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Congruences via umbral calculus

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Abstract: In this paper, we use the properties of the classical umbral calculus to give some congruences related to the Bell numbers and Bell polynomials. We also present a new congruence involving Appell polynomials with integer coefficients.
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1 Introduction

Recall that the n-th Bell polynomials $\mathcal{B}_n(x)$ are defined by

$$\mathcal{B}_n(x) = \sum_{k=0}^{n} \left\{ n \atop k \right\} x^k,$$

where $\left\{ n \atop k \right\}$ is the (n,k)-th Stirling number of the second kind which counts the number of partition of the set $[n] := \{1, ..., n\}$ into k non-empty subsets. In a similar way the n-th r-Bell polynomials $\mathcal{B}_{n,r}(x)$ are defined by [9]:

$$\mathcal{B}_{n,r}(x) = \sum_{k=0}^{n} \left\{ n+r \atop k+r \right\}_r x^k,$$

where $\left\{ n \atop k \right\}_r$ is the r-Stirling numbers of the second kind which counts the number of partition of the set $[n]$ into k non-empty subsets such that the numbers $1, ..., r$ are in distinct subsets, see [2]. For $x = 1$, $\mathcal{B}_n(1)$ is the n-th Bell number which counts the number of all partitions of the set $[n]$ and $\mathcal{B}_{n,r}(1)$ is the n-th r-Bell number which counts the number of all partitions of set $[n+r]$

such that the first r elements are in distinct subsets, see [9]. The n-th derangemant polynomials and the n-th Lah polynomials are defined as follows, respectively:

$$\mathcal{D}_n(x) = \sum_{k=0}^{n} \binom{n}{k} k!\,(x-1)^{n-k},\ \mathcal{L}_n(x) = \sum_{k=0}^{n} L(n,k)\,x^k,$$

where $\mathcal{D}_n = \mathcal{D}_n(0)$ is the n-th derangement number, counting the number of permutation of the set $[n]$ without a fixed point and $L(n,k)$ is the (n,k)-th Lah number counts partitions of the set $[n]$ into k ordered lists, see [3, 10]. Let $\mathcal{L}_n = \mathcal{L}_n(1)$, it is clear $\mathcal{L}_n$ is the number of all partitions of set $[n]$ into k ordered lists. The exponential generating series for $\mathcal{D}_n(x)$ and $\mathcal{L}_n(x)$ are as follows, respectively:

$$\sum_{n=0}^{\infty} \mathcal{D}_n(x) \frac{t^n}{n!} = \tfrac{e^{-t}}{1-t} e^{xt},\ \textstyle\sum_{n=0}^{\infty} \mathcal{L}_n(x) \frac{t^n}{n!} = e^{\left(\frac{t}{1-t}\right)x},$$

see [3]. The modifed Lah polynomials $\mathcal{M}_n(x)$ can be defined as follows:

$$\mathcal{M}_n(x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{L}_{n-k} x^k,\ \text{ for } x=0,\ \mathcal{M}_n(0) = \mathcal{L}_n.$$

The exponential generating function for $\mathcal{M}_n(x)$ is

$$\sum_{n=0}^{\infty} \mathcal{M}_n(x) \frac{t^n}{n!} = e^{\left(\frac{t}{1-t}\right)} e^{tx}.$$

For any non-negative integers n, r and any prime p, we have the following congruences, see [3, 15–17]

$$\mathcal{B}_{n+p} \equiv \mathcal{B}_{n+1} + \mathcal{B}_{n+1} \pmod p, \tag{1}$$

$$\mathcal{B}_{n+p,r} \equiv \mathcal{B}_{n+1,r} + \mathcal{B}_{n,r} \pmod p, \tag{2}$$

$$\mathcal{D}_{n+p} \equiv -\mathcal{D}_n \pmod p, \tag{3}$$

$$\mathcal{L}_{n+p} \equiv \mathcal{L}_n \pmod p, \tag{4}$$

$$(x)_p \equiv x^p - x \pmod{p\mathbb{Z}[x]}. \tag{5}$$

The number of partitions of set $[n]$ without singletons denoted by $\mathcal{V}_n$. For any non-negative integers n, r, we have [9, 16]:

$$\mathcal{V}_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \mathcal{B}_k,\ \mathcal{B}_{n,r} = \sum_{k=0}^{n} r^k \binom{n}{k} \mathcal{B}_{n-k},\ \mathcal{B}_n = \sum_{k=0}^{n} \binom{n}{k} \mathcal{V}_k. \tag{6}$$

Let $\mathbf{B}$ and $\mathbf{B_x}$, respectively, be the Bell and generalized Bell umbra introduced by Rota [11] and Sun [14], given by $\mathbf{B}^n = \mathcal{B}_n$ and $\mathbf{B}_{\mathbf{x}}^n = \mathcal{B}_n(x)$. We also have the following relations, [3, 6, 14]: For any integer $n \geq 0$ and any polynomial f:

$$(\mathbf{B})_n = 1,\ (\mathbf{B_x})_n = x^n \tag{7}$$

$$f(\mathbf{B_x}) = e^{-x} \sum_{j=0}^{\infty} f(j) \frac{x^j}{j!}, \tag{8}$$

where $(x)_n$ is the falling factorial defined by

$$(x)_n = x(x-1)\cdots(x-n+1), \text{when } n \geq 1 \text{ and } (x)_0 = 1.$$

Then the above numbers and polynomials can be represented umbrally as follows see [3, 16]:

$$\mathcal{B}_{n,r} = (\mathbf{B}+r)^n\,,\ \mathcal{V}_n = (\mathbf{B}-1)^n\,,\ \mathcal{D}_n = (-1)^n (\mathbf{B}-1)_n\,,\ \mathcal{L}_n = (\mathbf{B}+n-1)_n\,,$$

and

$$\mathcal{B}_{n,r}(x) = (\mathbf{B}_\mathbf{x}+r)^n\,,\ \mathcal{D}_n(1-x) = (-1)^n (\mathbf{B}_\mathbf{x}-1)_n\,,\ \mathcal{L}_n(x) = (\mathbf{B}_\mathbf{x}+n-1)_n\,.$$

For more information on the umbral calculus and its applications one can see [4–6, 12, 13]. We also use the notations [2]

$$(x)_n = \sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix} x^k,\quad x^n = \sum_{k=0}^{n}\begin{Bmatrix} n \\ k \end{Bmatrix}(x)_k\,, \tag{9}$$

where $\left[{n \atop k}\right]$ is the absolute Stirling number of the first kind which counts the number of permutations of the set $[n] := \{1,\ldots,n\}$ into k cycles. The paper is organized as follow: In the next section we give some congruences on the number of partitions of set without singletons. In the third section we present some congruences related to Bell polynomials. In the last section we give a new congruences concerning Appell polynomials with integer coefficients.

2 Congruences on the number of partitions of a set without singletons via Bell umbra

In this section, we give some congruences on the number of partitions of set $[n]$ without singletons by using the Bell umbra.

Lemma 2.1. *Let f be a polynomial in $\mathbb{Z}[x]$, let $s \geq 1$ be an integer and let p be a prime number. The following congruence holds*

$$f(\mathbf{B}-1)\left((\mathbf{B}-1)^{p^s} - (\mathbf{B}-1)\right) \equiv sf(\mathbf{B}-1)\,(mod\, p)\,.$$

Proof. It suffices to take $f(x) = x^n$. We proceed by induction on s. For $s = 1$, by setting $x = \mathbf{B}-1$ in (5), and by (7) and the congruence

$$\binom{p}{k} \equiv 0 \pmod p\,,\ 0 < k < p.$$

We have

$$\begin{aligned}(\mathbf{B}-1)^n((\mathbf{B}-1)^p - (\mathbf{B}-1)) &\equiv (\mathbf{B}-1)^n(\mathbf{B}-1)_p \\ &= (\mathbf{B}-1)^n \sum_{k=0}^{n}\binom{p}{k}(-1)_k(\mathbf{B})_{p-k} \\ &\equiv (\mathbf{B}-1)^n \pmod p\,.\end{aligned}$$

Assume it is true for s. Then

$$
\begin{aligned}
(\mathbf{B}-1)^n((\mathbf{B}-1)^{p^{s+1}}-(\mathbf{B}-1)) &= (((\mathbf{B}-1)^{p^s}-(\mathbf{B}-1)+(\mathbf{B}-1))^p-(\mathbf{B}-1)))\,(\mathbf{B}-1)^n \\
&\equiv (((\mathbf{B}-1)^{p^s}-(\mathbf{B}-1))^p+(\mathbf{B}-1)^p-(\mathbf{B}-1)))\,(\mathbf{B}-1)^n \\
&= ((\mathbf{B}-1)^{p^s}-(\mathbf{B}-1))^p(\mathbf{B}-1)^n+((\mathbf{B}-1)^p-(\mathbf{B}-1))\,(\mathbf{B}-1)^n \\
&\equiv s((\mathbf{B}-1)^{p^s}-(\mathbf{B}-1))^{p-1}\,(\mathbf{B}-1)^n+(\mathbf{B}-1)^n \\
&\;\;\vdots \\
&\equiv s^p\,(\mathbf{B}-1)^n+(\mathbf{B}-1)^n \\
&\equiv s\,(\mathbf{B}-1)^n+(\mathbf{B}-1)^n \\
&= (s+1)\,(\mathbf{B}-1)^n \,(\mathrm{mod}\; p)\,.
\end{aligned}
$$

Hence, the proof of the induction step is complete. □

Proposition 2.1. *For any integers $n \geq 0, m \geq 0, s \geq 1$ and any prime p, the following congruences holds*

$$
\mathcal{V}_{n+p^s} \equiv \mathcal{V}_{n+1}+s\mathcal{V}_n \;\;(mod\; p)\,, \tag{10}
$$

$$
\mathcal{V}_{n+mp^s} \equiv \sum_{k=0}^{m}\binom{m}{k}s^k\mathcal{V}_{n+m-k} \;\;(mod\; p)\,. \tag{11}
$$

Proof. For (10) take $f(x)=x^n$ in lemma 2.1. For (11), we have

$$
\begin{aligned}
\mathcal{V}_{n+mp^s} &= (\mathbf{B}-1)^n\left((\mathbf{B}-1)^{p^s}-(\mathbf{B}-1)+(\mathbf{B}-1)\right)^m \\
&= \sum_{k=0}^{m}\binom{m}{k}(\mathbf{B}-1)^{n+m-k}\left((\mathbf{B}-1)^{p^s}-(\mathbf{B}-1)\right)^k.
\end{aligned}
$$

□

Corollary 2.1. *For any integer $n \geq 0$ and any prime p, we have*

$$
\mathcal{V}_{n+p^{s+1}-p^s} \equiv \sum_{k=0}^{p-1}(-s)^k\left(\mathcal{V}_{n-k}+\mathcal{V}_{n-1-k}\right) \;\;(mod\; p)\,, \tag{12}
$$

$$
\mathcal{V}_{n+p^{s+1}-2p^s} \equiv \sum_{k=0}^{p-2}(-s)^k(1+k)\left(\mathcal{V}_{n-1-k}+\mathcal{V}_{n-2-k}\right) \;\;(mod\; p)\,, \tag{13}
$$

$$
\mathcal{V}_{n+p\mathcal{N}_p} \equiv \mathcal{V}_{n+\mathcal{N}_p} \;\;(mod\; p)\,, \tag{14}
$$

$$
\sum_{k=1}^{\mathcal{N}_p}\binom{\mathcal{N}_p}{k}\mathcal{V}_{n+\mathcal{N}_p-k} \equiv 0 \;\;(mod\; p)\,, \tag{15}
$$

where, $\mathcal{N}_p = 1+p+\cdots+p^{p-1}$.

Proof. To obtain (12) and (13) it suffices to replace m by $p-1$ or $p-2$ in (11) and to take $r=0$ or 1 in the following congruence

$$
\binom{p-r-1}{k} \equiv (-1)^k\binom{r+k}{r} \,(\mathrm{mod}\; p)\,.
$$

For (15), by setting $m=\mathcal{N}_p$ in (11), we get

$$\mathcal{V}_{n+\mathcal{N}_p p}=\mathcal{V}_{n+\mathcal{N}_p}+\sum_{k=1}^{\mathcal{N}_p}\binom{\mathcal{N}_p}{k}\mathcal{V}_{n+\mathcal{N}_p-k}. \tag{16}$$

On the other hand by (10), we have

$$\mathcal{V}_{n+\mathcal{N}_p p}=\mathcal{V}_{n+p+\cdots+p^p}\equiv\mathcal{V}_{n+\mathcal{N}_p}\pmod{p}. \tag{17}$$

This completes the proof. □

Corollary 2.2. *Let $n\geq 0$ be an integer and let p be a prime number. The following congruences holds*

$$\begin{aligned}
\mathcal{V}_n &= \sum_{k=0}^{n}(-1)^k\left\{ n \atop k\right\}\mathcal{D}_k,\\
\mathcal{D}_n &= \sum_{k=0}^{n}(-1)^k\left[n \atop k\right]\mathcal{V}_k,\\
\mathcal{V}_{n+1}+\mathcal{V}_n &\equiv \sum_{k=0}^{n+p}(-1)^k\left\{ n+p \atop k\right\}\mathcal{D}_k \pmod{p},\\
\mathcal{D}_n &\equiv -\sum_{k=0}^{n+p}(-1)^k\left[n+p \atop k\right]\mathcal{V}_k \pmod{p}.
\end{aligned}$$

Proof. Replace x by $\mathbf{B}-1$ in (9) and use the congruences (10) and (3). □

Corollary 2.3. *For any integers $n\geq 0, r\geq 1$ and any prime p, such that $p\nmid r$, we have*

$$\mathcal{B}_{n+1}+\mathcal{B}_n \equiv \sum_{k=0}^{n+p}\binom{n+p}{k}\mathcal{V}_k \pmod{p}, \tag{18}$$

$$\sum_{k=0}^{p-1}\frac{\mathcal{V}_k}{(-r)^k} \equiv \mathcal{B}_{p-1,r-1} \pmod{p}, \tag{19}$$

$$\mathcal{V}_{n+1}+\mathcal{V}_n \equiv \sum_{k=0}^{n+p}(-1)^k\binom{n+p}{k}(k+1)^{k-1}\left(\mathcal{B}_{n-k+1,k}+\mathcal{B}_{n-k,k}\right) \pmod{p}. \tag{20}$$

Proof. For (18) use the Touchard's congruence (1) and (6). For (19), we have

$$\begin{aligned}
\sum_{k=0}^{p-1}\frac{\mathcal{V}_k}{(-r)^k} &= \sum_{k=0}^{p-1}(-1)^k r^{-k}\mathcal{V}_k\\
&\equiv \sum_{k=0}^{p-1}\binom{p-1}{k}r^{p-1-k}(\mathbf{B}-1)^k\\
&= (\mathbf{B}-1+r)^{p-1}\\
&= \mathcal{B}_{p-1,r-1} \pmod{p}.
\end{aligned}$$

The last congruence (20), follows by siting $x=\mathbf{B}$ in the following identity [7]

$$(x-1)^n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+1)^{k-1}(x+k)^{n-k},$$

and use the congruences (2) and (10). □

3 Congruences via generalized Bell umbra

In this section we use the generalized Bell umbra to present some congruences related to Bell polynomials.

Lemma 3.1. *For any polynomial f and any non-negative integer r, there holds*

$$f\left(\mathbf{B_x}+r\right)=\sum_{i=0}^{r}\binom{r}{i}\frac{d^i}{dx^i}f\left(\mathbf{B_x}\right).$$

Proof. By (8), we have $e^x f\left(\mathbf{B_x}\right)=\sum_{j=0}^{\infty}f\left(j\right)\frac{x^j}{j!}$. Then

$$\begin{aligned}\frac{d^r}{dx^r}\left(e^x f\left(\mathbf{B_x}\right)\right)&=\frac{d^r}{dx^r}\left(\sum_{j=0}^{\infty}f\left(j\right)\frac{x^j}{j!}\right)\\&=\sum_{j=0}^{\infty}f\left(j+r\right)\frac{x^j}{j!}\\&=e^x f\left(\mathbf{B_x}+r\right).\end{aligned}$$

Hence

$$\begin{aligned}f\left(\mathbf{B_x}+r\right)&=e^{-x}\tfrac{d^r}{dx^r}\left(e^x f\left(\mathbf{B_x}\right)\right)\\&=\textstyle\sum_{i=0}^{r}\binom{r}{i}\frac{d^i}{dx^i}f\left(\mathbf{B_x}\right).\end{aligned}$$

□

Proposition 3.1. *Let n, r be non-negative integers. The following identities holds*

$$\begin{aligned}\mathcal{B}_{n,r}\left(x\right)&=\sum_{i=0}^{r}\binom{r}{i}\frac{d^i}{dx^i}\mathcal{B}_n\left(x\right),\\\mathcal{L}_r\left(x\right)&=\sum_{i=0}^{r}\left(-1\right)^r\binom{r}{i}\frac{d^i}{dx^i}\mathcal{D}_r\left(1-x\right),\\\mathcal{L}_r\left(x\right)&=\sum_{i=0}^{r}\left(-1\right)^{r+i}i!\binom{r}{i}^2\mathcal{D}_{r-i}\left(1-x\right),\text{ in this case for } x=1, \text{ we obtain}\\\mathcal{L}_r&=\sum_{i=0}^{r}\left(-1\right)^{r+i}i!\binom{r}{i}^2\mathcal{D}_{r-i}.\end{aligned}$$

Proof. Take in Lemma 3.1, $f\left(x\right)=x^n$ or $\left(x-1\right)_r$, and use

$$\frac{d^i}{dx^i}\mathcal{D}_n\left(x\right)=i!\binom{n}{i}\mathcal{D}_{n-i}\left(x\right).$$

□

Corollary 3.1. *For any non-negative integer n and for any prime number p, there holds*

$$\begin{aligned}\mathcal{L}_p\left(x\right)&\equiv\left(-1\right)^p\mathcal{D}_p\left(1-x\right)\ \left(mod\ p\mathbb{Z}[x]\right).\\\mathcal{B}_{n,p-1}\left(x\right)&\equiv\sum_{i=0}^{p-1}\left(-1\right)^i\frac{d^i}{dx^i}\mathcal{B}_n\left(x\right)\left(mod\ p\mathbb{Z}[x]\right).\\\mathcal{L}_{p-1}\left(x\right)&\equiv\sum_{i=0}^{p-1}\left(-1\right)^i\frac{d^i}{dx^i}\mathcal{D}_{p-1}\left(1-x\right)\left(mod\ p\mathbb{Z}[x]\right),\\\mathcal{L}_{p-1}\left(x\right)&\equiv\sum_{i=0}^{p-1}\left(-1\right)^i i!\mathcal{D}_{p-1-i}\left(1-x\right)\left(mod\ p\mathbb{Z}[x]\right).\end{aligned}$$

Proof. Take $r = p$ or $p-1$ in proposition 3.1 and use the following congruences:

$$\binom{p}{k} \equiv 0, \ \ 0 < k < p, \text{and} \ \ \binom{p-1}{k} \equiv (-1)^k, 0 \leq k \leq p. \qquad \square$$

Corollary 3.2. *Let $n, r, s \geq 1, m \geq 0$ be integres and let p be a prime number. The following congruences holds*

$$\begin{aligned}
\sum_{i=0}^{r+mp^s} \binom{r+mp^s}{i} \frac{d^i}{dx^i} \mathcal{B}_n(x) &\equiv \sum_{i=0}^{r} \binom{r}{i} \frac{d^i}{dx^i} \mathcal{B}_n(x) \left(mod\, p\mathbb{Z}[x]\right), \\
\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathcal{B}_{n-i}(x) &\equiv \sum_{i=0}^{p-1} (-1)^i \frac{d^i}{dx^i} \mathcal{B}_n(x) \left(mod\, p\mathbb{Z}[x]\right), \\
\sum_{i=0}^{r+\mathcal{N}_p} \binom{r+\mathcal{N}_p}{i} \frac{d^i}{dx^i} \mathcal{B}_n(x) &\equiv \sum_{i=0}^{r+1} \binom{r+1}{i} \frac{d^i}{dx^i} \mathcal{B}_n(x) \left(mod\, p\mathbb{Z}[x]\right),
\end{aligned}$$

where, $\mathcal{N}_p = 1 + p + \cdots + p^{p-1}$.

Proof. By using the following identity, [9]

$$\mathcal{B}_{n,r}(x) = \sum_{k=0}^{n} r^k \binom{n}{k} \mathcal{B}_{n-k}(x),$$

and by Lemma 3.1, we get the desired congruences. $\square$

4 Congruences involving Appell polynomials with integer coefficients

Let $(\mathcal{A}_n(x))_{n \geq 0}$ be a sequence of Appell polynomials [1], with integer coefficients defined by :

$$\sum_{n \geq 0} \mathcal{A}_n(x) \frac{t^n}{n!} = F(t) \exp(xt),$$

where

$$F(t) = 1 + \sum_{n \geq 1} \mathcal{A}_n \frac{t^n}{n!}, \quad \mathcal{A}_n = \mathcal{A}_n(0) \in \mathbb{Z}, \text{and } \mathcal{A}_0 = 1.$$

Let $\mathbf{A}$ be the Appell umbra defined by $\mathbf{A}^n = \mathcal{A}_n$. Then we can define the generalized Appell umbra $\mathbf{A_x}$ as follows:

$$\mathbf{A}_\mathbf{x}^n = \mathcal{A}_n(x) = (\mathbf{A} + x)^n,$$

see [8]. The n-th Appell polynomials $\mathcal{A}_n(x)$ can be given explicitly as

$$\mathcal{A}_n(x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{A}_{n-k} x^k.$$

The main result of this section is the following theorem.

Theorem 4.1. *Let f be a polynomial in $\mathbb{Z}[x]$, let $m \geq 0, s \geq 1$ be integers and let p be a prime number. Then if for any integer $n \geq 0$, there exists an integer t such that $\mathcal{A}_{n+p} \equiv t\mathcal{A}_n \,(mod\, p)$, we have the following congruence*

$$(\mathbf{A}+x)^{mp^s} f(\mathbf{A}+x) \equiv \left(x^{p^s}+t\right)^m f(\mathbf{A}+x) \ \ (mod\, p\mathbb{Z}[x]).$$

This congruence is equivalent when $f(x) = x^n$ to

$$\mathcal{A}_{n+mp^s}(x) \equiv \left(x^{p^s}+t\right)^m \mathcal{A}_n(x)\,(mod\, p\mathbb{Z}[x]).$$

In particular for $n = 0$, we have

$$\mathcal{A}_{mp^s}(x) \equiv \left(x^{p^s}+t\right)^m (mod\, p\mathbb{Z}[x]).$$

Proof. It suffices to take $f(x) = x^n$. For $m = 1$ we proceed by induction on s. Indeed, for $s = 1$ we have

$$\begin{aligned}
(\mathbf{A}_{\mathbf{x}}^p - t)\,\mathbf{A}_{\mathbf{x}}^n &= \mathbf{A}_{\mathbf{x}}^{n+p} - t\mathbf{A}_{\mathbf{x}}^n \\
&= (\mathbf{A}+x)^p (\mathbf{A}+x)^n - t\mathbf{A}_{\mathbf{x}}^n \\
&\equiv (\mathbf{A}^p + x^p)(\mathbf{A}+x)^n - t\mathbf{A}_{\mathbf{x}}^n \\
&= x^p\mathbf{A}_{\mathbf{x}}^n + \sum_{k=0}^{n}\binom{n}{k}\mathbf{A}^{n+p-k}x^k - t\mathbf{A}_{\mathbf{x}}^n \\
&\equiv x^p\mathbf{A}_{\mathbf{x}}^n + t\sum_{k=0}^{n}\binom{n}{k}\mathbf{A}^{n-k}x^k - t\mathbf{A}_{\mathbf{x}}^n \\
&= x^p\mathbf{A}_{\mathbf{x}}^n \ \ (\text{mod}\, p\mathbb{Z}[x]).
\end{aligned}$$

Assume it is true for $s \geq 1$. Then we have

$$\begin{aligned}
\mathbf{A}_{\mathbf{x}}^n\left(\mathbf{A}_{\mathbf{x}}^{p^{s+1}} - t\right) &= \mathbf{A}_{\mathbf{x}}^n\left(\left(\mathbf{A}_{\mathbf{x}}^{p^s} - t + t\right)^p - t\right) \\
&\equiv \mathbf{A}_{\mathbf{x}}^n\left(\left(\mathbf{A}_{\mathbf{x}}^{p^s} - t\right)^p + t - t\right) \\
&= \mathbf{A}_{\mathbf{x}}^n\left(\mathbf{A}_{\mathbf{x}}^{p^s} - t\right)^p \\
&= \left[\mathbf{A}_{\mathbf{x}}^n\left(\mathbf{A}_{\mathbf{x}}^{p^s} - t\right)\right]\left(\mathbf{A}_{\mathbf{x}}^{p^s} - t\right)^{p-1} \\
&\equiv x^{p^s}\mathbf{A}_{\mathbf{x}}^n\left(\mathbf{A}_{\mathbf{x}}^{p^s} - t\right)^{p-1} \\
&= x^{p^s}\left[\mathbf{A}_{\mathbf{x}}^n\left(\mathbf{A}_{\mathbf{x}}^{p^s} - t\right)\right]\left(\mathbf{A}_{\mathbf{x}}^{p^s} - t\right)^{p-2} \\
&\equiv x^{2p^s}\mathbf{A}_{\mathbf{x}}^n\left(\mathbf{A}_{\mathbf{x}}^{p^s} - t\right)^{p-2} \\
&\ \vdots \\
&\equiv \left(x^{p^s}\right)^p\mathbf{A}_{\mathbf{x}}^n \\
&= x^{p^{s+1}}\mathbf{A}_{\mathbf{x}}^n \ \ (\text{mod}\, p\mathbb{Z}[x]).
\end{aligned}$$

So, we proved that $(\mathbf{A}+x)^{p^s} f(\mathbf{A}+x) \equiv \left(x^{p^s}+t\right) f(\mathbf{A}+x)\,(\text{mod}\, p\mathbb{Z}[x])$. For $m \geq 1$ we use this last congruence to obtain

$$
\begin{aligned}
(\mathbf{A}+x)^{mp^s} f(\mathbf{A}+x) &= (\mathbf{A}+x)^{p^s} (\mathbf{A}+x)^{(m-1)p^s} f(\mathbf{A}+x) \\
&\equiv \left(x^{p^s}+t\right) (\mathbf{A}+x)^{(m-1)p^s} f(\mathbf{A}+x) \\
&\vdots \\
&\equiv \left(x^{p^s}+t\right)^m f(\mathbf{A}+x) \, (\text{mod } p\mathbb{Z}[x]) . \qquad \square
\end{aligned}
$$

Corollary 4.1. *Let f be a polynomial in $\mathbb{Z}[x]$, let p be a prime number and let $s \geq 1, m_1, \ldots, m_s \in \{0, \ldots, p-1\}$, be integers. Then if for any integer $n \geq 0$, there exists an integer t such that $\mathcal{A}_{n+p} \equiv t\mathcal{A}_n \, (mod\ p)$, there holds*

$$
(\mathbf{A}+x)^{m_1 p + \cdots + m_s p^s} f(\mathbf{A}+x) \equiv (x^p+t)^{m_1} \cdots \left(x^{p^s}+t\right)^{m_s} f(\mathbf{A}+x) \, (mod\ p\mathbb{Z}[x]) .
$$

This congruence is equivalent when $f(x) = x^n$ to

$$
\mathcal{A}_{n+m_1 p + \cdots + m_s p^s}(x) \equiv (x^p+t)^{m_1} \cdots \left(x^{p^s}+t\right)^{m_s} \mathcal{A}_n(x) \, (mod\ p\mathbb{Z}[x]) .
$$

In particular, for $n = 0$, we have

$$
\mathcal{A}_{m_1 p + \cdots + m_s p^s}(x) \equiv (x^p+t)^{m_1} \cdots \left(x^{p^s}+t\right)^{m_s} (mod\ p\mathbb{Z}[x]) .
$$

Proof. By Theorem 4.1, we can write

$$
\begin{aligned}
(\mathbf{A}+x)^{m_1 p + \cdots + m_s p^s} f(\mathbf{A}+x) &= (\mathbf{A}+x)^{m_1 p} (\mathbf{A}+x)^{m_2 p^2 + \cdots + m_s p^s} f(\mathbf{A}+x) \\
&\equiv (x^p+t)^{m_1} (\mathbf{A}+x)^{m_2 p^2 + \cdots + m_s p^s} f(\mathbf{A}+x) \\
&\vdots \\
&\equiv (x^p+t)^{m_1} \cdots \left(x^{p^s}+t\right)^{m_s} f(\mathbf{A}+x) \, (\text{mod } p\mathbb{Z}[x]) . \qquad \square
\end{aligned}
$$

4.1 Application

Now we give two applcations of Theorem 4.1.

Corollary 4.2. *For any integers $n \geq 0$, $s \geq 1$, $m \geq 0$ and for any prime number p, there holds*

$$
\mathcal{D}_{n+mp^s}(x) \equiv \left(x^{p^s}-1\right)^m \mathcal{D}_n(x) \, (mod\ p\mathbb{Z}[x]) . \tag{21}
$$

For $x = 0$, we obtain

$$
\mathcal{D}_{n+mp^s} \equiv (-1)^m \mathcal{D}_n \, (mod\ p) .
$$

Proof. The congruence (21) follows by setting $f(x) = x^n$ in Theorem 4.1 and $t = -1$, (3). $\square$

Corollary 4.3. *For any prime number p and any integers $n \geq 0, s \geq 1, m_1, \ldots, m_s \in \{0, \ldots, p-1\}$, there holds*

$$
\mathcal{D}_{n+m_1 p + \cdots + m_s p^s}(x) \equiv (x^p-1)^{m_1} \left(x^{p^2}-1\right)^{m_2} \cdots \left(x^{p^s}-1\right)^{m_s} \mathcal{D}_n(x) \, (mod\ p\mathbb{Z}[x]) .
$$

In particular, we have

$$
\begin{aligned}
\mathcal{D}_{m_1 p + \cdots + m_s p^s}(x) &\equiv (x^p-1)^{m_1} \left(x^{p^2}-1\right)^{m_2} \cdots \left(x^{p^s}-1\right)^{m_s} (mod\ p\mathbb{Z}[x]) , \\
\mathcal{D}_{m_1 p + \cdots + m_s p^s}(k) &\equiv (k-1)^{m_1+m_2+\cdots+m_s} (mod\ p) .
\end{aligned}
$$

Corollary 4.4. *For any integers $n \geq 1$, $s \geq 1$, $m \geq 0$ and for any prime number p, there holds*

$$\mathcal{M}_{n+mp^s}(x) \equiv \left(x^{p^s}+1\right)^m \mathcal{M}_n(x)\,(mod\; p\mathbb{Z}[x]).$$

For $x = 0$, we obtain

$$\mathcal{L}_{n+mp^s} \equiv \mathcal{L}_n\,(mod\; p).$$

Proof. Take $f(x) = x^n$ in Theorem 4.1 and $t = 1$, (4). □

Corollary 4.5. *For any prime number p and any integers*
$n \geq 0, s \geq 1, m_1, \ldots, m_s \in \{0, \ldots, p-1\}$, there holds

$$\mathcal{M}_{n+m_1p+\cdots+m_sp^s}(x) \equiv (x^p+1)^{m_1}\left(x^{p^2}+1\right)^{m_2}\cdots\left(x^{p^s}+1\right)^{m_s}\mathcal{M}_n(x)\,(mod\; p\mathbb{Z}[x]).$$

In particular, we have

$$\begin{aligned}
&\mathcal{M}_{m_1p+\cdots+m_sp^s}(x) \equiv (x^p+1)^{m_1}\left(x^{p^2}+1\right)^{m_2}\cdots\left(x^{p^s}+1\right)^{m_s}(mod\; p\mathbb{Z}[x]),\\
&\mathcal{M}_{m_1p+\cdots+m_sp^s}(k) \equiv (k+1)^{m_1+m_2+\cdots+m_s}\,(mod\; p).
\end{aligned}$$

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