-1}\left(\sum_{a=1}^{p-1}B_m\left(\frac{a}{p}\right)\right)\left(\sum_{b=1}^{p-1}B_n\left(\frac{b}{p}\right)\right)\Bigg\}. \end{aligned}$$

2. *If* $m \equiv n \equiv 1 \pmod 2$*, then*

$$M(p,l,m,n) = (-1)^{\frac{m-n}{2}}\frac{(2\pi)^{m+n}}{2\cdot p\cdot m!\cdot n!}\sum_{\substack{1\leq a,b\leq p-1\\ a\equiv bl \mod p}} B_m\left(\frac{a}{p}\right)B_n\left(\frac{b}{p}\right).$$

Noting that, for $l = 1$, our formulas give closed formulas for $M(p,1,m,n)$, since

$$\sum_{\substack{1\leq a,b\leq p-1\\ a\equiv b \mod p}} B_m\left(\frac{a}{p}\right)B_n\left(\frac{b}{p}\right) = \sum_{a=1}^{p-1}B_m\left(\frac{a}{p}\right)B_n\left(\frac{a}{p}\right).$$

In particular, if $m = n = 1$, we recover Walum's formula and if $m = n = 2$, we recover Liu's formula (see Example 1.1). On the other hand, for $l > 1$, the computation will be much more complicated, nevertheless the formulas of Theorem 1.1 are still useful because they make the numerical computation very easy and very quick for different values of p, l, m, and n.

Example 1.1. *It follows by taking $m = n = 2$ and $l = 1$ that: $B_2(x) = x^2 - x + \frac{1}{6}$ and*

$$\begin{aligned} M(p,1,2,2) &= \frac{\pi^4}{p}\left\{ 2\sum_{b=1}^{p-1}\left(\frac{b^2}{p^2} - \frac{b}{p} + \frac{1}{6}\right)^2 + \frac{p}{18(p-1)}\left(1 - \frac{1}{p^2}\right)^2 \right. \\ &\qquad \left. - \frac{2}{p-1}\left(\sum_{a=1}^{p-1}\left(\frac{a^2}{p^2} - \frac{a}{p} + \frac{1}{6}\right)\right)^2 \right\} \\ &= \frac{\pi^4}{90}\frac{(p^2-1)(p^2+11)}{p^4}, \end{aligned}$$

which is Formula (1).
Also, by taking $m = n = l = 1$ we have $B_1(x) = x - \frac{1}{2}$ and we get Walum's formula. Indeed

$$\begin{aligned} M(p,1,1,1) &= \frac{2\pi^2}{p}\sum_{b=1}^{p-1} B_1\left(\frac{b}{p}\right) B_1\left(\frac{b}{p}\right) \\ &= \frac{2\pi^2}{p}\sum_{b=1}^{p-1}\left(\frac{b^2}{p^2} - \frac{b}{p} + \frac{1}{4}\right) \\ &= \frac{\pi^2}{6}\frac{(p-1)(p-2)}{p^2}. \end{aligned}$$

2 Proof of Theorem 1.1

2.1 Tools and lemmas

In order to prove Theorem 1.1 we need the following tools and lemmas.

Let χ be a Dirichlet character modulo $k \geq 3$. Then the generalized Bernoulli numbers $B_m(\chi)$ $(m = 0, 1, 2, ...)$ are defined by using the generating function (see e.g., [2]):

$$\sum_{j=1}^{k}\chi(j)\frac{ze^{jz}}{e^{kz}-1} = \sum_{m=0}^{\infty}\frac{B_m(\chi)}{m!}z^m, \quad |z| < \frac{2\pi}{k}.$$

They can be expressed in terms of Bernoulli polynomials as:

$$B_m(\chi) = k^{m-1}\sum_{a=1}^{k-1}\chi(a)B_m\left(\frac{a}{k}\right). \tag{2}$$

Now, if we suppose that χ is a primitive character, then the following lemma gives the value of $L(s,\chi)$ at positive integers.

Lemma 2.1. [2, Theorem 9.6] *Let χ be a primitive character modulo $k \geq 3$. If $\chi(-1) = (-1)^n$ $(n \geq 1)$, then*

$$L(n, \chi) = (-1)^{n-1} \frac{\tau(\chi)}{2n!} \left(\frac{2\pi i}{k}\right)^n B_n(\overline{\chi}), \tag{3}$$

where $\tau(\chi) = \sum_{a=1}^{k} \chi(a) \exp\left(\frac{2\pi i a}{k}\right)$ is the Gaussian sum associated with χ.

We should point out that, due to our use of this lemma, our proof has to be restricted to characters of prime conductors, since it is well known that all non-principal characters modulo a prime number are primitive. The reader can find (e.g., in [4]) a formula that extends Walum's formula and is not necessarily associated with primitive characters.

Lemma 2.2. [1, Theorem 8.11] *Let χ be a primitive character modulo $k \geq 3$. Then*

$$|\tau(\chi)| = |\tau(\overline{\chi})| = \sqrt{k}. \tag{4}$$

Lemma 2.3. *Let $\widehat{G}\,(k)$ be the set of the Dirichlet characters modulo $k \geq 3$. Then for $n, m \in \mathbb{Z}$ with $\gcd(nm, k) = 1$ we have*

$$\sum_{\substack{\chi \in \widehat{G}(k) \\ \chi(-1)=+1}} \chi(n)\overline{\chi}\,(m) = \sum_{\substack{\chi \in \widehat{G}(k) \\ \chi(-1)=-1}} \chi(n)\overline{\chi}\,(m) = \begin{cases} \frac{\varphi(k)}{2}, & \text{if } n \equiv m \,(\text{mod } k)\,, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. This follows at once from [1, Theorem 6.16]. □

2.2 Proof of Theorem 1.1

Suppose that $m \equiv n \equiv 0$ (mod 2). Then

$$\begin{aligned} M(p, l, m, n) &= \frac{2}{p-1} \sum_{\substack{\chi \mod p \\ \chi(-1)=+1}} \chi(l) L(m, \chi) L(n, \overline{\chi}) \\ &= \frac{2}{p-1} \left(L(m, \chi_0) L(n, \chi_0) + \sum_{\substack{\chi \mod p \\ \chi(-1)=+1, \chi \neq \chi_0}} \chi(l) L(m, \chi) L(n, \overline{\chi}) \right), \end{aligned}$$

where χ_0 is the principal character modulo p. According to [1, p. 232] and to [1, Theorem 12.17], we have

$$L(m, \chi_0) = \left(1 - \frac{1}{p^m}\right) \zeta(m) = (-1)^{\frac{m+2}{2}} \left(1 - \frac{1}{p^m}\right) \frac{(2\pi)^m}{2m!} B_m(0),$$

from which

$$L(m, \chi_0) L(n, \chi_0) = (-1)^{\frac{m+n}{2}} (2\pi)^{m+n} \frac{B_m B_n}{4m!n!} \frac{(p^m - 1)(p^n - 1)}{p^{m+n}}. \tag{5}$$

On the other hand, it follows by using Lemma 2.1 and Lemma 2.2 that:

$$\begin{aligned} \sum_{\substack{\chi \mod p \\ \chi(-1)=+1, \chi \neq \chi_0}} \chi(l) L(m, \chi) L(n, \overline{\chi}) &= (-1)^{\frac{m+n}{2}} \frac{p}{4 \cdot m! \cdot n!} \left(\frac{2\pi}{p}\right)^{m+n} \\ &\quad \times \sum_{\substack{\chi \mod p \\ \chi(-1)=+1, \chi \neq \chi_0}} \chi(l) B_m(\overline{\chi}) B_n(\chi). \end{aligned} \tag{6}$$

Formula (2) allows us to write the last sum as:

$$p^{m+n-2}\sum_{\substack{a=1\\b=1}}^{p-1} B_m\left(\frac{a}{p}\right)B_n\left(\frac{b}{p}\right)\sum_{\substack{\chi \mod p\\ \chi(-1)=+1,\chi\neq\chi_0}}\chi(lb)\overline{\chi}(a) \tag{7}$$

Next, by using Lemma 2.3, Formula (7) becomes

$$p^{m+n-2}\left(\frac{p-1}{2}\sum_{\substack{1\leq a,b\leq p-1\\ a\equiv bl \mod p}} B_m\left(\frac{a}{p}\right)B_n\left(\frac{b}{p}\right)-\sum_{\substack{a=1\\b=1}}^{p-1} B_m\left(\frac{a}{p}\right)B_n\left(\frac{b}{p}\right)\right). \tag{8}$$

Finally, from (5), (6) and (8) we get the first formula of Theorem 1.1.

Now, suppose that $m \equiv n \equiv 1$ (mod 2). Then, similarly we can get the second formula. This completes the proof. □

Acknowledgements

The author would like to thank the referees and the editor-in-chief for useful comments and suggestions.

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