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Linear mappings in paraletrix spaces and their application to fractional calculus

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Abstract: This paper considers linear mappings in paraletrix spaces as an extension of the one given for rhotrix vector spaces. Furthermore, the adjoints of these mappings are given with their application in fractional calculus.

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1 Introduction

In [1], the idea of rhotrix was introduced as an object whose elements are arranged in a rhomboidal nature which of course was an extension of matrix-tertions and matrix-noitrets given by Atanassov and Shannon [5]. Suppose R and Q are two rhotrices such that

$$R=\left\langle\begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array}\right\rangle, Q=\left\langle\begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array}\right\rangle,$$

where $h(R)$ and $h(Q)$ are the hearts of these rhotrices.

It follows from Ajibade [1] that:

$$R+Q=\left\langle\begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array}\right\rangle+\left\langle\begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array}\right\rangle=\left\langle\begin{array}{ccc} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{array}\right\rangle$$

and

$$R\circ Q=\left\langle\begin{array}{ccc} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{array}\right\rangle$$

An alternative multiplication method was given by Sani [16] as follows:

$$R\circ Q=\left\langle\begin{array}{ccc} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bj+ek & \end{array}\right\rangle.$$

In [4], Aminu and Michael introduced the concept of paraletrix as an extension of rhotrix earlier given in [1]. In this case, the number of rows of the rhotrix is not equal to the number of columns. The addition and multiplication of two paraletrices presented in [4] are as follows:

$$p_1=\left\langle\begin{array}{ccccc} & a_1 & & & \\ a_2 & a_3 & a_4 & & \\ & a_5 & a_6 & a_7 & \\ & & a_8 & a_9 & a_{10} \\ & & & a_{11} & \end{array}\right\rangle, \quad p_2=\left\langle\begin{array}{ccccc} & b_1 & & & \\ b_2 & b_3 & b_4 & & \\ & b_5 & b_6 & b_7 & \\ & & b_8 & b_9 & b_{10} \\ & & & b_{11} & \end{array}\right\rangle$$

$$p_1+p_2=\left\langle\begin{array}{ccccc} & d_1 & & & \\ d_2 & d_3 & d_4 & & \\ & d_5 & d_6 & d_7 & \\ & & d_8 & d_9 & d_{10} \\ & & & d_{11} & \end{array}\right\rangle,$$

where $d_i=a_i+b_i$ is such that $i=1,2,3,\ldots,11$.

$$p_1\circ p_2=\left\langle\begin{array}{ccccc} & c_1 & & & \\ c_2 & c_3 & c_4 & & \\ & c_5 & c_6 & c_7 & \\ & & c_8 & c_9 & c_{10} \\ & & & c_{11} & \end{array}\right\rangle,$$

where each $c_i = a_i h(p_2) + b_i h(p_1)$ is such that $i = 1,2,3, \dots, 11,\ i \neq 6$ and $c_6 = h(p_1)h(p_2)$, $h(p_1) = a_6, h(p_2) = b_6$. It is important to note that the multiplication of the two matrices above follows Ajibade's multiplication of rhotrices [1]. However, the multiplication of paraletrix p_1 and p_2 using Sani [16] rhotrix approach is only possible whenever the number of columns of p_1 is equal to the rows of p_2.

Rhotrix vector spaces and rhotrix linear mappings have been studied by Aminu in [2, 3]. Oni and Aminu [12] presented paraletrix linear space as an extension of rhotrix vector space.

The concept of fractional calculus in paraletrix spaces, control theory, semigroup theory etc is one of the tremendous avenues for the analysis of real life problems. Fractional calculus is a branch of mathematical analysis that deals with construction and application of integrals and derivatives of arbitrary order [6].

Numerous definitions has been proposed for this concept. The definitions are divided into two main sources: the local (singular) kernel and non-singular kernels. The singular kernel such as Riemann–Liouville and Caputo fractional derivatives which uses power law as their kernels are the most popular used derivatives for researchers. Though Caputo derivatives is mostly used for modeling initial value problems, Riemann–Liouville derivative is the most natural and popular models of fractional calculus [7].

The non-singular kernel like Atangana–Baleanu and Caputo–Fabrizio fractional derivatives uses the Mittag-Leffler function and exponential decay function respectively as their kernels. These recent fractional derivatives are proposed to deal with real data which corresponds to complex models and requires sophisticated kernels.
Recent years have seen a considerable interest in the application of fractional calculus especially in the area of physics, chemistry, engineering, design of heat flux meters, signal and image processing, epidemiology [10, 11, 13, 15]. In this respect, a fractional order multi-vaccination model for Covid-19 with non-singular kernel was proposed by Omame et al. [14] and the references there in.

This work therefore gives a presentation of linear mappings in paraletrices as an extension of rhotrix linear mappings given in [3] and consequently considers their application to fractional calculus.

2 Preliminaries

In this section we recall some definitions as well as some known results which will be useful in this paper. For notation and terminologies not mentioned in this paper, the reader is referred to [4, 6, 9, 11, 12, 15].

Definition 2.1. Let p_1, p_2 and p_3 be 3×7-dimensional heart-oriented paraletrices such that $f_i \in p_3,\ i = 1,2,3, \dots, 11,\ f_6 = h(p_3)$ and let $\rho : p_1 \to p_2,\ \sigma : p_2 \to p_3$. The composition of ρ and σ denoted by $\sigma \circ \rho$ is the mapping $\sigma \circ \rho : p_1 \to p_3$ defined by $(\sigma \circ \rho)(a) = \sigma(\rho(a))$.

Lemma 2.2. *Let* $\rho : p_1 \to p_2,\ \sigma : p_2 \to p_3$ *and* $\pi : p_3 \to p_4$ *for* $g_i \in p_4, i = 1, 2, 3, \dots, 11$, $g_6 = h(p_4)$. *Then we have that* $\pi \circ (\sigma \circ \rho) = (\pi \circ \sigma) \circ \rho$.

Proof. Let $a \in p_1$. *Then we have that*

$$\big(\pi \circ (\sigma \circ \rho)\big)(a) = \pi\big((\sigma \circ \rho)(a)\big) = \pi\big(\sigma(\rho(a))\big).$$

Similarly,

$$\big((\pi \circ \sigma) \circ \rho\big)(a) = (\pi \circ \sigma)(\rho(a)) = \pi\big(\sigma(\rho(a))\big).$$

It follows that for every $a \in p_1$, $\big(\pi \circ (\sigma \circ \rho)\big)(a) = \big((\pi \circ \sigma) \circ \rho\big)(a)$ and the result follows. $\square$

Definition 2.3. Suppose P and Q are two be 3×7-dimensional heart-oriented paraletrix linear spaces over the same field K ($\mathbb{R}$ or $\mathbb{C}$). A mapping $\psi : P \to Q$ is said to be linear if it satisfies the following two conditions;

i) for $a, b \in P, \psi(a + b) = \psi(a) + \psi(b)$

ii) for $k \in K$ and $a \in P$, $\psi(ka) = k\psi(a)$.

Remark 2.4. It is important to note that for $k_1, k_2 \in K$ and $a, b \in P$, we have that

$$\psi(k_1 a + k_2 b) = \psi(k_1 a) + \psi(k_2 b) = k_1 \psi(a) + k_2 \psi(b).$$

A linear mapping $\psi : P \to Q$ is characterized by the condition that $\psi(k_1 a + k_2 b) = k_1 \psi(a) + k_2 \psi(b)$. We shall employ this condition subsequently and refer $\psi : P \to P$ to be a linear operator. In this case $\psi = \gamma$.

Example 2.5. It is shown in [9] that polynomial equations can be defined over paraletrices. Now consider the paraletrix linear space P of polynomials over the real field $\mathbb{R}$. Let $p_1(t)$ and $p_2(t)$ be any polynomial in P and let k be any scalar. Let $D : P \to P$ be the derivative mapping. It follows from [9] that $\frac{d}{dt}\langle p_1(t) + p_2(t)\rangle = \frac{d}{dt} p_1(t) + \frac{d}{dt} p_2(t)$ and $\frac{d}{dt}\langle k p_1(t)\rangle = k\frac{d}{dt} p_1(t)$. Thus $D(p_1 + p_2) = D(p_1) + D(p_2)$ and$D(kp_1) = kD(p_1)$. This shows that the derivative mapping is linear.

Definition 2.6. Let X be a vector space ($\mathbb{R}$ or $\mathbb{C}$) and K a scalar field ($\mathbb{R}$ or $\mathbb{C}$) of X. An inner product on X is a function $\langle\ ,\ \rangle : X \times X \to K$ such that for $x, y \in X$ and $k_1, k_2 \in K$, the following axioms are satisfied,

i) $\langle x, x\rangle \geq 0$, $\langle x, x\rangle = 0$ if and only if $x = 0$

ii) $\langle k_1 x + k_2\, y, z\rangle = k_1\langle x\, , z\rangle + k_2\langle y, z\rangle$

iii) $\langle x, y\rangle = \langle y, z\rangle$, when X is a complex vector space, then $\overline{\langle x, y\rangle} = \langle y, x\rangle$.

Remark. 2.7. It is worthy to note that when X is an inner product space, X defines a norm on X given by $\parallel x \parallel = \sqrt{\langle x, x\rangle}$. The properties of a norm can be found in [8]. A metric on X is given by $d(x, y) = \parallel x - y \parallel = \sqrt{\langle x - y, x - y\rangle}$. We say that a real or complex vector space is called a normed linear space or a normed vector space if there is a function $\parallel . \parallel$ satisfying the properties of a norm given in [8].

Example 2.8. Let $P[a, b]$ be the paraletrix linear space of all paraletrix functions on the closed interval $[a, b]$. The following defines an inner product on $P[a, b]$ where $p_1(t)$ and $p_2(t)$ are paraletrix functions (polynomials in t) in $P[a, b]$:

$$\langle p_1, p_2\rangle = \int_a^b p_1(t) p_2(t)\, dt.$$

Example 2.9 [8]. Let $M_{m,n}$ be the vector space of all real $m \times n$ matrices. An inner product is defined on $M_{m,n}$ where $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices in $M_{m,n}$:

$\langle A, B \rangle = tr(B^T A) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$ and $\| A \|^2 = \langle A, A \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2$.

Theorem 2.10 [17]. *Suppose* $A = [a_{ij}]$ *and* $B = [b_{ij}]$ *are matrices and* $k \in K$ *is a scalar. Then we have that:*

i) $(A + B)^T = A^T + B^T$

ii) $(kA)^T = kA^T$

iii) $(AB)^T = B^T A^T$

iv) $(A^T)^T = A$

It is important to note that Theorem 2.10 is analogous to the one given in Theorem 5.1 of [4].

Definition 2.11. Let P and Q be two paraletrix linear spaces and $\psi : P \to Q$ be a linear mapping. The adjoint of ψ is the function $\psi^* : Q \to P$ such that $\langle \psi(a), m \rangle = \langle a, \psi^*(m) \rangle$ for every $a \in P$ and $m \in Q$. If ψ is a linear operator, then $\psi = \gamma$ and $\langle \gamma(a), m \rangle = \langle a, \gamma^*(m) \rangle$ every $a, b \in P$.

Theorem 2.12 [12]. *Let* P *be the set of all* 3×7*-dimensional heart-oriented paraletrix. Then* P *is a vector* space.

Definition 2.13. Let $\{x_n\}$ be a sequence of real numbers. Then $\{x_n\}$ is a Cauchy sequence if for all $\varepsilon > 0$ there is N such that $n, m \geq N$

implies that $\| x_n - x_m \| < \varepsilon$.

Definition 2.14. Let W be an inner product space on $\{x_n\} \epsilon W$. Then $\{x_n\}$ is called a Cauchy sequence if for each $\varepsilon > 0$, there is an N such that $m, n \geq N$ implies $\| x_n - x_m \| < \varepsilon$. We say that $\{x_n\}$ converges to u if for each $\varepsilon > 0$, there is an N such that $n \geq N$ implies $\| x_n - u \| < \varepsilon$.

Definition 2.15. A normed linear space X is said to be complete if every Cauchy sequence in X converges.

Theorem 2.16. [12] *Let* P *be a paraletrix linear space and define a map* $\| \, \| : P \to \mathbb{R}$ *by* $\| p_1 \| = \max_{i \leq 11} |a_i|$ *where* p_1 *is a* 3×7*-dimensional heart-oriented paraletrices such that* $a_i \in p_1$, $i = 1,2,3, \ldots, 11$, $a_6 = h(p_1)$, *then* p_1 *is* a norm *and* $(p, \ \| p_1 \|)$ *is a normed linear space.*

Definition 2.17. Let P and Q be two paraletrix linear spaces and $\psi : P \to Q$ be a linear mapping. Then ψ is said to be bounded if there is a constant A such that $\| \psi(a) \| \leq A \| a \|$ for all $a \in P$.

Definition 2.18. [7]. Suppose $g \in C' ([a, b], \mathbb{R}^+)$ where $b > 0$and $q \in (0,1)$, then the Riemann–Liouville fractional integral is given by

$$_{a}J_t^q \, g(t) = \frac{1}{\Gamma(q)} \int_a^t (t - \varepsilon)^{q-1} \, g(\varepsilon) \, d\varepsilon, \qquad Re\ (q) > 0.$$

Definition 2.19. [6]. Suppose $g \in C'([a,b], \mathbb{R}^+)$ where $b > 0$ and $n-1 < q \leq n$, then the Riemann–Liouville fractional derivative is given by

$$ {}_{a}^{Rl}D_{t}^{q}\, g(t) = D^{n}\,{}_{a}J_{t}^{n-q}\, g(t) = \frac{d^n}{dt^n}\left[\frac{1}{\Gamma(n-q)}\int_a^t (t-\varepsilon)^{n-q-1}\, g(\varepsilon)\, d\varepsilon\right], Re(q) \geq 0, $$

$n \in \mathbb{N}$.

Definition 2.20. [6, 7]. Suppose $g \in C'([a,b], \mathbb{R}^+)$ where $b > 0$, then the Caputo fractional derivative is given by

$$ {}_{a}^{C}D_{t}^{q}\, g(t) = {}_{a}J_{t}^{n-q}D^{n}g(t) = {}_{a}J_{t}^{n-q}g^{n}g(t) = \frac{1}{\Gamma(n-q)}\int_a^t (t-\varepsilon)^{n-q-1}\, g(\varepsilon)\, d\varepsilon, $$

$Re(q) \geq 0, n-1 < q \leq n,\ n \in \mathbb{N}$.

Definition 2.21. [7]. Suppose $g \in C'([a,b], \mathbb{R}^+)$ where $b > 0$, then the Atangana–Baleanu derivative in Caputo sense is given by

$$ {}_{a}^{ABC}D_{t}^{q}\, g(t) = \frac{w(q)}{1-q}\int_0^t E_q\left(\frac{-q(t-\varepsilon)^q}{1-q}\right) g(\varepsilon)\, d\varepsilon, 0 < q < 1. $$

Definition 2.22. [10]. Suppose $g \in C'([a,b], \mathbb{R}^+)$ where $b > 0$, then the Caputo–Fabrizio fractional derivative of order q is given by

$$ {}_{a}^{CF}D_{t}^{q}\, g(t) = \frac{(z-q)w(q)}{z(1-q)}\int_a^t \exp\left(-\frac{q(t-\varepsilon)}{1-q}\right) g'(\varepsilon)\, d\varepsilon, 0 < q < 1, \qquad t \geq 0, $$

where $w(q)$ represents the normalization function such that $w(0) = w(1) = 1$ is satisfied.

3 Paraletrix linear mappings

In this section, we consider the linear mappings on paraletrices. Our interest will be on linear mappings embedded in a 3×7-dimensional heart-oriented paraletrix.

Now suppose $\psi : P \to Q$ is a linear mapping. The kernel of ψ which we usually denote as $\ker\psi$ is the set of elements in P that map into the zero vector $0 \in P$. In other words, we have that

$$ \ker\psi = \{a \in P :\ \psi(a) = 0\}. $$

The theorem below easily follows.

Theorem 3.1. *Let $\psi : P \to Q$ be a linear mapping. Then $\ker\psi$ is a subspace of P.*

Proof. The proof is a routine check. □

One can combine paraletrix linear mappings to obtain new paraletrix linear mappings. In our next result, we will show that the sum of two linear mappings is also a linear mapping.

Suppose that $\psi : P \to Q$ and $\theta : P \to Q$ are linear mappings over a field K. Then $\psi + \theta$ and $k\psi$ where $k \in K$ can be defined to be the mapping $(\psi+\theta)(a) \equiv \psi(a) + \theta(a)$ and $(k\psi)(a) \equiv k\psi(a)$. With this, we have the following Lemma.

Lemma 3.2. *Let ψ and θ be as defined above. Then $\psi+\theta$ and $k\psi$ are also linear.*
Proof. *Let $a,b \epsilon P$ and $k_1,k_2 \epsilon K$, then we have that*

$$\begin{aligned}(\psi+\theta)(k_1a+k_2b) &= \psi(k_1a+k_2b)+\theta(k_1a+k_2b)\\ &= k_1\,\psi(a)+k_2\,\psi(b)+k_1\theta(a)+k_2\theta(b)\\ &= k_1\big(\psi(a)+\theta(a)\big)+k_2\big(\psi(b)+\theta(b)\big)\\ &= k_1(\psi+\theta)(a)+k_2(\psi+\theta)(b).\end{aligned}$$

Similarly,

$$\begin{aligned}(k\psi)(k_1a+k_2b) &= k\psi(k_1a+k_2b)\\ &= k\big(k_1\,\psi(a)+k_2\,\psi(b)\big)\\ &= k_1k\psi(a)+k_2k\psi(b)\\ &= k_1(k\psi)(a)+k_2(k\psi)(b).\end{aligned}$$

Hence $\psi+\theta$ and $k\psi$ are linear. □

Theorem 3.3. *Let P and Q be two paraletrix linear spaces over a field K. Then the collection of all linear mappings from P to$Q$ with the operation of addition and scalar multiplication form a vector space over K.*

Proof. The proof is a routine check. □

We know from Lemma 2.2 that the composition of mappings is associative. The next result shows that the composition of paraletrix linear mappings is linear.

Proposition 3.4. *Suppose P,Q and R be any 3×7-heart oriented paraletrix linear spaces and let $\psi : P \to Q$, $\theta : Q \to R$ be linear mappings. Then the mapping $\theta\circ\psi : P \to R$ defined by $(\theta\circ\psi)(a) = \theta(\psi(a))$ is linear.*
Proof. *Let $a,b \epsilon P$ and $k_1,k_2 \epsilon K$, then we have that*

$$\begin{aligned}(\theta\circ\psi)(k_1a+k_2b) &= \theta(\psi(k_1a+k_2b))\\ &= \theta(k_1\psi(a)+k_2\psi(b))\\ &= k_1\theta\big(\psi(a)\big)+k_2\theta\big(\psi(b)\big)\\ &= k_1(\theta\circ\psi)(a)+k_2(\theta\circ\psi)(b).\end{aligned}$$

Hence $(\theta\circ\psi)$ is linear. □

Theorem 3.5. *Let P and Q be two paraletrix linear spaces and $\psi : P \to Q$ be a linear mapping. Then ψ is bounded if and only if it is continuous.*

Proof. *For the direct part of the proof, let ψ be bounded and suppose $\varepsilon > 0$, then for $a,b \epsilon P$ we have*

$$\| a-b \| < \frac{\varepsilon}{A} \text{ implies } \| Aa-Ab \| = \| A(a-b) \| \leq A \| a-b \| < \varepsilon.$$

Conversely, let ψ be continuous. For $\varepsilon > 0$, there exists $\delta > 0$ such that $\| a \| \leq \delta$ implies $\| Aa \| < \varepsilon$. For any arbitrary a, the linearity of ψ and the continuity precisely at 0 guarantee that we can write

$$\| A(a) \| = \| A\left(\frac{b\delta}{\| a \|}\right) \| \frac{\| b \|}{\delta} \leq \frac{\varepsilon}{\delta} \| b \|.$$

Thus the constant A exists. □

Remark 3.6. We can have unbounded linear mapping but they cannot exist in the whole paraletrix space. It is important to note that the collection of bounded linear mappings in P is a vector space (this can be considered as an analogous result of Theorem 2.12). Simply put, it is a non-commutative algebra, that is to say that we can have two linear mappings ψ and θ such that $\psi\theta \neq \theta\psi$.

The Lemma below presents the norm of the composition of linear mappings since we know from Proposition 3.4 that the composition of linear mappings in paraletrix spaces exists.

Lemma 3.7. *Let P, Q and R be any 3×7-heart oriented paraletrix linear spaces and let $\psi : P \to Q$, $\theta : Q \to R$ be linear mappings. Then $\| \theta \circ \psi \| \leq \| \theta \| \| \psi \|$.*

Proof. Let $a \in P$, then we have that

$$\| (\theta \circ \psi)(a) \| = \| \theta \circ (\psi(a) \| \leq \| \theta \| \| \psi(a) \|$$
$$= \| \theta \| \| \psi \| \| a \| .$$

Consequently, $\| \theta \circ \psi \| \leq \| \theta \| \| \psi \|$ and the proof is complete. ☐

We will conclude this section by considering a special case of linear mappings from P into its field K of scalars. Obviously, our previous results for mappings from P to Q necessarily hold for this special case.

Let P be a 3×7 dimensional heart-oriented paraletrix linear space over a field K. A mapping $\Lambda : P \to K$ is said to be a linear functional if for every $a, b \in P$ and $k_1, k_2 \in K$,

$$\Lambda(k_1 a + k_2 b) = k_1 \Lambda(a) + k_2 \Lambda(b).$$

Example 3.8. Let $P(t)$ be a paraletrix linear space of polynomials in t over the real field $\mathbb{R}$. Let $\Phi : P \to \mathbb{R}$ be the integral mapping defined by $\Phi\big(P(t)\big) = \int_0^1 P(t)dt$. It follows from [9] that Φ is linear so that it is a linear functional in P.

The following Theorem is analogous to Theorem 3.3.

Theorem 3.9. *Let P be a paraletrix linear space over a field K. Then the collection of all linear functional from P to K with the operation of addition and scalar multiplication form a vector space over K.*

Proof. Similar to that of Theorem 3.3. ☐

The vector space of linear functional in Theorem 3.9 will denoted by $P^{\sim}$ and referred to as the dual space of P.

Remark 3.10. It is important to note that every 3×7 dimensional heart-oriented paraletrix linear space P has a dual space $P^{\sim}$ and this space consists of all the linear functions in P. This implies that $P^{\sim}$ itself has a dual space $P^{\sim\sim}$. This space will be referred to as the second dual of P and it consists of all the linear functional in $P^{\sim}$.

Theorem 3.11. *Let P be a 3×7 dimensional heart-oriented paraletrix linear space and $P^{\sim}$ be the dual space of P. Then each $a \in P$ determines a unique element $\bar{a} \in P^{\sim\sim}$ such that for any $\Lambda \in P^{\sim}$ we have that $\bar{a}(\Lambda) = \Lambda(a)$.*

Proof. We are to prove that the map $\bar{a} : P^{\sim} \to K$ is linear. Now for any $k_1, k_2 \in K$ and any linear functionals $\Lambda, \omega \in P^{\sim}$, we have that

$$\bar{a}(k_1 \Lambda + k_2 \omega) = (k_1 \Lambda + k_2 \omega)(a) = k_1 \Lambda(a) + k_2 \omega(a) = k_1 \bar{a}(\Lambda) + k_2 \bar{a}(\omega).$$

Thus $\bar{a}$ is linear and so $\bar{a} \in P^{\sim\sim}$. ☐

4 Adjoints of linear mappings in paraletrix spaces

This section considers adjoints of linear mappings in paraletrix spaces. It is known from Example 2.8 that an inner product can be defined in paraletrix spaces. The results obtained in this section are analogous to the ones given on inner product spaces, precisely on matrix spaces $M = M_{m,n}$.We start by showing that the adjoint of a linear map in paraletrix spaces is linear.

Lemma 4.1. *Let P and Q be two paraletrix linear spaces and let $\psi : P \to Q$ be a linear mapping. Then the adjoint ψ^* is linear.*

Proof. Let $m, n \in Q$. Then we have from Definition 2.3 and Definition 2.6 that

$$\begin{aligned}\langle a,\ \psi^*(m+n)\rangle &= \langle \psi(a), m+n\rangle \\ &= \langle \psi(a), m\rangle + \langle \psi(a), n\rangle & \text{(by Definition 2.6 (ii))} \\ &= \langle a, \psi^*(m)\rangle + \langle a, \psi^*(n)\rangle \\ &= \langle a, \psi^*(m) + \psi^*(n)\rangle\ .\end{aligned}$$

It follows that $\psi^*(m+n) = \psi^*(m) + \psi^*(n)$.
Now let $k \in K$ and $m \in Q$. Then we have that

$$\begin{aligned}\langle a,\ \psi^*(km)\rangle &= \langle \psi(a), km\rangle \\ &= k\langle \psi(a), m\rangle & \text{(by Definition 2.6 (ii))} \\ &= k\langle a,\ \psi^*(m)\rangle \\ &= \langle a,\ k\psi^*(m)\rangle\ .\end{aligned}$$

Consequently, $\psi^*(km) = k\psi^*(m)$. Thus ψ^* is linear. □

The following theorem gives the properties of the adjoint which is analogous to that of Theorem 2.10 of [17] and Theorem 5.1 of [4].

Theorem 4.2. *Let P, Q and R be paraletrix linear spaces and let $\psi : P \to Q$ and $\theta : Q \to R$ be linear mappings. Then we have that*

i) $(\psi + \theta)^* = \psi^* + \theta^*$,
ii) $(k\psi)^* = k\psi^*$ *for all* $k \in K$,
iii) $(\psi \circ \theta)^* = \psi^* \circ \theta^*$,
iv) $(\psi^*)^* = \psi$.

Proof. (i) For any $a \in P$ and $m \in Q$ we have that

$$\begin{aligned}\langle (\psi+\theta)(a), m\rangle &= \langle \psi(a) + \theta(a), m\rangle = \langle \psi(a), m\rangle + \langle \theta(a), m\rangle \\ &= \langle a, \psi^*(m)\rangle + \langle a, \theta^*(m)\rangle = \langle a, \psi^*(m) + \theta^*(m)\rangle \\ &= \langle a, (\psi^* + \theta^*)(m)\rangle\ .\end{aligned}$$

Consequently, it follows that $(\psi + \theta)^* = \psi^* + \theta^*$.

ii) The proof is obvious.

iii) Let $a \in P$ and $r \in R$, then we have that

$$\begin{aligned}\langle a, (\theta\circ\psi)^*(r)\rangle &= \langle (\theta\circ\psi)(a), r\rangle = \langle \theta(\psi(a)), r\rangle \\ &= \langle \psi(a), \theta^*(r)\rangle = \langle a,\ \psi^*(\theta^*(r))\rangle\ .\end{aligned}$$

Consequently, it follows that

$$(\theta\circ\psi)^*(r) = \psi^*(\theta^*(r)) = (\psi^*\circ\theta^*)(r).$$

Thus $(\theta\circ\psi)^* = \psi^*\circ\theta^*$.

iv) The proof is a routine check. □

We know from Lemma 2.2 that a paraletrix linear map is a semigroup. The theorem below shows that the adjoint of linear mappings on paraletrix spaces is also a semigroup.

Theorem 4.3. *Let* P, Q, R *and* S *be paraletrix linear spaces and* $\psi : P \to Q, \theta : Q \to R$ *and* $\beta : R \to S$ *be linear mappings. Then* $\psi^* \circ (\theta^* \circ \beta^*) = (\psi^* \circ \theta^*) \circ \beta^*$.

Proof. Let $a \in P, m \in Q, r \in R$ and $s \in S$, then we have that

$$\begin{aligned}\psi^* \circ \theta^* &= \langle \psi(a), m\rangle \circ \langle \theta(m), r\rangle = \langle a, \psi^*(m)\rangle \circ \langle m, \theta^*(r)\rangle \\ &= \langle a + m, \psi^*(m) + \theta^*(r)\rangle,\\ (\psi^* \circ \theta^*) \circ \beta^* &= \langle a + m, \psi^*(m) + \theta^*(r)\rangle \circ \langle r, \beta^*(s)\rangle \\ &= \langle (a + m) + r, \big(\psi^*(m) + \theta^*(r)\big) + \beta^*(r)\rangle.\end{aligned}$$

Similarly,

$$\begin{aligned}\theta^* \circ \beta^* &= \langle \theta(m), r\rangle \circ \langle \beta(r), s\rangle = \langle m, \theta^*(r)\rangle \circ \langle r, \beta^*(s)\rangle \\ &= \langle m + r, \theta^*(r) + \beta^*(s)\rangle,\\ \psi^* \circ (\theta^* \circ \beta^*) &= \langle a, \psi^*(m)\rangle \circ \langle m + r, \theta^*(r) + \beta^*(s)\rangle \\ &= \langle a + (m + r), \psi^*(m) + \big(\theta^*(r) + \beta^*(s)\big)\rangle.\end{aligned}$$

Consequently, we have that

$$(a + m) + r = a + (m + r),\ \big(\psi^*(m) + \theta^*(r)\big) + \beta^*(r) = \psi^*(m) + \big(\theta^*(r) + \beta^*(s)\big).$$

Thus $\psi^* \circ (\theta^* \circ \beta^*) = (\psi^* \circ \theta^*) \circ \beta^*$. □

Below is an example of an adjoint.

Example 4.4. Let P be a paraletrix linear space and $\gamma : P \to P$ be a linear operator defined by the rule: $\gamma(p_1)(x) = \int_0^x p_1(t)dt$ where p_1 is a 3×7 dimensional heart-oriented paraletrix. It is known from Example 3.8 that γ is a linear functional and that integrals can be defined over p_1 . Now suppose that p_2 is also a 3×7 dimensional heart-oriented paraletrix and γ^* is such that $\gamma^*(p_2)(y) = \int_y^1 p_2(s)ds$. Then we have that

$$\begin{aligned}\langle \gamma(p_1), p_2\rangle &= \int_0^1 \int_0^x p_1(t)dt\, \overline{p_2(x)}\, dx = \int_0^1 \int_t^1 \overline{p_2(x)}\, ds\ p_1(t)dt \\ &= \langle p_1, \gamma^*(p_2)\rangle.\end{aligned}$$

5 Application of adjoints in fractional calculus

In this section, we look at an evolutionary equation arising from fractional calculus and then present an adjoint problem corresponding to this equation.

Consider a mathematical model of some physical process that is governed by the fractional partial differential equation of the type:

$$\frac{\partial^q u}{\partial t^q} + Au = f, \quad u(x, 0) = g(x), \quad t = 0, \tag{5.1}$$

where A is a linear operator acting in the paraletrix space P with the domain $D(A)$, $g(x) \in P$, $u = u(x, t) \in D(A)$, $f = f(x, t) \in P$ for each $t \in [0, T]$, $T < \infty$. In this case, $u = u(x, t)$ is an unknown continuous function with fractional order $0 < q \leq 1$ while the continuous function f denotes the external force or source function. The quantity $\frac{\partial^q u(x,t)}{\partial t^q}$ represents the Caputo fractional partial derivative defined by

$$\frac{\partial^q u(x,t)}{\partial t^q} = \frac{1}{\Gamma(1-q)} \int_0^t (t-\varepsilon)^{-q} u_t(x,\varepsilon) d\varepsilon\,, \quad t > 0. \tag{5.2}$$

Now the adjoint problem corresponding to Equation (5.1) is as follows:

$$-\frac{\partial^q u^*}{\partial t^q} + A^* u^* = f^*, \quad u^*(x,0) = g^*(x), \ \ t = T, \tag{5.3}$$

where the function f^* and g^* are yet to be determined and A^* is the adjoint operator of $A. A^*$ is such that at each t:

$$\langle Au, u^* \rangle = \langle u, A^* u^* \rangle, \quad u \in D(A), \quad u^* \in D(A^*).$$

From (5.1) and (5.3) we have that

$$\int_0^T \frac{\partial^q}{\partial t^q}(u^*, u) dt + \int_0^T [(Au, u^*) - (u, \quad A^* u^*)] dt = \int_0^T [(f, u^*) - (f^*, u)] dt.$$

Thus,

$$(g, u_T) - (g, \ u_0^*) = \int_0^T [(f, \ u^*) - (f^*, u)] dt, \tag{5.4}$$

where $u_T = u(T), \ u_0^* = u^*(0)$.

The exact solution of (5.1) using Laplace transform is given by

$$u(x,t) = g(x) E_q(-At^q) + \int_0^t (t-\varepsilon)^{q-1} E_{q,q}(-A(t-\varepsilon)^q) f(x,\varepsilon) \, d\varepsilon,$$

where $E_q(-At^q) = \sum_{k=0}^{\infty} \frac{(-At^q)^k}{\Gamma(qk+1)}$, $q > 0$ is the well-known Mittag-Leffler function.

Remark 5.1. One can also find a linear functional of the solution of (5.1). In fact, sensitivity of this functional depends on the solution of the adjoint problem (5.3). Subsequent work will consider generalizations of the above results to fractional calculus with general analytic kernels.

6 Conclusion

We have used the known results in rhotrix vector spaces to study linear mappings in paraletrix spaces and obtained some interesting results. In section 4, we showed that the adjoints of these mappings exist and they are analogous to that of matrix spaces. Finally, we applied these adjoints to a mathematical model in fractional calculus.

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