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On a generalization of a function of J. Sándor

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Abstract: Using a strictly increasing function $\alpha : [1, \infty) \to [1, \infty)$, we define below (see(1.1) and (1.2)) two functions $S_\alpha : [1, \infty) \to \mathbb{N}$ and $S_\alpha^* : [1, \infty) \to \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers. The functions S_α and S_α^* respectively generalize the functions S and S_* introduced and studied by J. Sándor [5] as well as the functions G and G_* considered by N. Anitha [1]. In this paper we obtain several properties of S_α and S_α^*- some of which give the results of Sándor [5] and of Anitha [1] as special cases.
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1 Introduction

Suppose $\alpha : [1, \infty) \to [1, \infty)$ is a strictly increasing function. For any real number x, define the functions

$$S_\alpha(x) = \min\{m \in \mathbb{N} : \alpha(m) \geq x\} \tag{1.1}$$

and

$$S_\alpha^*(x) = \max\{m \in \mathbb{N} : \alpha(m) \leq x\}, \tag{1.2}$$

where $\mathbb{N}$ is the set of all natural numbers.

Clearly, both $S_\alpha(x)$ and $S^*_\alpha(x)$ are defined for $x \geq 1$, with their values in $\mathbb{N}$. Also if $x = \alpha(k)$ for some $k \in \mathbb{N}$, then $S_\alpha(x) = S^*_\alpha(x) = k$; and if $\alpha(k) < x < \alpha(k+1)$ for some $k \in \mathbb{N}$, then $S_\alpha(x) = k+1$ while $S^*_\alpha(x) = k$, so that

$$S_\alpha(x) = \begin{cases} S^*_\alpha(x), & \text{if } x = \alpha(k) \text{ for some } k \in \mathbb{N} \\ S^*_\alpha(x)+1, & \text{if } \alpha(k) < x < \alpha(k+1) \text{ for some } k \in \mathbb{N}. \end{cases}$$

Hence

$$S^*_\alpha(x) \leq S_\alpha(x) \leq S^*_\alpha(x) + 1 \quad \text{for } x \in [1, \infty). \tag{1.3}$$

Examples:

(a) If $F(x) = [x]!$ for $x \geq 1$, then note that

$$S_F(x) = S(x) \text{ and } S^*_F(x) = S_*(x), \tag{1.4}$$

the two functions introduced and studied by J. Sándor [5], who called $S(x)$ as "the additive analogue of the Smarandache function" and $S_*(x)$ as the "dual" of $S(x)$.

(b) If $E(x) = e^x$ for $x \geq 1$, then we observe that

$$S_E(x) = G(x) \text{ and } S^*_E(x) = G_*(x), \tag{1.5}$$

the two functions considered by N. Anitha [1].

(c) If $I(x) = x$ for $x \geq 1$, then

$$S^*_I(x) = [x], \tag{1.6}$$

showing that $S^*_\alpha(x)$ is a generalization of the greatest integer function.

(d) If $I_r(x) = x^r$ for $x \geq 1$ and $r > 0$, then

$$S^*_{I_r}(x) = [x^{\frac{1}{r}}]. \tag{1.7}$$

In Section 2 of this note, we show that the basic properties proved for $S_*(x)$ in [5] and for $G_*(x)$ in [1] also hold for $S^*_\alpha(x)$, in addition to some new discussed here. An asymptotic result proved for $S^*_\alpha(x)$ (in Section 3) gives an easy proof for the similar result for $G_*(x)$ established in [1].

2 Properties of $S^*_\alpha(x)$

In the rest of the paper $\alpha : [1, \infty) \to [1, \infty)$ is a strictly increasing function and $S^*_\alpha(x)$ is the corresponding function defined in (1.2).

Note that, in view of (1.3), it suffices to study $S^*_\alpha(x)$.

Theorem 2.1. *(A) $S^*_\alpha : [1, \infty) \to \mathbb{N}$ is surjective and monotonic increasing.*
(B) For $x \geq \alpha(9)$ the interval $\left(S^*_\alpha(x), \frac{4}{3}S^*_\alpha(x)\right)$ has at least one prime. In other words,

$$\pi\left(\frac{4}{3}S^*_\alpha(x)\right) - \pi\left(S^*_\alpha(x)\right) \geq 1 \textit{ for } x \geq \alpha(9),$$

where $\pi(t)$ is the number of primes not exceeding $t \geq 1$

Proof. (A) Given $k \in \mathbb{N}$, then for any $x \in [\alpha(k), \alpha(k+1))$ we have $S^*_\alpha(x) = k$, proving S^*_α is a surjection.

If $x, y \in [1, \infty)$ with $x < y$, then there exist $k, l \in \mathbb{N}$ such that $x \in [\alpha(k), \alpha(k+1))$ and $y \in [\alpha(l), \alpha(l+1))$. Since α is strictly increasing it follows that $k \leq l$ giving $S^*_\alpha(x) \leq S^*_\alpha(y)$.

(B) If $x \geq \alpha(9)$, then, by (A) of the theorem, there is a $n \in \mathbb{N}$ such that $S^*_\alpha(x) = n$ and that $n = S^*_\alpha(x) \geq S^*_\alpha(\alpha(9)) = 9$. That is,

$$\text{if } x \geq \alpha(9), \text{ then there is } n \in \mathbb{N} \text{ with } S^*_\alpha(x) = n \text{ and } n \geq 9. \tag{2.1}$$

Rohrbach and Weiss [4] showed that for any integer $n \geq 118$ the open interval $\left(n, \frac{14}{13}n\right)$ has at least one prime number. Also, by a direct verification, one finds that for any integer n with $9 \leq n < 118$ the open interval $\left(n, \frac{4}{3}n\right)$ has at least one prime. Since $\dfrac{14}{13} < \dfrac{4}{3}$, it follows that

$$\left(n, \frac{4}{3}n\right) \text{ has at least one prime number for any integer } n \geq 9. \tag{2.2}$$

From (2.1) and (2.2), part (B) of the theorem follows. □

Remark 2.2. We note that in the special case of $\alpha(x) = F(x) = [x]!$, it has been proved in [3], using estimates for $\ln(n!)$ that the open interval $(S(x), S(x^2))$ has at least one prime for $x > \sqrt{13!}$.

Theorem 2.3. *(A) If $D = \{\alpha(k) : k \in \mathbb{N}\}$, then at each point of D the function $S^*_\alpha(x)$ is right-continuous but not left-continuous; also $S^*_\alpha(x)$ is contiuous elsewhere in $[1, \infty)$. More precisely, $S^*_\alpha(x)$ is continuous on $[1, \infty) - D$.*

(B) On any bounded interval $[a, b]$, the function $S^*_\alpha(x)$ is Riemann integrable and that $\displaystyle\int_a^b S^*_\alpha(x)dx$ is as given in (2.4).

Proof. (A) Observe that

$$S^*_\alpha(\alpha(k) + 0) = \lim_{\substack{x \to \alpha(k) \\ x > \alpha(k)}} S^*_\alpha(x) = k \text{ and } S^*_\alpha(\alpha(k) - 0) = \lim_{\substack{x \to \alpha(k) \\ x < \alpha(k)}} S^*_\alpha(x) = k - 1,$$

which shows that $S^*_\alpha(x)$ is discontinuous at each $\alpha(k)$. Also since $S^*_\alpha(\alpha(k)) = k$, we get $S^*_\alpha(\alpha(k) + 0) = S^*_\alpha(\alpha(k)) = k$ and $S^*_\alpha(\alpha(k) - 0) \neq S^*_\alpha(\alpha(k))$. That is, $S^*_\alpha(x)$ is right-continuous at each $\alpha(k)$ and not left-continuous at any $\alpha(k)$. Clearly, it is continuous elsewhere.

(B) If $[a, b]$ is a bounded interval, then it has only a finite number of $\alpha(k)$, (since α is strictly increasing) which are discontinuities of $S^*_\alpha(x)$. Therefore, by Lebesgue's criterion (Theorem 7.48 of [2], p. 171), S^*_α is Riemann integrable on $[a, b]$.

If $[a, b] \subseteq [\alpha(k), \alpha(k+1))$ for some $k \in \mathbb{N}$, then $S^*_\alpha(x) = k$ for all $x \in [a, b]$ so that

$$\int_a^b S^*_\alpha(x)dx = k(b - a). \tag{2.3}$$

If $[a, b]$ is any bounded interval with $a \in [\alpha(k), \alpha(k+1))$ and $b \in [\alpha(l), \alpha(l+1))$ for $k, l \in \mathbb{N}$, then

$$\int_a^b S_\alpha^*(x)dx = \int_a^{\alpha(k+1)} S_\alpha^*(x)dx + \sum_{t=1}^{l-k-1} \int_{\alpha(k+t)}^{\alpha(k+t+1)} S_\alpha^*(x)dx + \int_{\alpha(l)}^b S_\alpha^*(x)dx$$
$$= k\{\alpha(k+1) - a\} + \sum_{t=1}^{l-k-1} (k+t)\{\alpha(k+t+1) - \alpha(k+t)\} + l\{b - \alpha(l)\}, \quad (2.4)$$

wherein we used (2.3). □

Now we show that certain real functions are Riemann–Stieltjes integrable with respect to S_α^* on any bounded interval $[a, b]$.

Theorem 2.4. *Suppose f is a bounded real-valued function defined on $[a, b]$ such that it is left-continuous at each $\alpha(k) \in D_0 \doteq D \cap [a, b]$. Then f is Riemann–Stieltjes integrable with respect to S_α^* on $[a, b]$ and*

$$\int_a^b f(x)d\left(S_\alpha^*(x)\right) = \sum_{\alpha(k) \in D_0} f\left(\alpha(k)\right).$$

Trivially, the theorem holds if f is continuous on $[a, b]$.

Proof. Function f is left-continuous at each $\alpha(k) \in D_0$ (by hypothesis) and S_α^* is right-continuous at each $\alpha(k) \in D_0$ (by Theorem 2.3 (A)). Therefore, by Theorem 7.11 of [2] (p. 148–149), f is Riemann–Stieltjes integrable with respect to S_α^* on $[a, b]$ and the identity of the theorem holds. □

Corollary 2.5. *If f is a bounded real-valued function defined on $[a, b]$ and is left-continuous at each integer in $[a, b]$, then f is Riemann–Stieltjes integrable with respect to $[x]$ and*

$$\int_a^b f(x)d\left([x]\right) = \sum_{k \in \mathbb{N} \cap [a,b]} f(k).$$

Proof. Taking $\alpha(x) = I(x) \equiv x$ in Theorem 2.4, the Corollary follows in view of (1.6). □

Remark 2.6. Corollary 2.5 is well-known (for instance, see [2], Theorem 7.12., p. 149).

Remark 2.7. If α is also continuous on $[a, b]$, then by Theorem 2.4, α is Riemann–Stieltjes integrable with respect to S_α^* on $[a, b]$ and

$$\int_a^b \alpha(x)d\left(S_\alpha^*(x)\right) = \sum_{\alpha(k) \in D_0} \alpha\left(\alpha(k)\right). \quad (2.5)$$

Taking, for instance $\alpha(x) = I_r(x) = x^r$ in (2.5), we get in view of (1.7), that

$$\int_a^b x^r d\left([x^{\frac{1}{r}}]\right) = \sum_{\substack{a \le k^r \le b \\ k \in \mathbb{N}}} k^{r^2} = \sum_{\substack{a^{\frac{1}{r}} \le k \le b^{\frac{1}{r}} \\ k \in \mathbb{N}}} k^{r^2}. \quad (2.6)$$

As a particular case of (2.6), we have

$$\int_1^9 x^2 d\left([\sqrt{x}]\right) = \sum_{1 \le k \le 3} k^4 = 98. \quad (2.7)$$

Again, taking $\alpha(x) = E(x) = e^x$ for $x \in [e, \infty)$ in (2.5), we get for any $[a, b] \subseteq [e, \infty)$, that

$$\int_a^b e^x d\left(G_*(x)\right) = \sum_{\substack{a \le e^k \le b \\ k \in \mathbb{N}}} e^{e^k}. \tag{2.8}$$

3 An asymptotic result

In addition to the strictly increasing nature, if the function α is also continuous on $[1, \infty)$ with range $R = \alpha\left([1, \infty)\right)$, then it is easy to see that $\alpha : [1, \infty) \to R$ is a bijection and hence has inverse α^{-1}. Moreover, by Theorem 4.53 of [2] (p. 95), α^{-1} is also continuous and strictly increasing on R.

Theorem 3.1. *If α is continuous and strictly increasing on $[1, \infty)$, then $S^*_\alpha(x) \sim \alpha^{-1}(x)$ as $x \to \infty$.*

Proof. For any $x \geq 1$ there is a unique $n \in \mathbb{N}$ such that $\alpha(n) \leq x < \alpha(n+1)$. Then $S^*_\alpha(x) = n$ and $n \leq \alpha^{-1}(x) < n + 1$. Therefore

$$\frac{n}{n+1} < \frac{S^*_\alpha(x)}{\alpha^{-1}(x)} \leq \frac{n}{n},$$

which gives

$$\lim_{n \to \infty} \frac{n}{n+1} < \lim_{x \to \infty} \frac{S^*_\alpha(x)}{\alpha^{-1}(x)} \leq 1,$$

since $n \to \infty$ as $x \to \infty$. This proves the theorem. □

Corollary 3.2. *If α is continuous and strictly increasing on $[1, \infty)$, then $S_\alpha(x) \sim \alpha^{-1}(x)$ as $x \to \infty$.*

Proof. Immediate from (1.3) and Theorem 3.1. □

Corollary 3.3. ([1], Equation (2.1)) $G_*(x) \sim \ln x$ *as* $x \to \infty$.

Proof. Taking $\alpha(x) = E(x) = e^x$ for $x \geq 1$ in Theorem 3.1, the Corollary follows in view of (1.5) and the fact $E^{-1}(x) = \ln x$ for $x \geq e$. □

Remark 3.4. Since $I^{-1}(x) = x$ and $I_r^{-1}(x) = x^{\frac{1}{r}}$ for $x \geq 1$, we get two trivial asymptotic results $S^*_I(x) = [x] \sim x$ and $S^*_{I_r}(x) = [x^{\frac{1}{r}}] \sim x^{\frac{1}{r}}$ as $x \to \infty$, from Theorem 3.1.

Remark 3.5. Since $F(x) = [x]!$ for $x \geq 1$, is not continuous, Theorem 3.1 is not applicable for $S_*(x)$, given in (1.4). However Sándor [5] proved that $S_*(x) \sim \frac{\log x}{\log \log x}$ as $x \to \infty$, using Stirling formula.

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References

[1] Anitha, N., (2005). A note on Sándor Type functions. *Journal of Inequalities in Pure and Applied Mathematics*, 6(4), Article 127.

[2] Apostol, T. M. (1974). *Mathematical Analysis* (Second Edition). Addison Wesley Publishing Company.

[3] Mincu, G., & Panaitopol, L. (2006). Properties of the Sándor function. *Notes on Number Theory and Discrete Mathematics*, 12(1), 21–24.

[4] Rohrbach, H., & Weiss, J. (1964). Zum finiten Fall des Bertrandschen Postulats. *Journal für die reine und angewandte Mathematik*, 214/215, 432–440.

[5] Sándor, J. (2001). On an additive analogue of the function S. *Notes on Number Theory and Discrete Mathematics*, 7(3), 91–95.