Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2022, Volume 28, Number 4, 677–691
DOI: 10.7546/nntdm.2022.28.4.677-691

Equations of two sets of consecutive square sums

P. J. Bush[1,*] **and K. V. Murphy**[2]

[1] Department of Mathematics, Valley City State University
101 College St. SE, Valley City, ND, 58072, United States

[2] Department of Mathematics, Rogue Community College
101 S. Bartlett St., Medford, OR, 97501, United States
e-mail: kmurphy@roguecc.edu

Received: 3 April 2021 **Revised:** 22 August 2022
Accepted: 25 October 2022 **Online First:** 27 October 2022

Abstract: In this paper we investigate equations featuring sums of consecutive square integers, such as $3^2 + 4^2 = 5^2$, and $108^2 + 109^2 + 110^2 = 133^2 + 134^2$. In general, for a sum of $m + 1$ consecutive square integers, $x^2 + (x+1)^2 + \cdots + (x+m)^2$, there is a distinct set of m consecutive squares, $(x + n)^2 + (x + (n + 1))^2 + \cdots + (x + (n + (m - 1)))^2$, to which these are equal. We present a bootstrap method for constructing these equations, which yields solutions comprising an infinite two-dimensional array. We apply a similar method to constructing consecutive square sum equations involving $m + 2$ terms on the left, and m terms on the right, formed from two distinct sets of consecutive squares separated one term to the left of the equals sign, such as $2^2 + 3^2 + 6^2 = 7^2$.
Keywords: Consecutive squares, Sums of squares, Perfect squares, Finite sums.
2010 Mathematics Subject Classification: 11D09, 11C08.

1 Introduction

The Pythagorean equation ($a^2 + b^2 = c^2$) having the "Pythagorean triple" solution $a = 3, b = 4$, $c = 5$ is an equation involving three consecutive squares: $3^2 + 4^2 = 5^2$. The Pythagorean equation can be written as

$$x^2 + (x + 1)^2 = (x + 2)^2, \tag{1}$$

an equation of three consecutive, squared terms.

[*] Prof. Preston Bush passed away on January 10, 2022.

The only positive integer solution to this equation, $x = 3$, gives rise to the well-known Pythagorean triple, but the negative integer solution, $x = -1$, is also of interest in this paper.

Another well-known equation involving sums of consecutive squares for which integer solutions can be easily found is

$$x^2 + (x+1)^2 + (x+2)^2 = (x+3)^2 + (x+4)^2. \tag{2}$$

The positive solution $x = 10$ produces $10^2 + 11^2 + 12^2 = 13^2 + 14^2$. The negative solution will be of interest, too.

In these familiar cases, the consecutive numbers form one set in one equation. Several publications have investigated sums of consecutive integer squares equalling a perfect square, including the work of Brother Alfred [1]. Bremner, Stroeker, and Tzanakis treat the problem in terms of its relation to finding integral points on elliptic curves [4]. Beeckmans [2] and Freitag [5] consider the problem in terms of Diophantine equations of degree two. Patel and Siksek have recently published their work [6, 3] on the more general problem of consecutive perfect powers equalling a perfect power, solved with the help of Bernoulli polynomials.

In this paper we will consider equations of *two* sets of consecutive integer squares in one equation, where there is a break in the sequence of consecutive bases at either the equals symbol or one term to its left. The infinite two-dimensional array of equations having two distinct sets of consecutive squares will be called *ensembles*. We present the method of construction of two such ensembles in this paper.

2 First ensemble

Equations (1) and (2) are cases of what we will call the *first ensemble equation*:

$$x^2 + (x+1)^2 + \cdots + (x+m)^2 = (x+n)^2 + (x+(n+1))^2 + \cdots + (x+(n+(m-1)))^2. \tag{3}$$

The term *ensemble* is chosen because eq. (3) represents an "ensemble" of different consecutive square sums equations in which there are m terms on the right and $m+1$ terms on the left. (A *second ensemble equation* will be introduced in the next section.)

We will say that any particular consecutive square sums equation having value m belongs to *family* m. The initial example in eq. (1) belongs to family 1, because $m = 1$. The equation of five consecutive integers shown in eq. (2) belongs to family 2.

In section 2.1, we demonstrate a bootstrap method of solving eq. (3) for all family 2 equations. Then, in section 2.2, we discuss the solution of eq. (3) for equations of any family m. In section 2.3, we provide a recurrence relation equivalent to the bootstrap method, and conclude the section in 2.4 by discussing some features of the first ensemble.

2.1 Family $m = 2$: Bootstrap method

By letting $m = 2$ in eq. (3), we have the general consecutive square sums equation for family 2:

$$x^2 + (x+1)^2 + (x+2)^2 = (x+n)^2 + (x+(n+1))^2. \tag{4}$$

It is important to note that, in eq. (4), the consecutive squares on the right may not necessarily be a continuation of the consecutive squares on the left – that is, n may or may not equal three. While eq. (4) is easily solved as

$$x = 2(n-1) \pm \sqrt{6n(n-1)} \tag{5}$$

the challenge lies in finding values of n that yield x as integers. A simple bootstrapping method involving the negation of an already known solution facilitates this discovery, as we now illustrate.

When $n = 3$ then one solution is $x = 10$. This gives the familiar equation $10^2 + 11^2 + 12^2 = 13^2 + 14^2$, which we will call the *first generation* ($k = 1$) solution. Since each term is squared, then each term on the left-hand side of the equation may be negated without altering the validity of the equation: $(-12)^2 + (-11)^2 + (-10)^2 = 13^2 + 14^2$. As eq. (5) indicates, there is a positive solution x for $n = 25$ in addition to this negative one. It is $x = 2(25-1) + \sqrt{6(25)(25-1)} = 108$. Then $x + n = 108 + 25 = 133$, which forms the *second generation* ($k = 2$) solution: $108^2 + 109^2 + 110^2 = 133^2 + 134^2$. This procedure may be repeated again – negate $108, 109, 110$, and so on – to obtain the *third generation* ($k = 3$) solution. The procedure may be iterated indefinitely.

The procedure may be applied in a reverse iteration to generate the *zeroth generation* ($k = 0$). We saw that family 2, generation 1 ($10^2 + 11^2 + 12^2 = 13^2 + 14^2$) satisfies eq. (5) with $n = 3$ and the positive solution $x = 10$. The reverse iteration applies the negative solution $x = -2$ and $n = 3$ to eq. (4), producing $(-2)^2 + (-1)^2 + (0)^2 = 1^2 + 2^2$, the "trivial" zeroth generation solution.

2.2 Family m: Bootstrap method

In the previous subsection, we considered the family of equations featuring $m = 2$ consecutive square sums on the right, and $m + 1 = 3$ consecutive squares on the left. We now generalize further to any positive integer m terms on the right and $m + 1$ terms on the left, to form any family m of consecutive square sums equations.

Theorem 2.1. *The solutions to the first ensemble equation, eq. (3), are given by*

$$x = m(n-1) \pm \sqrt{m(m+1)n(n-1)}. \tag{6}$$

Proof. The left-hand side of eq. (3) may be written

$$\sum_{k=0}^{m}(x+k)^2 = x^2 + \sum_{k=1}^{m} x^2 + \sum_{k=1}^{m} 2kx + \sum_{k=1}^{m} k^2. \tag{7}$$

Similarly, the right-hand side of eq. (3) is

$$\begin{aligned}\sum_{k=0}^{m-1}(x+n+k)^2 &= (x+n)^2 + \sum_{k=1}^{m}(x+n+k)^2 - (x+n+m)^2 \\ &= -2mx - 2mn - m^2 + \sum_{k=1}^{m} x^2 + \sum_{k=1}^{m} 2nx \\ &\quad + \sum_{k=1}^{m} 2kx + \sum_{k=1}^{m} 2kn + \sum_{k=1}^{m} k^2 + \sum_{k=1}^{m} n^2. \end{aligned} \tag{8}$$

Equating the two sides, simplifying, and employing standard summation formulas, we have

$$x^2 = -2mx - 2mn - m^2 + 2mnx + m(m+1)n + mn^2. \tag{9}$$

This equation may be rearranged and the base x solved for using the quadratic formula, giving the solutions in eq. (6). □

The members of the first ensemble are presented in Table 1.

	Family 1	**Family 2**	**Family 3**	
Gen. 0	$(0,1),(1)$	$(0,1,2),(1,2)$	$(0,1,2,3),(1,2,3)$	$\cdots$
Gen. 1	$(3,4),(5)$	$(10,11,12),(13,14)$	$(21,22,23,24),(25,26,27)$	$\cdots$
Gen. 2	$(20,21),(29)$	$(108,109,110),(133,134)$	$(312,313,314,315),(361,362,363)$	$\cdots$
Gen. 3	$(119,120),(169)$	$(1078,1079,1080),(1321,1322)$	$(4365,4366,4367,4368),(5041,5042,5043)$	$\cdots$
Gen. 4	$(696,697),(985)$	$(10680,10681,10682),(13081,13082)$	$(60816,\ldots,60819),(70225,\ldots,70227)$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	

Table 1. The first ensemble is a two-dimensional array made up of families m and generations k constructed by applying the iterative procedure to the solution of eq. (3) shown in eq. (6). The solutions $(a, a+1, a+2, \ldots), (b, b+1, \ldots)$ represent $a+(a+1)^2+(a+2)^2+\cdots = b^2+(b+1)^2+\cdots$. For example, the generation 3, family 2 entry represents $1078^2+1079^2+1080^2 = 1321^2+1322^2$.

2.3 Family m: Recurrence relation

The validity of the bootstrap method may be shown by proving the corresponding recurrence relation.

Theorem 2.2. *For any given family m, generation $k+1$ of the first ensemble may be constructed from generation k by the following recurrence relation:*

$$n_{k+1} = 2x_k + n_k + m, \quad k \geq 1. \tag{10}$$

Proof. Consider eq. (6), rewritten here for convenience:

$$x = m(n-1) \pm \sqrt{m(m+1)n(n-1)}. \tag{(6)}$$

In order that x be an integer, we must have that the radicand be a perfect square. So let $m(m+1)n(n-1) = w^2$. We now define $X = 2n-1$ and $Y = 2w/(m(m+1))$. It follows that

$$X^2 - m(m+1)Y^2 = 1,$$

which is the Pell equation. It is easy to verify that the pair $(X_1, Y_1) = (2m+1, 2)$ is the fundamental solution of the Pell equation. Hence, all positive integer solutions are given by

$$X_k + \sqrt{m(m+1)}Y_k = \left((2m+1) + 2\sqrt{m(m+1)}\right)^k, \quad k \geq 1.$$

Define $\alpha = (2m+1) + 2\sqrt{m(m+1)}$ and $\beta = (2m+1) - 2\sqrt{m(m+1)}$. Then

$$X_k = \frac{\alpha^k + \beta^k}{2}, \quad Y_k = \frac{\alpha^k - \beta^k}{2\sqrt{m(m+1)}}, \quad k \geq 1.$$

Since $X_k = 2n_k - 1$ and $Y_k = 2w_k/(m(m+1))$, where $w_k = \sqrt{m(m+1)n_k(n_k-1)}$, then

$$n_k = \frac{\alpha^k + \beta^k + 2}{4}, \quad w_k = \frac{\sqrt{m(m+1)}(\alpha^k - \beta^k)}{4}, \quad k \geq 1.$$

From eq. (6) we know that $x_k = m(n_k - 1) + \sqrt{m(m+1)n_k(n_k-1)}$, which may be rewritten

$$x_k = m(n_k - 1) + w_k.$$

Thus, for $k \geq 1$, the right-hand side of eq. (10) equals the left-hand side, as the following string of equalities shows:

$$\begin{aligned}
2x_k + n_k + m &= 2\big(m(n_k-1) + w_k\big) + n_k + m \\
&= (2m+1)n_k + 2w_k - m \\
&= (2m+1)\left(\frac{\alpha^k + \beta^k + 2}{4}\right) + 2\left(\frac{\sqrt{m(m+1)}(\alpha^k - \beta^k)}{4}\right) - m \\
&= \frac{\big((2m+1) + 2\sqrt{m(m+1)}\big)\alpha^k}{4} + \frac{\big((2m+1) - 2\sqrt{m(m+1)}\big)\beta^k}{4} \\
&\quad + (2m+1)\frac{1}{2} - m \\
&= \frac{\alpha \cdot \alpha^k}{4} + \frac{\beta \cdot \beta^k}{4} + m + \frac{1}{2} - m \\
&= \frac{\alpha^{k+1} + \beta^{k+1} + 2}{4}.
\end{aligned}$$

Therefore, the recurrence relation given by eq. (10) holds. □

2.4 Discussion of the first ensemble

The first ensemble exhibits some intriguing patterns.

Theorem 2.3. *For any given family m in generation 1, the base x of the left-most term of eq. (3) is a triangular number.*

Proof. We show that the base x of the left-most term of eq. (3), labelled here x_1 with 1 denoting generation, is a triangular number. We proceed by showing that

$$x_1 = j(j+1)/2,$$

where $j = 2m$.

By eq. (10), we have that $n_1 = 2x_0 + n_0 + m$. Since 0 is the base x_0 of all members of generation 0, it follows $n_1 = n_0 + m$. Thus, eq. (3) simplifies to

$$(1)^2 + (2)^2 + \cdots + (m)^2 = (n_0)^2 + (n_0+1)^2 + \cdots + (n_0 + (m-1))^2,$$

where there are m terms on both the left- and right-hand sides of the equation. In order that the equation hold, we must have that $n_0 = 1$. Hence, $n_1 = 1 + m$. By eq. (6),

$$\begin{aligned}
x_1 &= m(n_1 - 1) + \sqrt{m(m+1)n_1(n_1 - 1)} \\
&= m((1+m) - 1) + \sqrt{m(m+1)(1+m)((1+m) - 1)} \\
&= m^2 + \sqrt{m^2(m+1)^2} \\
&= m^2 + m(m+1) \\
&= m(2m+1) \\
&= \frac{2m(2m+1)}{2} \\
&= \frac{j(j+1)}{2}, \quad j = 2m,
\end{aligned}$$

as we set out to show.

Therefore, for any given family m in generation 1, the base x of the left-most term of eq. (3) is a triangular number. □

Coupled with the bootstrap method to build "vertical" entries of the generations within a family, Theorem (2.3) facilitates the construction of the ensemble horizontally across families.

In generation 1, terms in the left-set $(a, a+1, a+2, \ldots)$ and right set $(b, b+1, \ldots)$ are consecutive. That is, with respect to eq. (3), $n = m + 1$. Between families m and $m + 1$, where family m has $2m + 1$ terms and family $m + 1$ has $2(m + 1) + 1$ terms, there are $[(2m+1) + (2(m+1)+1)]/2 = 2m + 2$ terms, which one may envision as consecutive terms "linking" the consecutive terms in successive families. For example, between the three terms of family 1 $(3, 4, 5)$ and the five terms of family 2 $(10, 11, 12, 13, 14)$ are the four "discarded" terms linking them: $6, 7, 8, 9$.

While Table 1 displays the integer solutions of eq. (3) derived from positive integers n, implicit in the table are those solutions corresponding to negative integers n. For example, there is a hidden generation 1, family $m = 8$ solution having base $x = 20$, which is given by $n = -49$. The corresponding equation is:

$$20^2+21^2+22^2+23^2+24^2+25^2+26^2+27^2+28^2 = 22^2+23^2+24^2+25^2+26^2+27^2+28^2+29^2.$$

Intriguingly, seven terms cancel from each side of this equation, and it thereby collapses from family $m = 8$ to the generation 1, family $m = 8 - 7 = 1$ solution

$$20^2 + 21^2 = 29^2,$$

which is seen in Table 1.

2.5 Sparseness factor

There is a general sparseness to the generations of solutions of each family. For example, in family 2, the value of the leftmost term of the equation initially increases by nearly an order of magnitude for each successive generation (see Table 1). In general, the amount by which the base x_k of generation k increases from the previous generation's base x_{k-1} is given by the sparseness factor α. That is,

$$x_k \approx \alpha x_{k-1}. \tag{11}$$

This approximation becomes exact in the limit as $k \to \infty$.

Theorem 2.4. *In the positive infinite limit of generation k, the amount by which base x_k of family m increases from the previous generation x_{k-1} is given by the sparseness factor α, where*

$$\alpha = 2m + 1 + 2\sqrt{m(m+1)}.$$

Proof. Let $\alpha = 2m + 1 + 2\sqrt{m(m+1)}$ and $\beta = 2m + 1 - 2\sqrt{m(m+1)}$. We prove that base x_k of family m increases from the previous generation x_{k-1} by a factor of α by showing that

$$\lim_{k\to\infty} \frac{x_{k+1}}{x_k} = \alpha.$$

Recall the formula for x_k, given in eq. (6) and rewritten here for convenience:

$$x_k = m(n_k - 1) \pm \sqrt{m(m+1)n_k(n_k-1)}. \tag{(6)}$$

From the proof of Theorem (2.2), we have that, for $k \geq 1$,

$$n_k = \frac{\alpha^k + \beta^k + 2}{4}, \quad w_k = \frac{\sqrt{m(m+1)}(\alpha^k - \beta^k)}{4} = \sqrt{m(m+1)n_k(n_k-1)}.$$

Thus, for $k \geq 1$,

$$\begin{aligned} x_k &= m(n_k - 1) + \sqrt{m(m+1)n_k(n_k-1)} \\ &= m(n_k - 1) + w_k \\ &= m\left(\frac{\alpha^k + \beta^k + 2}{4} - 1\right) + \frac{\sqrt{m(m+1)}(\alpha^k - \beta^k)}{4} \\ &= \frac{(m + \sqrt{m(m+1)})\alpha^k + (m - \sqrt{m(m+1)})\beta^k - 2m}{4} \\ &= \frac{(\alpha-1)\alpha^k + (\beta-1)\beta^k - 4m}{8}. \end{aligned}$$

It follows

$$\lim_{k\to\infty} \frac{x_{k+1}}{x_k} = \lim_{k\to\infty} \frac{(\alpha-1)\alpha^{k+1} + (\beta-1)\beta^{k+1} - 4m}{(\alpha-1)\alpha^k + (\beta-1)\beta^k - 4m}.$$

Consider the limit of β^k as k approaches infinity:

$$\begin{aligned} \lim_{k\to\infty} \beta^k &= \lim_{k\to\infty} (2m + 1 - 2\sqrt{m(m+1)})^k \\ &= \lim_{k\to\infty} (\exp(\ln(2m + 1 - 2\sqrt{m(m+1)})^k)) \\ &= \lim_{k\to\infty} (\exp(k \ln(2m + 1 - 2\sqrt{m(m+1)}))) \\ &= \exp(\lim_{k\to\infty} k \ln(2m + 1 - 2\sqrt{m(m+1)})) \\ &= \exp(\ln(2m + 1 - 2\sqrt{m(m+1)}) \cdot \lim_{k\to\infty} k) \\ &= \exp(\ln(2m + 1 - 2\sqrt{m(m+1)}) \cdot \infty). \end{aligned}$$

Given m is a positive integer, then

$$0 < 2m + 1 - 2\sqrt{m(m+1)} < 1$$

such that

$$\ln(2m + 1 - 2\sqrt{m(m+1)}) < 0.$$

Consequently,

$$\begin{aligned}\lim_{k\to\infty}\beta^k &= \exp(\ln(2m+1-2\sqrt{m(m+1)})\cdot\infty)\\ &= \exp(-\infty)\\ &= 0.\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{k\to\infty}\frac{x_{k+1}}{x_k} &= \lim_{k\to\infty}\frac{(\alpha-1)\alpha^{k+1}+(\beta-1)\beta^{k+1}-4m}{(\alpha-1)\alpha^k+(\beta-1)\beta^k-4m}\\ &= \lim_{k\to\infty}\frac{(\alpha-1)\alpha^{k+1}-4m}{(\alpha-1)\alpha^k-4m}\\ &= \lim_{k\to\infty}\frac{(\alpha-1)\alpha^{k+1}}{(\alpha-1)\alpha^k}\\ &= \lim_{k\to\infty}\frac{\alpha\cdot\alpha^k}{\alpha^k}\\ &= \lim_{k\to\infty}\alpha\\ &= \alpha.\end{aligned}$$

Therefore, in the limit of k at infinity, the ratio of x_{k+1} to x_k is equal to the sparseness factor α, as we set out to show. □

3 Second ensemble

Like the first ensemble, the *second ensemble* is a two-dimensional array of infinite families, each made up of likely infinite generations. The second ensemble equation is

$$x^2+(x+1)^2+\cdots+(x+m)^2+(x+n)^2=(x+(n+1))^2+(x+(n+2))^2+\cdots+(x+(n+m))^2. \quad (12)$$

There are $m+1$ terms making up the left set of consecutive squares, which are partitioned from the right set of consecutive squares one term to the left of the equality symbol. There are $m+1$ consecutive squares making up the right set.

We first demonstrate the application of the bootstrap method to the second ensemble with family $m=1$.

3.1 Family $m = 1$: Bootstrap method

The Pythagorean triples $(3, 4, 5)$ and $(5, 12, 13)$ are members of the first ensemble, family $m=1$. Since $3^2+4^2=5^2$ and $5^2+12^2=13^2$, then of course $(3^2+4^2)+12^2=13^2$. This equation has two sets of consecutive numbers: 3 and 4, as well as 12 and 13. The "breaking point" between the two consecutive bases is no longer at the equal sign, but one term to its left. This particular equation,

$$3^2+4^2+12^2=13^2, \quad (13)$$

is a member of the second ensemble, family $m=1$ and generation $k=1$. Eq. (13) is of the form

$$x^2+(x+1)^2+(x+n)^2=(x+(n+1))^2. \quad (14)$$

Note that eq. (14) is simply eq. (12) with $m = 1$, and has the simple solution

$$x = \pm\sqrt{n}. \tag{15}$$

Within this and any other family of the second ensemble we can increase the generation k by employing the negation technique introduced for the first ensemble. In eq. (13) we have $x = 3$, and $n = 9$. So we consider the negation of the bases of the left set, $(3, 4)$. With -3 and -4 as bases, then upon reordering we have

$$(-4)^2 + (-3)^2 + 12^2 = 13^2,$$

such that $x = -4$ and $n = 16$. Since -4 is the negative solution of eq. (14), then its positive counterpart ($x = 4$, $n = 16$) initiates the next generation:

$$4^2 + 5^2 + 20^2 = 21^2.$$

The procedure may be continued indefinitely.

We may also decrease the generation from eq. (13). While $x = 3$ corresponds to the positive solution of $x = \pm\sqrt{9}$, $x = -3$ is its negative solution. Thus, with $x = -3$, the left two terms of the left hand side of eq. (14) become

$$\begin{aligned}(-3)^2 + ((-3) + 1)^2 &= (-3)^2 + (-2)^2 \\ &= (-2)^2 + (-3)^2 \\ &= 2^2 + 3^2.\end{aligned}$$

Hence, we have found $x = 2$, which corresponds to $n = 4$. Therefore, the generation below eq. (13) is

$$2^2 + 3^2 + 6^2 = 7^2.$$

The downward iteration may continue twice more until the zeroth generation $0^2 + 1^2 + 0^2 = 1^2$ is found.

3.2 Family m: Bootstrap method

Eq. (14) is an instance of the second ensemble equation, eq. (12), with $m = 1$. There are also families $m = 2$, $m = 3$, and so on, characterized by increasing numbers of consecutive terms.

Theorem 3.1. *The solutions to the second ensemble equation, eq. (12), are given by*

$$x = \frac{n(m-1) \pm \sqrt{n(m+1)(n(m-1)+2m)}}{2}. \tag{16}$$

Proof. The left-hand side of the second-ensemble equation, eq. (12), may be rewritten

$$\sum_{k=0}^{m}(x+k)^2 + (x+n)^2 = x^2 + \sum_{k=1}^{m} x^2 + \sum_{k=1}^{m} 2kx + \sum_{k=1}^{m} k^2 + x^2 + 2nx + n^2. \tag{17}$$

The right-hand side of eq. (12) may be written as

$$\sum_{k=1}^{m}(x+n+k)^2 = \sum_{k=1}^{m}x^2 + \sum_{k=1}^{m}2nx + \sum_{k=1}^{m}2kx + \sum_{k=1}^{m}2kn + \sum_{k=1}^{m}k^2 + \sum_{k=1}^{m}n^2. \tag{18}$$

Equating the two sides, simplifying, and employing conventional summation formulas produces

$$2x^2 + 2nx + n^2 = 2mnx + m(m+1)n + mn^2. \tag{19}$$

The base x may be solved for using the quadratic formula, producing the solution in eq. (16). □

A good starting place for determining integer solutions in family 2 is with base $x = 0$. It follows that the zeroth generation of family 2 is

$$0^2 + 1^2 + 2^2 + 0^2 = 1^2 + 2^2.$$

To iterate upward, we negate the bases of the left set.

$$(-2)^2 + (-1)^2 + (0)^2 + (0)^2 = 1^2 + 2^2$$

With $n = 2$ and $x = -2$ as the negative solution of eq. (16), then $m = 2$. But the positive solution for $n = 2$ and $m = 2$ is $x = 4$, which yields generation 1:

$$4^2 + 5^2 + 6^2 + 6^2 = 7^2 + 8^2.$$

This procedure may be continued indefinitely to find the remaining generations.

Higher-order families can be generated in a similar way. The second ensemble is given in Table 2.

	Family 1	**Family 2**	**Family 3**	
Gen. 0	$(0,1),(0,1)$	$(0,1,2),(0,1,2)$	$(0,1,2,3),(0,1,2,3)$	$\cdots$
Gen. 1	$(1,2),(2,3)$	$(4,5,6),(6,7,8)$	$(9,10,11,12),(12,13,14,15)$	$\cdots$
Gen. 2	$(2,3),(6,7)$	$(18,19,20),(30,31,32)$	$(60,61,62,63),(84,85,86,87)$	$\cdots$
Gen. 3	$(3,4),(12,13)$	$(70,71,72),(120,121,122)$	$(357,358,359,360),(504,505,506,507)$	$\cdots$
Gen. 4	$(4,5),(20,21)$	$(264,265,266),(456,457,458)$	$(2088,\ldots,2091),(2952,\ldots,2955)$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	

Table 2. The second ensemble is a two-dimensional array made up of families m and generations k constructed by applying the negation iterative procedure to the solution of eq. (12) shown in eq. (16). The solutions $(a, a+1, a+2, \ldots), (b, b+1, b+2, \ldots)$ represent $a + (a+1)^2 + (a+2)^2 + \cdots + b^2 = (b+1)^2 + (b+2)^2 + \cdots$. For example, the generation 3, family 2 entry represents $70^2 + 71^2 + 72^2 + 120^2 = 121^2 + 122^2$.

3.3 Family m: Recurrence relation

The recurrence relation for the second ensemble is identical to that for the first ensemble.

Theorem 3.2. *For any given family m, generation $k+1$ of the second ensemble may be constructed from generation k by the following recurrence relation:*

$$n_{k+1} = 2x_k + n_k + m, \quad k \geq 1. \tag{20}$$

Proof. Consider eq. (16), rewritten here for convenience:

$$x = \frac{n(m-1) \pm \sqrt{n(m+1)(n(m-1)+2m)}}{2} \qquad ((16))$$

In order that x be an integer, we must have that the radicand be a perfect square. So let $n(m+1)(n(m-1)+2m) = w^2$. We now define $X = (n(m-1)+m)/m$ and $Y = w/(m(m+1))$. It follows that

$$X^2 - (m^2-1)Y^2 = 1,$$

which is the Pell equation. It is easy to verify that the pair $(X_1, Y_1) = (m, 1)$ is the fundamental solution of the Pell equation. Hence, all positive integer solutions are given by

$$X_k + \sqrt{m^2-1}Y_k = (m + \sqrt{m^2-1})^k, \quad k \geq 1.$$

Define $\zeta = m + \sqrt{m^2-1}$ and $\eta = m - \sqrt{m^2-1}$. Then

$$X_k = \frac{\zeta^k + \eta^k}{2}, \quad Y_k = \frac{\zeta^k - \eta^k}{2\sqrt{m^2-1}}, \quad k \geq 1.$$

Since $X_k = (n_k(m-1)+m)/m$ and $Y_k = w_k/(m(m+1))$, where

$$w_k = \sqrt{n_k(m+1)(n_k(m-1)+2m)},$$

then

$$n_k = \frac{m(\zeta^k + \eta^k - 2)}{2(m-1)}, \quad w_k = \frac{m(m+1)(\zeta^k - \eta^k)}{2\sqrt{m^2-1}}, \quad k \geq 1.$$

From eq. (16) we know that

$$x_k = \frac{n_k(m-1) + \sqrt{n_k(m+1)(n_k(m-1)+2m)}}{2},$$

which may be rewritten

$$x_k = \frac{n_k(m-1) + w_k}{2}.$$

Thus, for $k \geq 1$, the right-hand side of eq. (20) equals the left-hand side, as the following string of equalities shows:

$$\begin{aligned}
2x_k + n_k + m &= 2\big(n_k(m-1) + w_k\big)/2 + n_k + m \\
&= n_k(m-1) + n_k + m + w_k \\
&= mn_k + w_k + m \\
&= m\left(\frac{m(\zeta^k + \eta^k - 2)}{2(m-1)}\right) + \frac{m(m+1)(\zeta^k - \eta^k)}{2\sqrt{m^2-1}} + m \\
&= \frac{m^2\sqrt{m+1}(\zeta^k + \eta^k - 2) + m(m^2-1)(\zeta^k - \eta^k)}{2(m-1)\sqrt{m^2-1}} + m \\
&= \frac{m(m + \sqrt{m^2-1})\zeta^k + m(m - \sqrt{m^2-1})\eta^k - 2m}{2(m-1)} \\
&= \frac{m(\zeta \cdot \zeta^k + \eta \cdot \eta^k - 2)}{2(m-1)} \\
&= \frac{m(\zeta^{k+1} + \eta^{k+1} - 2)}{2(m-1)} \\
&= n_{k+1}.
\end{aligned}$$

Therefore, the recurrence relation given by eq. (20) holds. □

3.4 Discussion of the second ensemble

The second ensemble holds some intriguing patterns.

Theorem 3.3. *For any given family m in generation 1, the base x of the left-most term of eq. (3) is a perfect square. Specifically, it is m^2.*

Proof. We show that the base x of the left-most term of eq. (12), labelled here x_1 with 1 denoting generation, is a perfect square. We proceed by showing that $x_1 = m^2$.

By eq. (20), we have that $n_1 = 2x_0 + n_0 + m$. Since 0 is the base x_0 of all members of generation 0, it follows $n_1 = n_0 + m$. Thus, eq. (12) simplifies to

$$(1)^2 + (2)^2 + \cdots + (m)^2 + (n_0)^2 = (n_0 + 1)^2 + (n_0 + 2)^2 + \cdots + (n_0 + m)^2,$$

where there are $m + 1$ terms on the left-hand side and m terms on the right-hand side of the equation. In order that the equation hold, we must have that $n_0 = 0$. Hence, $n_1 = m$. By eq. (16),

$$\begin{aligned} x_1 &= \frac{n_1(m-1) + \sqrt{n_1(m+1)(n_1(m-1)+2m)}}{2} \\ &= \frac{m^2 - m + \sqrt{(m^2-m)(m^2-m)}}{2} \\ &= \frac{m^2 - m + m^2 + m}{2} \\ &= m^2, \end{aligned}$$

as we set out to show.

Therefore, for any given family m in generation 1, the base x of the left-most term of eq. (12) is a perfect square. □

Additionally, in family 1 the base $x + n$ of the right set of consecutive squares increases by successive multiples of two: From the zeroth to the first generation, $x + n$ increases by 2; from the first to the second, by 4; and so on. Equivalently, $x + n = k(k + 1)$, since $x = k$.

3.5 Sparseness factor

As with the first ensemble, there is a general sparseness to the generations of solutions of each family in the second ensemble. Specifically,

$$x_k \approx \zeta x_{k-1}. \tag{21}$$

This approximation becomes exact in the limit as $k \to \infty$.

Theorem 3.4. *In the positive infinite limit of generation k, the amount by which base x_k of family m increases from the previous generation x_{k-1} is given by the sparseness factor ζ, where*

$$\zeta = m + \sqrt{m^2 - 1}.$$

Proof. Let $\zeta = m + \sqrt{m^2-1}$ and $\eta = m - \sqrt{m^2-1}$. We prove that base x_k of family m increases from the previous generation x_{k-1} by a factor of ζ by showing that

$$\lim_{k\to\infty} \frac{x_{k+1}}{x_k} = \zeta.$$

Recall the formula for x_k, given in eq. (16) and rewritten here for convenience:

$$x_k = \frac{n_k(m-1) \pm \sqrt{n_k(m+1)(n_k(m-1)+2m)}}{2}. \qquad ((16))$$

From the proof of Theorem (3.2), we have that, for $k \geq 1$,

$$n_k = \frac{m(\zeta^k+\eta^k-2)}{2(m-1)}, \quad w_k = \frac{m(m+1)(\zeta^k-\eta^k)}{2\sqrt{m^2-1}} = \sqrt{n_k(m+1)(n_k(m-1)+2m))}.$$

Thus, for $k \geq 1$,

$$\begin{aligned}
x_k &= \frac{n_k(m-1) + \sqrt{n_k(m+1)(n_k(m-1)+2m)}}{2} \\
&= \frac{n_k(m-1)+w_k}{2} \\
&= \frac{m(m-1)(\zeta^k+\eta^k-2)}{4(m-1)} + \frac{m(m+1)(\zeta^k-\eta^k)}{4\sqrt{m^2-1}} \\
&= \frac{m\sqrt{m^2-1}(\zeta^k+\eta^k-2) + m(m+1)(\zeta^k-\eta^k)}{4\sqrt{m^2-1}} \\
&= \frac{m(m+\sqrt{m^2-1}+1)\zeta^k - m(m-\sqrt{m^2-1}+1)\eta^k - 2m\sqrt{m^2-1}}{4\sqrt{m^2-1}} \\
&= \frac{m(\zeta+1)\zeta^k - m(\eta+1)\eta^k - 2m\sqrt{m^2-1}}{4\sqrt{m^2-1}}
\end{aligned}$$

It follows

$$\lim_{k\to\infty} \frac{x_{k+1}}{x_k} = \lim_{k\to\infty} \frac{m(\zeta+1)\zeta^{k+1} - m(\eta+1)\eta^{k+1} - 2m\sqrt{m^2-1}}{m(\zeta+1)\zeta^k - m(\eta+1)\eta^k - 2m\sqrt{m^2-1}}.$$

Consider the limit of η^k as k approaches infinity:

$$\begin{aligned}
\lim_{k\to\infty} \eta^k &= \lim_{k\to\infty} (m-\sqrt{m^2-1})^k \\
&= \lim_{k\to\infty} (\exp(\ln(m-\sqrt{m^2-1})^k)) \\
&= \lim_{k\to\infty} (\exp(k\ln(m-\sqrt{m^2-1}))) \\
&= \exp(\lim_{k\to\infty} k\ln(m-\sqrt{m^2-1})) \\
&= \exp(\ln(m-\sqrt{m^2-1}) \cdot \lim_{k\to\infty} k) \\
&= \exp(\ln(m-\sqrt{m^2-1}) \cdot \infty).
\end{aligned}$$

Given m is a positive integer, then

$$0 < m - \sqrt{m^2-1} < 1,$$

such that

$$\ln(m-\sqrt{m^2-1}) < 0.$$

Consequently,

$$\begin{aligned}\lim_{k\to\infty}\beta^k &= \exp(\ln(m-\sqrt{m^2-1})\cdot\infty)\\ &= \exp(-\infty)\\ &= 0.\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{k\to\infty}\frac{x_{k+1}}{x_k} &= \lim_{k\to\infty}\frac{m(\zeta+1)\zeta^{k+1}-m(\eta+1)\eta^{k+1}-2m\sqrt{m^2-1}}{m(\zeta+1)\zeta^{k}-m(\eta+1)\eta^{k}-2m\sqrt{m^2-1}}\\ &= \lim_{k\to\infty}\frac{m(\zeta+1)\zeta^{k+1}-2m\sqrt{m^2-1}}{m(\zeta+1)\zeta^{k}-2m\sqrt{m^2-1}}\\ &= \lim_{k\to\infty}\frac{m(\zeta+1)\zeta^{k+1}}{m(\zeta+1)\zeta^{k}}\\ &= \lim_{k\to\infty}\frac{\zeta\cdot\zeta^k}{\zeta^k}\\ &= \lim_{k\to\infty}\zeta\\ &= \zeta.\end{aligned}$$

Therefore, in the limit of k at infinity, the ratio of x_{k+1} to x_k is equal to the sparseness factor ζ, as we set out to show. □

4 Postscript: Equations with an even number of terms

The first ensemble equations described in Section 2.1 feature an odd number of terms. One might wonder whether consecutive squares equations can also be constructed from an even number of terms. To demonstrate that they can, consider the following equation featuring three terms on the left and three on the right:

$$x^2+(x+1)^2+(x+2)^2=(x+n)^2+(x+(n+1))^2+(x+(n+2))^2.$$

This equation trivially holds when $n=0$. It also holds for nontrivial cases. For example, if $n=4$, then $x=-3$, as in $(-3)^2+(-2)^2+(-1)^2=(1)^2+(2)^2+(3)^2$.

However, it is not universally true that consecutive squares equations can be constructed from an even number of terms. Consider the case where there are three terms on the left and one on the right. Any attempt to find a sum of three consecutive squares equaling one square, as in

$$x^2+(x+1)^2+(x+2)^2=(x+n)^2,$$

will fail. This is well-known in the literature [1], but what follows is a brief proof.

Theorem 4.1. *The sum of three consecutive squares is not a perfect square.*

Proof. Consider the sum of consecutive squares

$$\sum_{k=0}^{m}(x+k)^2=x^2+(x+1)^2+(x+2)^2+\cdots+(x+(m-1))^2+(x+m)^2. \tag{22}$$

In the case that $m = 2$ to give the sum of three consecutive squares $x^2 + (x+1)^2 + (x+2)^2 = 3x^2 + 6x + 5$, then under division modulo 3 we have: $3x^2 + 6x + 5 = \ 3(x^2 + 2x + 1) + 2 \equiv 2 \pmod 3$. But all perfect squares $x^2 \equiv 0, 1 \pmod 3$. Therefore, the sum of three consecutive squares is never a perfect square. □

Similar proofs can be constructed to show that four ($n = 3$) and several other higher consecutive square sums are also never equal to a perfect square. The values of n that do give rise to consecutive squares equalling a perfect square include $11, 23, 24, 26, \ldots$, [1].

Acknowledgements

The authors thank the referees for their comments and suggestions and helping improve the quality of this paper. K. V. Murphy would also like to thank his previous affiliation, Valley City State University in Valley City, North Dakota, USA, for providing an intellectually stimulating environment in which this scholarship was originally conceived. In addition, K. V. Murphy would like to thank his wife for her unending support, careful proofreading of the manuscript, and mathematical insight.

References

[1] Alfred, U. (1964). Consecutive integers whose sum of squares is a perfect square. *Mathematic Magazine*, 37(1), 19–32.

[2] Beeckmans, L. (1994). Squares expressible as a sum of consecutive squares. *The American Mathematical Monthly*, 101, 437–442.

[3] Bennett, M. A., Patel, V., & Siksek, S. (2016). Perfect powers that are sums of consecutive cubes. *Mathematika*, 63, 230–249.

[4] Bremner, A., & Stroeker, R. J., & Tzanakis, N. (1997). On sums of consecutive squares. *Journal of Number Theory*, 62, 39–70.

[5] Freitag, H. T., & Phillips, G. M. (1996). On the sum of consecutive squares. *Applications of Fibonacci Numbers*, 6, 137–142.

[6] Patel, V., & Siksek, S. (2017). On powers that are sums of consecutive like powers. *Research in Number Theory*, 3, Article 2.