Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2022, Volume 28, Number 4, 710–718
DOI: 10.7546/nntdm.2022.28.4.710-718

On Vandiver's arithmetical function – II

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Received: 25 March 2021 **Revised:** 15 October 2022
Accepted: 3 November 2022 **Online First:** 7 November 2022

Abstract: We study more properties of Vandiver's arithmetical function

$$V(n) = \prod_{d|n}(d+1),$$

introduced in [2].
Keywords: Arithmetic function, Inequalities, Asymptotic results.
2020 Mathematical Subject Classification: 11A25, 11A41, 11N37, 26D15.

1 Introduction

This is a continuation of the first part [2], where we have considered the Vandiver arithmetic function

$$V(n) = \prod_{d|n}(d+1),$$

where d runs through all distinct divisors of n. In the first part we have proved inequalities related to this function; connections with notions of perfect numbers; equations related to $V(n)$, as well as some open problems and conjectures. The aim of this second part is to offer more properties of this function, and particularly to deduce also certain asymptotic results.

2 Main results

Theorem 2.1. *One has*

$$\sum_{d|n} \frac{1}{d+1} < \log \frac{V(n)}{T(n)} < \frac{\sigma(n)}{n} \quad (n \geq 1), \tag{2.1}$$

where $T(n)$ denotes (as in [2]) the product of divisors of n (see Theorem 7 of [2]).

Proof. Write

$$\log V(n) = \log \prod_{d|n}(d+1) = \sum_{d|n} \log(d+1)$$
$$= \sum_{d|n}\left(\log d + \log\left(1+\frac{1}{d}\right)\right) = \sum_{d|n}\log d + \sum_{d|n}\log\left(1+\frac{1}{d}\right).$$

Here the first term is $\log T(n)$. For the second one apply the double inequality

$$\frac{x}{1+x} < \log(1+x) < x \quad (x>0) \tag{2.2}$$

for $x = \frac{1}{d}$, implying

$$\frac{1}{d+1} < \log\left(1+\frac{1}{d}\right) < \frac{1}{d}. \tag{2.3}$$

Since

$$\sum_{d|n}\frac{1}{d} = \frac{\sigma(n)}{n},$$

relation (2.1) follows. □

Remark 2.2. As

$$\sum_{d|n}\frac{1}{d+1} = \sum_{d|n}\frac{1}{n/d+1} = \sum_{d|n}\frac{d}{n+d} \le \sum_{d|n}\frac{d}{n+1} = \frac{\sigma(n)}{n+1},$$

and

$$\sum_{d|n}\frac{1}{d+1} \ge \sum_{d|n}\frac{1}{n+1} = \frac{d(n)}{n+1},$$

we get

$$\frac{d(n)}{n+1} \le \sum_{d|n}\frac{1}{d+1} \le \frac{\sigma(n)}{n+1} \le \frac{d(n)}{2}, \tag{2.4}$$

where the last inequality is a consequence of relation (9) of [2]. The lower bound in (2.4) may be improved, by using the following inequality due to P. Henrici (see e.g. [1]):

$$\sum_{i=1}^{k}\frac{1}{1+x_i} \ge \frac{k}{1+\sqrt[k]{x_1\cdots x_k}}, \quad \text{where } x_i \ge 1 (i=\overline{1,k}). \tag{2.5}$$

By letting $x_i = d_i =$ divisors of n, and $k = d(n)$, as $x_1\cdots x_k = T(n) = n^{d(n)/2}$, we get from (2.5):

$$\sum_{d|n}\frac{1}{d+1} \ge \frac{d(n)}{\sqrt{n}+1}, \tag{2.6}$$

which clearly improves the left-hand side of (2.4). We note that a similar result to (2.6) may be obtained by the combined use of the arithmetic-geometric mean inequality, and the right-hand side of (5) from [2]:

$$\sum_{d|n} \frac{1}{d+1} \geq d(n) \frac{1}{\sqrt[d(n)]{\prod_{d|n}(d+1)}} = \frac{d(n)}{(V(n))^{1/d(n)}} \tag{2.7}$$

$$\sum_{d|n} \frac{1}{d+1} \geq \frac{d(n)}{(V(n))^{1/d(n)}} \geq \frac{d(n)}{\sigma(n)/d(n)+1} = \frac{(d(n))^2}{\sigma(n)+d(n)}. \tag{2.8}$$

However, the second inequality in (2.8) is weaker than (2.6), according to the known result (see [4])

$$\frac{\sigma(n)}{d(n)} \geq \sqrt{n}. \tag{2.9}$$

Corollary 2.3. *There exists a positive constant* $c > 0$ *such that*

$$\frac{V(2^n-1)}{T(2^n-1)} < (\log n)^c \quad (n \geq 3). \tag{2.10}$$

Proof. This follows by the right-hand side of (2.1), and an inequality due to P. Erdős [4]:

$$\frac{\sigma(2^n-1)}{2^n-1} < c \log\log n, \quad n \geq 3, \tag{2.11}$$

which completes the proof. □

Corollary 2.4. *The right-hand side of* (2.1) *gives a new proof of Theorem 4 of [2], written equivalently:*

$$\log V(n) \sim \log T(n) \text{ as } n \to \infty. \tag{2.12}$$

Proof. It is sufficient to prove that

$$\frac{\sigma(n)}{n d(n) \log n} \to 0 \text{ as } n \to \infty. \tag{2.13}$$

Indeed, as $T(n) = n^{d(n)/2}$, one has $\log T(n) = (d(n) \log n)/2$. Relation (2.13) follows, e.g., by selection (9) of [2], as

$$\frac{\sigma(n)}{d(n)} \cdot \frac{1}{n \log n} \leq \frac{n+1}{2n \log n} \to 0$$

as $n \to \infty$. □

Theorem 2.5.

$$\log \frac{V(n)}{T(n)} = \frac{\sigma(n)}{n} + O(1). \tag{2.14}$$

Proof. By the proof of Theorem 2.1 one has

$$\log \frac{V(n)}{T(n)} = \sum_{d|n} \log\left(1+\frac{1}{d}\right).$$

Since $\log(1+x) = x + O(x^2)$ $(x > 0)$, we get

$$\sum_{d|n} \log\left(1+\frac{1}{d}\right) = \sum_{d|n} \frac{1}{d} + O\left(\sum_{d|n} \frac{1}{d^2}\right).$$

By

$$\sum_{d|n}\frac{1}{d^2}<\sum_{d=1}^{\infty}\frac{1}{d^2}=\frac{\pi^2}{6},$$

the result follows.

A more concrete proof is based on the double inequality

$$\frac{\sigma(n)}{n}-\frac{\pi^2}{12}<\log\frac{V(n)}{T(n)}<\frac{\sigma(n)}{n}. \tag{2.15}$$

The left-hand side inequality follows by the logarithmic inequality

$$\log(1+x)>x-\frac{x^2}{2}\quad (x>0) \tag{2.16}$$

(which is stronger than the left-hand side of (2.2)); applied to $x:=\frac{1}{d}$.

Since

$$\frac{1}{2}\sum_{d|n}\frac{1}{d^2}<\frac{1}{2}\sum_{d=1}^{\infty}\frac{1}{d^2}=\frac{\pi^2}{12},$$

(2.15) follows. □

Corollary 2.6. *It holds true that*

$$\lim_{n\to\infty}\sup\frac{1}{\log\log n}\cdot\log\frac{V(n)}{T(n)}=e^{\gamma}, \tag{2.17}$$

where γ is Euler's constant.

Indeed, by (2.14) one has

$$\lim_{n\to\infty}\sup\frac{1}{\log\log n}\cdot\log\frac{V(n)}{T(n)}=\lim_{n\to\infty}\sup\frac{\sigma(n)}{n\log\log n}=e^{\gamma},$$

the last equality is a famous result due to T. H. Gronwall (see e.g. [4]).

Theorem 2.7. *It holds true that*

$$\lim_{n\to\infty}\frac{V(\sigma(n))}{T(\sigma(n))}=+\infty,\ \text{ on a set of density one,} \tag{2.18}$$

and

$$e^{1-\frac{\pi^2}{12}}\le\lim_{n\to\infty}\inf\frac{V(\sigma(n))}{T(\sigma(n))}\le e. \tag{2.19}$$

Proof. Applying the left-hand side of (2.15) for $n:=\sigma(n)$, we get

$$\log\frac{V(\sigma(n))}{T(\sigma(n))}>\frac{\sigma(\sigma(n))}{\sigma(n)}-\frac{\pi^2}{12}. \tag{2.20}$$

Now, by a result of P. Erdős and M. V. Subbarao (see [3]) one has

$$\frac{\sigma(\sigma(n))}{\sigma(n)}\to\infty\ \text{ (as } n\to\infty\text{) on a set of density one,} \tag{2.21}$$

so this combined with (2.20) yields (2.18). Particularly

$$\limsup_{n\to\infty} \frac{V(\sigma(n))}{T(\sigma(n))} = +\infty. \tag{2.22}$$

For the proof of (2.19) apply again (2.15), and the following limit

$$\liminf_{n\to\infty} \frac{\sigma(\sigma(n))}{\sigma(n)} = 1. \tag{2.23}$$

This follows by the inequality $\frac{\sigma(m)}{m} > 1$ for any $m > 1$, and the limit due to R. Bojanić (see [4]).

$$\lim_{p\to\infty,\ p \text{ prime}} \frac{\sigma(2^p - 1)}{2^p - 1} = 1. \tag{2.24}$$

As for $n = 2^{p-1}$ one has $\frac{\sigma(\sigma(n))}{\sigma(n)} = \frac{\sigma(2^p-1)}{2^p-1}$, relation (2.23) follows. □

Remark 2.8. Relation (2.19) shows that the $\liminf$ of $\frac{V(\sigma(n))}{T(\sigma(n))}$ is finite, and lies in the interval $[e^{1-\pi^2/12}, e]$. The exact determination of this value is not known to the author.

Theorem 2.9. *It holds true that*

$$\limsup_{n\to\infty} \frac{V(\varphi(n))}{T(\varphi(n))} = +\infty. \tag{2.25}$$

Let

$$k = \liminf_{n\to\infty} \frac{\sigma(\varphi(n))}{n}. \tag{2.26}$$

Then

$$e^{k-\pi^2/12} \le \lim_{n\to\infty} \frac{V(\varphi(n))}{T(\varphi(n))} \le e^k. \tag{2.27}$$

Proof. Apply (2.15) for $n := \varphi(n)$. Then one has

$$\limsup_{n\to\infty} \frac{\sigma(\varphi(n))}{\varphi(n)} = +\infty. \tag{2.28}$$

Relation (2.28) follows by

$$\limsup_{n\to\infty} \frac{\sigma(\varphi(n))}{n} = +\infty \tag{2.29}$$

due to L. Alaoglu and P. Erdős [4] and the remark that

$$\frac{\sigma(\varphi(n))}{\varphi(n)} \ge \frac{\sigma(\varphi(n))}{n}.$$

Therefore, (2.25) is true. Inequalities (2.27) are consequences of (2.15). □

Remark 2.10. The exact value of k is not known. A. Makovski and A. Schinzel [3] have shown that

$$k \le \frac{1}{2} + \frac{1}{2^{34} - 4}, \tag{2.30}$$

and conjectured an inequality (which is the famous Makowski–Schinzel conjecture; see, e.g., [3, 4]), namely:

$$\frac{\sigma(\varphi(n))}{n} \geq \frac{1}{2} \text{ for all } n \geq 1. \tag{2.31}$$

If (2.31) is true, then we get $k \geq \frac{1}{2}$. We know that $k > 0$, and even that $k > \frac{1}{39.4}$, due to K. Ford (see [3]).

The next result involves the normal order of magnitude of functions. Recall that we say that the the normal order of magnitude of arithmetical function $f(n)$ is $g(n)$ if for every $\varepsilon > 0$, one has

$$(1-\varepsilon)g(n) < f(n) < (1+\varepsilon)g(n) \tag{2.32}$$

for almost all integer n (i.e. the set of integers not satisfying (2.32) has density zero). Thus $f(n) \sim g(n)$ as $n \to \infty$, excepting $o(n)$ integers.

Theorem 2.11. *The normal order of magnitude of* $\log\log V(n)$ *is*

$$(1+\log 2)\cdot \log\log n \tag{2.33}$$

The same is true for $\log\log T(n)$.

Proof. We shall use the following lemma. □

Lemma 2.12. *If* $x_n, y_n > 0$ *and* $x_n \sim y_n$ $(n \to \infty)$, *where* $y_n \to \infty$, *then* $\log x_n \sim \log y_n$. *Indeed, as* $\frac{x_n}{y_n} \to 1$, *we get* $\log\left(\frac{x_n}{y_n}\right) \to 0$, *so* $\log x_n - \log y_n \to 0$, *giving* $\frac{\log x_n - \log y_n}{\log y_n} \to 0$. $0 = 0$, *so* $\frac{\log x_n}{\log y_n} \to 1$.

By Theorem 4 of [2], the above Lemma implies

$$\log\log V(n) \sim \log\log T(n). \tag{2.34}$$

Therefore, it will be sufficient to prove the result for $\log\log T(n)$. As $T(n) = n^{d(n)/2}$, we get

$$\log\log T(n) = \log\frac{d(n)}{2} + \log\log n. \tag{2.35}$$

By a classical result of G. H. Hardy and S. Ramanujan (see [4]), the normal order of magnitude of $\log d(n)$ is $(\log 2)\cdot\log\log n$. Clearly, the same is true for $\log\frac{d(n)}{2}$, therefore by (2.35) the result follows.

Theorem 2.13. *i) If* $n \geq 6$ *is even, then*

$$V(n) \geq 3(n+1)(n+2). \tag{2.36}$$

There is equality only if $n = 2p$, *where* $p \geq 3$ *is a prime.*

ii) If $n \geq 12$ *is divisible by* 4, *then*

$$V(n) \geq \frac{15}{4}\cdot(n+1)(n+2)(n+4). \tag{2.37}$$

There is equality only if $n = 4p$, *where* $p \geq 3$ *is a prime.*

Proof. i) If $n \geq 6$ is even, then $1, 2, \frac{n}{2}, n$ are distinct divisors of n, so

$$V(n) \geq (1+1)(2+1) \cdot \left(\frac{n}{2}+1\right)(n+1) = 3(n+1)(n+2).$$

There is equality iff there are no other divisors, i.e., when $\frac{n}{2} = p$ is a prime.

ii) $1, 2, \frac{n}{2}, \frac{n}{4}, n$ are distinct divisors. The proof is similar to the case i). □

Corollary 2.14. *It holds true that:*

$$1) \quad \lim_{n\to\infty} \sup \frac{V(n-1)}{V(n)} = +\infty; \quad \lim_{n\to\infty} \sup \frac{V(n+1)}{V(n)} = +\infty \tag{2.38}$$

$$2) \quad \lim_{\substack{p\to\infty \\ p \text{ prime}}} \sup \frac{V(p-1)}{p^2} = +\infty; \quad \lim_{p\to\infty} \sup \frac{V(p+1)}{p^2} = +\infty \tag{2.39}$$

$$3) \quad \lim_{p\to\infty} \frac{V(p-1)}{p^2} = 3, \tag{2.40}$$

if one assumes the existence of infinitely many primes p *of the form* $p = 2q+1$, *where q is a prime.*

$$4) \quad \lim_{n\to\infty} \sup \frac{V(p+1)}{p^2} = 3, \tag{2.41}$$

if one assumes the existence of infinitely many primes p *of the from* $p = 2q-1$, q *prime.*

Proof. 1) Let $n = p \geq 7$ be a prime. Then $V(p) = 2 \cdot (p+1)$, while by (2.36) one has $V(p-1) \geq 3p(p+1)$. Similarly, for $p \geq 5$ one has $V(p+1) \geq 3(p+2)(p+3)$, so (2.38) follows.

2) Let p be a prime of the form $4k+1$. Then, by (2.37) one has

$$V(p-1) = V(4k) \geq \frac{15}{4} \cdot (k+1)(k+2)(k+4), \quad (k \geq 3),$$

so

$$\frac{V(p-1)}{p^2} \geq \frac{15(k+1)(k+2)(k+4)}{4 \cdot (4k+1)^2} \to \infty \text{ as } k \to \infty.$$

The similar proof applies to $\frac{V(p+1)}{p^2}$.

3) As

$$\frac{V(p-1)}{p^2} \geq \frac{3p(p+1)}{p^2} = 3 \cdot \left(1 + \frac{1}{p}\right),$$

clearly

$$\lim_{p\to\infty} \frac{V(p-1)}{p^2} \geq 3.$$

On the other hand, if $p = 2q+1$, then $V(p-1) = 3 \cdot (2q+1)(2q+2)$, so

$$\frac{V(p-1)}{p^2} = \frac{3 \cdot (2q+1)(2q+2)}{2(q+1)^2} \to 3$$

as $q \to \infty$. This proves (2.40).

4) The proof of (2.41) is similar. □

Remark 2.15. The existence of infinitely many primes p of the form $p = 2q + 1$ (or $p = 2q - 1$) is one of the difficult open problems of Number theory (see [3, 4]).

The number $V(n-1)$ behaves differently as $V(n)$, it was shown by the case $n =$ prime.

As

$$\frac{\log\log(n-1)}{\log(n-1)} \sim \frac{\log\log n}{\log n},$$

by (2.24) of [2], we can write also

$$\lim_{n\to\infty} \sup \log\log V(n-1)\frac{\log\log n}{\log n} = \log 2. \tag{2.42}$$

One has:

Theorem 2.16. *It holds true that:*

$$\lim_{\substack{p\to\infty \\ p \text{ prime}}} \log\log V(p-1)\cdot\frac{\log\log p}{\log p} > 0. \tag{2.43}$$

Proof. By the left-hand side of relation (5) of [2] we can write

$$\log\log V(p-1) > \log d(p-1) + \log\log(\sqrt{p-1}+1). \tag{2.44}$$

Now, by a result of K. Prachar [3] there exists $c > 0$ such that

$$\log d(q-1) > c\cdot\frac{\log q}{\log\log q} \tag{2.45}$$

for infinitely many primes q. This combined with (2.44) implies (2.43). □

Theorem 2.17. *The series* $\sum_{n=1}^{\infty}\frac{1}{V(n)}$ *is divergent, while* $\sum_{n=1}^{\infty}\frac{1}{(V(n))^{1+a}}$ *is convergent for any* $a > 0$.

One has the asymptotic formula

$$\sum_{n\le x}\frac{1}{V(n)} = \frac{1}{2}\log\log x + O(1). \tag{2.46}$$

Proof. As $V(p) = 2(p+1)$, and

$$\sum_{n=1}^{\infty}\frac{1}{V(n)} > \frac{1}{2}\sum_{p \text{ prime}}\frac{1}{p+1} > \frac{1}{4}\sum_{p \text{ prime}}\frac{1}{p},$$

the divergence follows by the known divergence of the series $\sum_p 1/p$.

Let now consider the series of general term $\frac{1}{(V(n))^{1+a}}$. Clearly,

$$\sum_{n\ge 1}\frac{1}{(V(n))^{1+a}} = \sum_{p \text{ prime}}\frac{1}{(2(p+1))^{1+a}} + \sum_{n \text{ composite}}\frac{1}{(V(n))^{1+a}}.$$

Now, as $\frac{1}{(2(p+1))^{1+a}} < \frac{1}{p^{1+a}}$, remark that the series $\sum_{p \text{ prime}}\frac{1}{p^{1+a}}$ is known to be convergent. This follows, e.g., by the remark that, if p_k denotes the k-th prime, then $p_k > k$, so

$$\sum_{k\ge 1}\frac{1}{p_k^{1+a}} < \sum_{k\ge 1}\frac{1}{k^{1+a}} = \zeta(1+a) < \infty$$

for $a > 0$.

On the other hand,

$$\sum_{n \text{ composite}} \frac{1}{(V(n))^{1+a}} \leq \sum_{n \text{ composite}} \frac{1}{(\sqrt{n}+1)^{d(n)(1+a)}} \leq \sum_{n} \frac{1}{(\sqrt{n}+1)^{3(1+a)}}$$
$$< \sum_{n} \frac{1}{n^{3(1+a)/2}} < \infty \text{ as } 3(1+a)/2 > 1.$$

For the proof of (2.46) remark that

$$\sum_{n\leq x} \frac{1}{V(n)} = \frac{1}{2} \sum_{\substack{p\leq x \\ p \text{ prime}}} \frac{1}{p+1} + \sum_{\substack{n\leq x \\ n \text{ composite}}} \frac{1}{V(n)}. \tag{2.47}$$

As above, it is immediate that the second term of (2.47) is $< C$, where C is a positive constant. For the first term of (2.47) however, we will use the known fact that

$$\sum_{p\leq x} \frac{1}{p} = \log\log x + O(1) \tag{2.48}$$

(see [4]). Now

$$\sum_{p\leq x} \left(\frac{1}{p} - \frac{1}{p+1}\right) = \sum_{p\leq x} \frac{1}{(p(p+1))} < \sum_{p\leq x} \frac{1}{p^2} < \sum_{p\leq x} \frac{1}{n^2} < \pi^2/6,$$

so

$$\sum_{p\leq x} \frac{1}{p+1} = \sum_{p\leq x} \frac{1}{p} + O(1),$$

and by (2.47) the result follows. □

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