Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2022, Volume 28, Number 4, 666–676
DOI: 10.7546/nntdm.2022.28.4.666-676

New type degenerate Stirling numbers and Bell polynomials

Hye Kyung Kim

Department of Mathematics Education, Daegu Catholic University
Gyeongsan 38430, Republic of Korea
e-mail: hkkim@cu.ac.kr

Received: 19 July 2022 **Revised:** 22 September 2022
Accepted: 24 October 2022 **Online First:** 27 October 2022

Abstract: In this paper, we introduce a new type degenerate Stirling numbers of the second kind and their degenerate Bell polynomials, which is different from degenerate Stirling numbers of the second kind studied so far. We investigate the explicit formula, recurrence relation and Dobinski-like formula of a new type degenerate Stirling numbers of the second kind. We also derived several interesting expressions and identities for bell polynomials of these new type degenerate Stirling numbers of the second kind including the generating function, recurrence relation, differential equation with Bernoulli number as coefficients, the derivative and Riemann integral, so on.

Keywords: Stirling numbers of the first and second kind, Degenerate Stirling numbers of the second kind, Bell polynomials, Bernoulli polynomials.
2020 Mathematics Subject Classification: 05A15, 05A18, 11B68.

1 Introduction

Special functions and polynomials appear in mathematical physics, electrodynamics, quantum mechanics, and even in statistics and biology to find explicit solutions to the most important problems. Among them, one of the most important sets of special numbers is the class of Stirling numbers (of the first and second kind), introduced in 1730 by the Scottish mathematician James Stirling (1692, 1770). The Stirling numbers of the second kind are generally denoted by $S_2(n,k)$ and are often denoted by $\left\{ n \atop k \right\}$ in combinatorics problems.

The degenerate version of these special numbers and polynomials were stared by Carlitz [3]. In recent years, some mathematicians have explored degenerate versions of many special polynomials and numbers, including the degenerate Stirling numbers of the first and second kinds, the degenerate Bell numbers and polynomials, the degenerate Bernoulli polynomials, the degenerate Euler polynomials, degenerate Hermite polynomials, degenerate Lah–Bell polynomials and so on (see [3–6, 8, 10, 12, 13, 15–17, 19]). Many scholars have found many interesting results for these degenerate versions by using several other tools such as combinatorial methods, function generations, p-adic analysis, umbral calculus techniques, differential equations, probability theory and operator theory, so on (see [7, 9, 11, 14, 18, 20]).

In this paper, we introduce a new type degenerate Stirling numbers of the second kind and their degenerate Bell polynomials, which is different from degenerate Stirling numbers of the second kind studied so far. We investigate several interesting expressions and identities for these numbers and polynomials. In more detail, we derive the explicit formula of a new type degenerate Stirling numbers of the second kind, the generating function and recurrence relation of degenerate Bell polynomials, the derivative and Riemann integral expressions for their degenerate Bell polynomials, so on. In addition, we investigate the Dobinski-like formula of the degenerate Bell polynomials for new type of degenerate String number of second kind by using the series transformation formula proved by Boyadzhiev in [1].

First, we introduce several definitions and properties needed in this paper.

The Stirling numbers of the second kind $S_2(n,k)$ are the number of ways in which n-labelled objects can be subdivided among k disjoint and non-empty subsets. The Stirling numbers of the second kind $S_2(n,k)$ are given by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l \tag{1}$$

(see [2, 3, 10]), where $(x)_0 = 1$ and $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$.
From (1), the generating function of $S_2(n,k)$ is

$$\frac{1}{k!}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!} \tag{2}$$

(see [2, 3, 10]).

For $n \geq 0$, the Stirling numbers of the first kind $S_1(n,k)$ are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \tag{3}$$

(see [2, 3]), and the generating function of $S_1(n,k)$ is

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!} \tag{4}$$

(see [2, 3]).

The ordinary Bell polynomials are given by

$$\mathrm{bel}_n(x) = \sum_{k=0}^{n} S_2(n,k)x^k \tag{5}$$

(see [2, 3]).

By (5), it is well known that the generating function of $\mathrm{bel}_n(x)$ are given by

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \mathrm{bel}_n(x)\frac{t^n}{n!} \tag{6}$$

(see [2, 3, 13, 17]).

The Bernoulli polynomials are given by the generating function to be

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}. \tag{7}$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers.
For any $\lambda \in \mathbb{R}$, the degenerate exponential function $e_\lambda^x(t)$

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!} \tag{8}$$

(see [6–19]), where $(x)_{0,\lambda} = 1$ and $(x)_{n,\lambda} = x(x-\lambda)\cdots\big(x-(n-1)\lambda\big)$, $(n \geq 1)$.

The degenerate Stirling numbers of the second kind are given by as follows:

$$(x)_{n,\lambda} = \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l, \quad (n \geq 0) \tag{9}$$

(see [6–10, 13, 16, 17]).

As an inversion formula of the degenerate Stirling numbers of the second kind, the degenerate Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda}, \quad (n \geq 0) \tag{10}$$

(see [6–10, 13, 17]).

From (9) and (10), it is well known that

$$\frac{1}{k!}\big(e_\lambda(t)-1\big)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!} \quad (k \geq 0) \tag{11}$$

(see [10, 11, 13, 17]) and

$$\frac{1}{k!}\big(\log_\lambda(1+t)\big)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} \quad (k \geq 0) \tag{12}$$

(see [6, 15, 16]).

The degenerate partial Bell polynomials were introduced by Kim et al. in [13] as follows:

$$\mathrm{bel}_{n,\lambda}(x) = \sum_{j=1}^{n} S_{2,\lambda}(n,j)x^j, \quad (n \geq 0). \tag{13}$$

When $x = 1$, $\mathrm{bel}_{n,\lambda} = \mathrm{bel}_{n,\lambda}(1)$ are called the degenerate partial Bell numbers.
When $\lim_{\lambda\to 0}\mathrm{bel}_{n,\lambda}(x) = \mathrm{bel}_n(x)$.

From (13), the generating function of $\mathrm{bel}_{n,\lambda}(x)$ is

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \mathrm{bel}_{n,\lambda}(x)\frac{t^n}{n!} \tag{14}$$

(see [13]).

In this paper, we introduce a new type degenerate Stirling numbers of the second kind and degenerate Bell polynomials, which is different from degenerate Stirling numbers of the second kind studied so far. We investigate several interesting expressions and identities for these numbers and polynomials.

2 New type of degenerate Stirling numbers and its Bell polynomials

Let t be a real variable, λ be a real number, and let n be a nonnegative integer.

In view of (2), we introduce a new type degenerate Stirling numbers of the second kind as

$$\frac{1}{k!}(e^t-1)_{k,\lambda} = \sum_{n=k}^{\infty} S_2^*(n,k|\lambda)\frac{t^n}{n!} \quad \text{and} \quad S_2^*(n,0|\lambda)=0, \quad (n\geq 1). \tag{15}$$

When $\lim_{\lambda\to 0} S_2^*(n,k|\lambda) = S_2(n,k)$.

Theorem 2.1. *For $n \geq k \geq 0$, we have*

$$S_2^*(n,k|\lambda) = \frac{1}{k!}\sum_{j=0}^{n}\sum_{l=j}^{k} j!\, S_{2,\lambda}(k,l)S_1(l,j)S_2(n,j),$$

where $S_{2,\lambda}(n,l)$ are the degenerate Stirling numbers of the second kind.

Proof. From (2), (3), (9) and (15), we note that

$$\begin{aligned}\sum_{n=k}^{\infty} S_2^*(n,k|\lambda)\frac{t^n}{n!} &= \frac{1}{k!}(e^t-1)_{k,\lambda} = \frac{1}{k!}\sum_{l=0}^{k} S_{2,\lambda}(k,l)(e^t-1)_l \\ &= \frac{1}{k!}\sum_{l=0}^{k} S_{2,\lambda}(k,l)\sum_{j=0}^{l} S_1(l,j)(e^t-1)^j \\ &= \frac{1}{k!}\sum_{j=0}^{k}\sum_{l=j}^{k} S_{2,\lambda}(k,l)S_1(l,j)j!\sum_{n=j}^{\infty} S_2(n,j)\frac{t^n}{n!} \\ &= \frac{1}{k!}\sum_{n=k}^{\infty}\sum_{j=0}^{n}\sum_{l=j}^{k} j!\, S_{2,\lambda}(k,l)S_1(l,j)S_2(n,j)\frac{t^n}{n!}.\end{aligned} \tag{16}$$

By comparing the coefficients of both sides of (16), we have the desired identity. □

Remark 1. *Replacing* t *by* $\log(1+t)$ *in* (15)*, we note that*

$$\frac{1}{k!}(t)_{k,\lambda}=\sum_{m=k}^{\infty}S_2^*(m,k|\lambda)\frac{1}{m!}\big(\log(1+t)\big)^m$$
$$=\sum_{m=k}^{\infty}S_2^*(m,k|\lambda)\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!}$$
$$=\sum_{n=k}^{\infty}\left(\sum_{m=k}^{n}S_2^*(m,k|\lambda)S_1(n,m)\right)\frac{t^n}{n!}.$$

Theorem 2.2. *For* $n\geq\ k\ \geq 0$*, we have*

$$S_2^*(n,k+1|\lambda)=\frac{1}{k+1}\sum_{l=k}^{n-1}\binom{n}{l}S_2^*(l,k|\lambda)-\frac{1-k\lambda}{k+1}S_2^*(n,k|\lambda),\quad \text{if}\ \ n\geq k+1$$

and

$$\frac{k\lambda}{k+1}S_2^*(k,k|\lambda)=0.$$

Proof. From (15), we observe that

$$\sum_{n=k+1}^{\infty}S_2^*(n,k+1|\lambda)\frac{t^n}{n!}=\frac{1}{k+1}\cdot\frac{1}{k!}(e^t-1)_{k,\lambda}(e^t-1-k\lambda)$$
$$=\frac{1}{k+1}\sum_{l=k}^{\infty}S_2^*(l,k|\lambda)\frac{t^l}{l!}\left(\sum_{j=1}^{\infty}\frac{t^j}{j!}-k\lambda\right) \tag{17}$$
$$=\frac{1}{k+1}\sum_{n=k+1}^{\infty}\left(\sum_{l=k}^{n}\binom{n}{l}S_2^*(l,k|\lambda)-k\lambda S_2^*(n,k|\lambda)\right)\frac{t^n}{n!}-\frac{k\lambda}{k+1}S_2^*(k,k|\lambda).$$

By comparing the coefficients of both sides of (17), we have the desired result. □

In view of (5), let us consider the new type degenerate Bell polynomials for $S_2^*(n,k|\lambda)$ as

$$\phi_n^*(x|\lambda)=\sum_{k=0}^{n}S_2^*(n,k|\lambda)x^k. \tag{18}$$

Note that $\lim_{\lambda\to 0}\phi_n^*(x|\lambda)=\mathrm{bel}_n(x)$.
When $x=1$, $\phi_n^*(\lambda)=\phi_n^*(1|\lambda)$ are called the degenerate Bell numbers.

Theorem 2.3. *For* $n\geq 0$*, we have the generating function of* $\phi_n^*(x|\lambda)$ *as*

$$\sum_{n=0}^{\infty}\phi_n^*(x|\lambda)\frac{t^n}{n!}=e_\lambda^{e^t-1}(x).$$

Proof. From (15) and (18), we observe that

$$\sum_{n=0}^{\infty}\phi_n^*(x|\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}S_2^*(n,k|\lambda)x^k\right)\frac{t^n}{n!}$$
$$=\sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty}S_2^*(n,k|\lambda)\frac{t^n}{n!}\right)x^k \tag{19}$$
$$=\sum_{k=0}^{\infty}\frac{1}{k!}(e^t-1)_{k,\lambda}x^k=e_\lambda^{e^t-1}(x).$$

By (19), we have the generating function of $\phi_n^*(n|\lambda)$. □

Remark 2. *Replacing t by $\log(1+t)$ in Theorem 2.3, we have*

$$e_\lambda^t(x) = \sum_{l=0}^{\infty} \phi_l^*(x|\lambda) \frac{(\log(1+t))^l}{l!} = \sum_{l=0}^{\infty} \phi_l^*(x|\lambda) \sum_{n=l}^{\infty} S_1(n,l) \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \phi_l^*(x|\lambda) S_1(n,l) \right) \frac{t^n}{n!}.$$

Theorem 2.4. *For $n \geq 0$, we have*

$$\phi_n^*(x|\lambda) = \sum_{k=0}^{n} S_2(n,k) \left(\frac{\log(1+\lambda x)}{\lambda} \right)^k = \mathrm{bel}_n \left(\frac{\log(1+\lambda x)}{\lambda} \right),$$

where, $\mathrm{bel}_n(x)$ *are the ordinary Bell polynomials.*

Proof. From (2), (8) and (18), we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} \phi_n^*(x|\lambda) \frac{t^n}{n!} &= e_\lambda^{e^t-1}(x) = (1+\lambda x)^{\frac{e^t-1}{\lambda}} = e^{\frac{e^t-1}{\lambda} \log(1+\lambda x)} \\ &= \sum_{k=0}^{\infty} \frac{1}{\lambda^k} (\log(1+\lambda x))^k \frac{1}{k!} (e^t-1)^k \\ &= \sum_{k=0}^{\infty} \left(\frac{\log(1+\lambda x)}{\lambda} \right)^k \sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{\log(1+\lambda x)}{\lambda} \right)^k S_2(n,k) \frac{t^n}{n!}. \end{aligned} \tag{20}$$

By comparing the coefficients of the both sides of (20), we get the first identity. From (5) and (20), we have the second identity. □

Theorem 2.5. *For $n \geq 0$, we have*

$$\phi_{n+1}^*(x|\lambda) = \frac{1}{\lambda} \log(1+\lambda x) \sum_{l=0}^{n} \binom{n}{l} \phi_l^*(x|\lambda).$$

Proof. Let $f(t) = e_\lambda^{e^t-1}(x) = (1+\lambda x)^{\frac{e^t-1}{\lambda}}$. Then $\log f(t) = \frac{1}{\lambda}(e^t-1)\log(1+\lambda x)$.

From Theorem 2.3, we have

$$\begin{aligned} \frac{d}{dt} f(t) &= \frac{1}{\lambda} e^t \log(1+\lambda x) e_\lambda^{e^t-1}(x) \\ &= \frac{1}{\lambda} \log(1+\lambda x) \sum_{j=0}^{\infty} \frac{t^j}{j!} \sum_{l=0}^{\infty} \phi_l^*(x|\lambda) \frac{t^l}{l!} \\ &= \frac{1}{\lambda} \log(1+\lambda x) \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} \phi_l^*(x|\lambda) \frac{t^n}{n!}. \end{aligned} \tag{21}$$

On the other hand, we observe that

$$\frac{d}{dt} \sum_{n=0}^{\infty} \phi_n^*(x|\lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \phi_{n+1}^*(x|\lambda) \frac{t^n}{n!}. \tag{22}$$

From (21) and (22), we have the desired identity. □

Theorem 2.6. *For $n \geq 1$, we have*

$$\frac{d}{dx}\phi_n^*(x|\lambda) = (\phi_n^*(x|\lambda))' = \frac{1}{1+\lambda x}\sum_{l=0}^{n-1}\binom{n}{l}\phi_l^*(x|\lambda), \qquad (n \geq 1).$$

Proof. We note that

$$\frac{\partial}{\partial x}e_\lambda^{e^t-1}(x) = \frac{\partial}{\partial x}(1+\lambda x)^{\frac{e^t-1}{\lambda}} = \frac{e^t-1}{1+\lambda x}\, e_\lambda^{e^t-1}(x). \tag{23}$$

By Theorem 2.3 and (23), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{d}{dx}\phi_n^*(x|\lambda)\frac{t^n}{n!} &= \frac{1}{1+\lambda x}\Big\{(e^t-1)e_\lambda^{e^t-1}(x)\Big\}\\ &= \frac{1}{1+\lambda x}\Big\{(e^t-1)\sum_{l=0}^{\infty}\phi_l^*(x|\lambda)\frac{t^l}{l!}\Big\}\\ &= \frac{1}{1+\lambda x}\sum_{n=0}^{\infty}\Big\{\sum_{l=0}^{n}\binom{n}{l}\phi_l^*(x|\lambda) - \phi_n^*(x|\lambda)\Big\}\frac{t^n}{n!}\\ &= \frac{1}{1+\lambda x}\sum_{n=0}^{\infty}\Big\{\sum_{l=0}^{n-1}\binom{n}{l}\phi_l^*(x|\lambda)\Big\}\frac{t^n}{n!}.\end{aligned} \tag{24}$$

By comparing the coefficients on both sides of (24), we have □

Theorem 2.7. *For $n \geq 1$, we have*

$$\sum_{m=0}^{n}\binom{n}{m}B_{n-m}\frac{d}{dx}\phi_m^*(x|\lambda) = \frac{n}{1+\lambda x}\phi_{n-1}^*(x|\lambda),$$

where B_n are the Bernoulli numbers.

In particular, we have

$$\frac{d}{dx}\phi_0^*(x|\lambda) = 0.$$

Proof. By multiple $\frac{t}{e^t-1}$ at both sides of the first equality of (24), we get

$$\frac{t}{e^t-1}\sum_{n=0}^{\infty}\frac{d}{dx}\phi_n^*(x|\lambda)\frac{t^n}{n!} = \frac{t}{1+\lambda x}e_\lambda^{e^t-1}(x). \tag{25}$$

The right-hand side of (25) is

$$\frac{t}{1+\lambda x}e_\lambda^{e^t-1}(x) = \frac{1}{1+\lambda x}\sum_{n=1}^{\infty}n\phi_{n-1}^*(x|\lambda)\frac{t^n}{n!}. \tag{26}$$

On the other hand, by (7), the left-hand side of (25) is

$$\begin{aligned}\frac{t}{e^t-1}\sum_{m=0}^{\infty}\frac{d}{dx}\phi_m^*(x|\lambda)\frac{t^m}{m!} &= \sum_{j=0}^{\infty}B_j\frac{t^j}{j!}\sum_{m=0}^{\infty}\frac{d}{dx}\phi_m^*(x|\lambda)\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}B_{n-m}\frac{d}{dx}\phi_m^*(x|\lambda)\frac{t^n}{n!}.\end{aligned} \tag{27}$$

By comparing the coefficients on (26) and (27), we have the desired identity. □

Theorem 2.8. *For $n \geq 1$, we have*

$$\int_0^x \phi_{n-1}^*(x|\lambda)dx = \frac{1+\lambda x}{n} \sum_{l=0}^{n} \binom{n}{l} B_{n-l}\phi_l^*(x|\lambda) - \frac{1}{n}B_n,$$

where B_n are the Bernoulli numbers.

In addition, we have $(1+\lambda x)\phi_0^*(x|\lambda) = 1$.

Proof. From Theorem 2.3, we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} \int_0^x \phi_n^*(x|\lambda)dx \frac{t^n}{n!} &= \int_0^x e_\lambda^{e^t-1}(x)dx \\ &= \left[\frac{1+\lambda x}{e^t-1} e_\lambda^{e^t-1}(x)\right]_0^x \\ &= \frac{1}{e^t-1}\Big((1+\lambda x)e_\lambda^{e^t-1}(x) - 1\Big). \end{aligned} \tag{28}$$

By multiplying t at both sides of (28), we have

$$\begin{aligned} \sum_{n=1}^{\infty} \int_0^x n\phi_{n-1}^*(x|\lambda)dx \frac{t^n}{n!} &= \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}\left((1+\lambda x)\sum_{l=0}^{\infty}\phi_l^*(x|\lambda)\frac{t^l}{l!} - 1\right) \\ &= (1+\lambda x)\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}B_{n-l}\phi_l^*(x|\lambda)\right)\frac{t^n}{n!} - \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \end{aligned} \tag{29}$$

By comparing the coefficients of both sides of (29), we have the desired identity. □

Let $D = \frac{d}{dx}$, and let f be analytic on an open set U in $\mathbb{C}$. Then, for each $x \in U$, we have

$$(xD)^n f(x) = \sum_{k=0}^{n} S_2(n,k)x^k D^k f(x), \quad (n \geq 0) \tag{30}$$

(see [11]).

K. N. Boyadzhiev [1] showed a formula that turns power series into series of functions by using (30) as follows: Assume that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{k=0}^{\infty} c_k x^k$ are power series convergent on some open disks centered at the origin. Then we have

$$\sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} f(k)x^k = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} S_2(n,k)g^{(k)}(x)x^k \tag{31}$$

(see [1, 12]).

Theorem 2.9. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} S_2^*(n,m|\lambda)f(m)x^m = \sum_{m=k}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\frac{f^{(n)}(0)k!}{m!}S_2(m,k)S_2^*(j,k|\lambda)\phi_{n-j}^*(x|\lambda)\left(\frac{x}{1+\lambda x}\right)^k,$$

where $S_2(n,k)$ are the Stirling numbers of the second kind.

Proof. Let $g(x)=e_\lambda^{e^t-1}(x)$. Then, we observe that

$$g^{(k)}(x)=\left(\frac{d}{dx}\right)^k e_\lambda^{e^t-1}(x)=\frac{(e^t-1)_{k,\lambda}}{(1+\lambda x)^k}e_\lambda^{e^t-1}(x). \tag{32}$$

From (15), (31) and (32), we note that

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{(e^t-1)_{m,\lambda}}{m!}f(m)x^m&=\sum_{m=0}^{\infty}\frac{f^{(m)}(0)}{m!}\sum_{k=0}^{m}S_2(m,k)x^k\frac{(e^t-1)_{k,\lambda}}{(1+\lambda x)^k}e_\lambda^{e^t-1}(x)\\&=e_\lambda^{e^t-1}(x)\sum_{m=0}^{\infty}\frac{f^{(m)}(0)}{m!}\sum_{k=0}^{m}S_2(m,k)\left(\frac{x}{1+\lambda x}\right)^k k!\sum_{j=k}^{\infty}S_2^*(j,k|\lambda)\frac{t^j}{j!}\\&=e_\lambda^{e^t-1}(x)\sum_{j=0}^{\infty}\sum_{m=k}^{\infty}\sum_{k=0}^{j}\frac{f^{(m)}(0)}{m!}S_2(m,k)\left(\frac{x}{1+\lambda x}\right)^k k!S_2^*(j,k|\lambda)\frac{t^j}{j!}\\&=\sum_{l=0}^{\infty}\phi_l^*(x|\lambda)\frac{t^l}{l!}\sum_{j=0}^{\infty}\sum_{m=k}^{\infty}\sum_{k=0}^{j}\frac{f^{(m)}(0)}{m!}S_2(m,k)\left(\frac{x}{1+\lambda x}\right)^k k!S_2^*(j,k|\lambda)\frac{t^j}{j!}\\&=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{m=k}^{\infty}\sum_{k=0}^{j}\binom{n}{j}\frac{f^{(n)}(0)k!}{m!}S_2(m,k)S_2^*(j,k|\lambda)\phi_{n-j}^*(x|\lambda)\left(\frac{x}{1+\lambda x}\right)^k\frac{t^n}{n!}.\end{aligned}\tag{33}$$

On the other hand, by (15), we have

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{(e^t-1)_{m,\lambda}}{m!}f(m)x^m&=\sum_{m=0}^{\infty}f(m)x^m\sum_{n=m}^{\infty}S_2^*(n,m|\lambda)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}S_2^*(n,m|\lambda)f(m)x^m\frac{t^n}{n!}.\end{aligned}\tag{34}$$

By (33) and (34), we have the desired identity. □

Theorem 2.10. *For* $n\geq 0$, *we have*

$$\left((1+\lambda x)\frac{d}{dx}\right)^k\phi_n^*(x|\lambda)=\begin{cases}k!\displaystyle\sum_{l=0}^{n}\sum_{j=0}^{l}\binom{n}{l}S_2^*(n-l,k|\lambda)S_2^*(l,j|\lambda)x^j, & \text{if } n\geq 0,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. From (15), we observe that

$$\begin{aligned}\left((1+\lambda x)\frac{d}{dx}\right)^k e_\lambda^{e^t-1}(x)&=(e^t-1)_{k,\lambda}e_\lambda^{e^t-1}(x)\\&=k!\sum_{m=k}^{\infty}S_2^*(m,k|\lambda)\frac{t^m}{m!}\sum_{l=0}^{\infty}\sum_{j=0}^{l}S_2^*(l,j|\lambda)x^j\frac{t^l}{l!}\\&=k!\sum_{n=k}^{\infty}\sum_{l=0}^{n}\sum_{j=0}^{l}\binom{n}{l}S_2^*(n-l,k|\lambda)S_2^*(l,j|\lambda)x^j\frac{t^n}{n!}.\end{aligned}\tag{35}$$

On the other hand, from Theorem 2.3, we have

$$\left((1+\lambda x)\frac{d}{dx}\right)^k e_\lambda^{e^t-1}(x)=\sum_{n=0}^{\infty}\left((1+\lambda x)\frac{d}{dx}\right)^k\phi_n^*(x|\lambda)\frac{t^n}{n!}. \tag{36}$$

By (35) and (36), we have the desired result. □

We consider

$$\left(x\frac{d}{dx}\right)^n e_\lambda^{e^t-1}(x) = \left(x\frac{d}{dx}\right)^n \sum_{k=0}^{\infty}(e^t-1)_{k,\lambda}\frac{x^k}{k!} = \sum_{k=0}^{\infty}(e^t-1)_{k,\lambda}\frac{k^n}{k!}x^k. \tag{37}$$

Let us put $\sum_{k=0}^{\infty}(e^t-1)_{k,\lambda}\frac{k^n}{k!}x^k = \chi_{n,\lambda}(x)$. Then, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\chi_{n,\lambda}(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(e^t-1)_{k,\lambda}\frac{k^n}{k!}x^k\frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty}(e^t-1)_{k,\lambda}\frac{1}{k!}x^k\sum_{n=0}^{\infty}k^n\frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty}(e^t-1)_{k,\lambda}\frac{1}{k!}x^k e^{kt} = e_\lambda^{e^t-1}(xe^t).\end{aligned} \tag{38}$$

Therefore, by (38), the generating function of new type polynomials $\chi_{n,\lambda}(x)$ is $e_\lambda^{e^t-1}(xe^t)$.

Acknowledgements

The author thanks Jangjeon Institute for Mathematical Science for the support of this research. This work was supported by the Basic Science Research Program, the National Research Foundation of Korea, (NRF-2021R1F1A1050151).

References

[1] Boyadzhiev, K. N. (2005). A series transformation formula and related polynomials. *International Journal of Mathematics and Mathematical Sciences*, 23, 3849–3866.

[2] Comtet, L. (1974). *Advanced Combinatorics. The Art of Finite and Infinite Expansions*. (Revised and enlarged edition) D. Reidel Publishing Co., Dordrecht.

[3] Carlitz, L. (1979). Degenerate Stirling, Bernoulli and Eulerian numbers. *Utilitas Mathematica*, 15, 51–88.

[4] Khan, W. A., & Kamarujjama, M. (2021). Some identities on type 2 degenerate Daehee polynomials and numbers. *Indian Journal of Mathematics*, 63(3), 433–447.

[5] Khan, W. A., & Kamarujjama, M. (2022). A note on type 2 degenerate multi poly-Bernoulli polynomials of the second kind. *Jangjeon Mathematics*, 25(1), 59–68.

[6] Kim, D. S., & Kim, T. (2020). A note on a new type of degenerate Bernoulli numbers. *Russian Journal of Mathematical Physics*, 27(2), 227–235.

[7] Kim, D. S., Kim, T., Kim, H. Y., & Lee, H. (2020). Two variable degenerate Bell polynomials associated with Poisson degenerate central moments. *Proceedings of the Jangjeon Mathematical Society*, 23(4), 587–596.

[8] Kim, H. K. (2020). Degenerate Lah–Bell polynomials arising from degenerate Sheffer sequences. *Advances in Difference Equations*, 2020, Article 687.

[9] Kim, H. K. (2021). Fully degenerate Bell polynomials associated with degenerate Poisson random variables. *Open Mathematics*, 19, 284–296.

[10] Kim, T. (2017). A note on degenerate Stirling polynomials of the second kind. *Proceedings of the Jangjeon Mathematical Society*, 20(3), 319–331.

[11] Kim, T., & Kim, D. S. (2022). On some degenerate differential and degenerate difference operators. *Russian Journal of Mathematical Physics*, 29(1), 37–46.

[12] Kim, T., & Kim, D. S. (2022). Some identities on degenerate Bell polynomials and their relatd identities. *Proceedings of the Jangjeon Mathematical Society*, 25(1), 1–11.

[13] Kim, T., Kim, D. S., & Dolgy, D. V. (2017). On partially degenerate Bell numbers and polynomials. *Proceedings of the Jangjeon Mathematical Society*, 20(3), 337–345.

[14] Kim, T., Kim, D. S., Jang L.-C., & Kim, H. Y. (2020). A note on discrete degenerate random variables. *Proceedings of the Jangjeon Mathematical Society*, 49(2), 521–538.

[15] Kim, T., Kim, D. S., Jang, L.-C., Lee, H., & Kim, H. (2022). Representations of degenerate Hermite polynomials. *Advances in Applied Mathematics*, 2139, Article 102359.

[16] Kim, T., Kim, D. S., & Kim, H. K. (2022). Some identities involving degenerate Stirling numbers arising from normal ordering. *AIMS Mathematics*, 7(9), 17357–17368.

[17] Kim, T., Kim, D. S., Kim H. Y., & Kwon, J. (2020). Some identities of degenerate Bell polynomials. *Mathematics*, 8(1), Article 40.

[18] Kim, T., Kim, D. S., Kwon J., & Lee, H. (2020). A note on degenerate gamma random variables. *Revista de Educación*, 388(4), 29–44.

[19] Kim, T., Kim, D. S., Lee, H., Park, S., & Kwon, J. (2021). New properties on degenerate Bell polynomials. *Complexity*, 2021, Article 7648994.

[20] Roman, S. (1984). *The Umbral Calculus*. Pure and Applied Mathematics. Vol. 111. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York.