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Arithmetical functions associated with conjugate pairs of sets under regular convolutions

Pentti Haukkanen

Faculty of Information Technology and Communication Sciences,
FI-33014 Tampere University, Finland
e-mail: pentti.haukkanen@tuni.fi

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Abstract: Two subsets P and Q of the set of positive integers is said to form a conjugate pair if each positive integer n possesses a unique factorization of the form $n = ab,\ a \in P,\ b \in Q$. In this paper we generalize conjugate pairs of sets to the setting of regular convolutions and study associated arithmetical functions. Particular attention is paid to arithmetical functions associated with k-free integers and k-th powers under regular convolution.
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1 Introduction

Two subsets P and Q of the set $\mathbb{Z}^+$ of positive integers is said to form a conjugate pair if each $n \in \mathbb{Z}^+$ posesses a unique factorization of the form

$$n = ab, \quad a \in P,\ b \in Q.$$

It is clear that $P \cap Q = \{1\}$. A simple example of a conjugate pair of sets is $\{1\}$, $\mathbb{Z}^+$. Another example consists of the set of k-free integers and the set of k-th powers. Cohen [1] studied arithmetical functions associated with conjugate pairs of sets, such as Möbius functions, totient functions, Ramanujan sums and relative partitions. Particular attention was paid to the example of k-free integers and k-th powers. It should be noted that Cohen [1] assumed that the sets in a conjugate pair possess a certain multiplicativity property. We do not need this property here.

In this paper we generalize conjugate pairs of sets to the setting of regular convolutions (for definition of regular convolution, see Section 2) and study associated arithmetical functions. We confine ourselves to Möbius functions and totient functions. Particular attention is paid to arithmetical functions associated with k-free integers and k-th powers under regular convolution. We note that the Dirichlet convolution and the unitary convolution are examples of regular convolutions. The Dirichlet convolution gives the usual conjugate pairs of sets and associated arithmetical functions. Another approach to conjugate pairs and regular convolution is presented in [7].

2 Regular convolution

In this section we introduce the concept of Narkiewicz's regular convolution. Background material on regular convolutions can be found e.g. in [5, Chapter 4] and [6,8]. We here review the concepts and notations, which are needed in this paper.

For each n let $A(n)$ be a subset of the set of positive divisors of n. The elements of $A(n)$ are said to be the A-divisors of n. The A-convolution of two arithmetical functions f and g is defined by

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d)g(n/d).$$

Narkiewicz [6] defines an A-convolution to be regular if:

(a) the set of arithmetical functions forms a commutative ring with unity with respect to the ordinary addition and the A-convolution,

(b) the A-convolution of multiplicative functions is multiplicative,

(c) the constant function 1 has an inverse μ_A with respect to the A-convolution, and $\mu_A(n) = 0$ or -1 whenever n is a prime power.

It can be proved [6] that an A-convolution is regular if and only if:

(i) $A(mn) = \{de : d \in A(m), e \in A(n)\}$ whenever $(m, n) = 1$,

(ii) for each prime power p^a (> 1) there exists a divisor $t = t_A(p^a)$ of a such that

$$A(p^a) = \{1, p^t, p^{2t}, \ldots, p^{rt}\},$$

where $rt = a$, and

$$A(p^{it}) = \{1,\ p^t, p^{2t}, \ldots, p^{it}\}, 0 \le i < r.$$

The positive integer $t = t_A(p^a)$ in Part (ii) is said to be the A-type of p^a. A positive integer n is said to be A-primitive if $A(n) = \{1, n\}$. The A-primitive numbers are 1 and p^t, where p runs through the primes and t runs through the A-types of the prime powers p^a with $a \ge 1$.

For all n let $D(n)$ denote the set of all positive divisors of n and let $U(n)$ denote the set of all unitary divisors of n, that is,

$$U(n) = \{d > 0 : d \mid n, (d, n/d) = 1\} = \{d > 0 : d \parallel n\}.$$

The D-convolution is the classical Dirichlet convolution and the U-convolution is the unitary convolution [2]. These convolutions are regular with $t_D(p^a) = 1$ and $t_U(p^a) = a$ for all prime powers p^a (> 1).

The A-analogue of the Möbius function μ_A is the multiplicative function given by

$$\mu_A(p^a) = \begin{cases} -1 & \text{if } p^a (> 1) \text{ is } A\text{-primitive}, \\ 0 & \text{if } p^a \text{ is non-}A\text{-primitive}. \end{cases}$$

In particular, $\mu_D = \mu$, the classical Möbius function, and $\mu_U = \mu^{\oplus}$, the unitary analogue of the Möbius function [2].

For a positive integer k the A_k-convolution is defined by

$$A_k(n) = \{d : \ d^k \in A(n^k)\}.$$

It is known [9] that the A_k-convolution is regular whenever the A-convolution is regular.

The symbol $(m, n)_{A,k}$ denotes the greatest k-th power divisor of m, which belongs to $A(n)$. In particular, $(m, n)_{A,1} = (m, n)_A$ and $(m, n)_{D,k} = (m, n)_k$. Note that $(m, n)_{D,1}$ is the usual greatest common divisor (m, n) of m and n.

Note. *Throughout the rest of the paper A will be an arbitrary but fixed regular convolution.*

3 Conjugate pair of sets

Definition 3.1. Two subsets P and Q of the set of positive integers form a conjugate pair under the A-convolution if every positive integer n has a unique factorization of the form

$$n = ab, \quad a \in P, \ b \in Q, \ a, b \in A(n).$$

It is clear that $P \cap Q = \{1\}$ in Definition 3.1. The pair $\{1\}$, $\mathbb{Z}^+$ is again an example of a conjugate pair of sets under the A-convolution. We below translate the concepts of k-free integers and k-th powers into the language of regular convolutions.

For a positive integer k we define n (> 1) to be a k-th power under the A-convolution if for each $p^a \parallel n$ $(p^a \neq 1)$ we have $a = kut$, $t = t_A(p^a)$. In addition, we define the number 1 to be a k-th power under any regular convolution for all k. The k-th powers under the Dirichlet convolution are the usual k-th powers. For $k = 1$ each positive integer is a k-th power under the unitary convolution and for $k > 1$ the number 1 is the only k-th power under the unitary convolution.

For a positive integer k we define n (> 1) to be k-th power free (or k-free) under the A-convolution if for each $p^a \parallel n$ we have $s < k$, where $a = st$, $t = t_A(p^a)$. In addition, we define the number 1 to be k-free under any regular convolution for all k. The k-free numbers under the Dirichlet convolution are the usual k-free numbers. For $k = 1$ the number 1 is the only k-free number under the unitary convolution and for $k > 1$ each positive integer is k-free under the unitary convolution.

Let $P_k(A)$ denote the set of k-free integers under the A-convolution, and let $Q_k(A)$ denote the set of k-th powers under the A-convolution. The pair $P_k(A)$, $Q_k(A)$ is a conjugate pair under the A-convolution.

4 Characteristic functions and Möbius functions

Let S be *any* subset of $\mathbb{Z}^+$ and let χ_S denote the characteristic function of S, that is,

$$\chi_S(n) = \begin{cases} 1, & \text{if } n \in S, \\ 0, & \text{if } n \notin S. \end{cases}$$

We define the Möbius function $\mu_{S,A}$ associated with S as

$$\mu_{S,A} *_A \zeta = \chi_S \tag{1}$$

or

$$\mu_{S,A} = \chi_S *_A \mu_A \tag{2}$$

(see [3]), where $\zeta(n) = 1$ for all $n \in \mathbb{Z}^+$.

Example 4.1. If $S = \{1\}$, then $\chi_S = \delta$ and $\mu_{S,A} = \mu_A$, where δ is the identity under the A-convolution, that is, $\delta(1) = 1$ and $\delta(n) = 0$ for $n > 1$. Further, if A is the Dirichlet convolution and the unitary convolution, then μ_A becomes μ and $\mu^{\oplus}$, respectively.

Example 4.2. Let $S = P_k(A)$, the set of all k-free integers under the A-convolution. Then $\mu_{S,A} = \mu_{A,k}$, where $\mu_{A,k}$ is the multiplicative function given as

$$\mu_{A,k}(p^a) = \mu_{A,k}(p^{st}) = \begin{cases} -1, & \text{if } s = k, \\ 0, & \text{otherwise}, \end{cases}$$

where $a \geq 1$. Denoting $\mu_{D,k} = \mu_k$ we have $\mu_k(n) = \mu(n^{1/k})$, $k \geq 1$. In particular, $\mu_1 = \mu$. Further, $\mu_{U,k} = \delta$ for $k > 1$ and $\mu_{U,1} = \mu^{\oplus}$. More generally, $\mu_{A,1} = \mu_A$.

Example 4.3. Let $S = Q_k(A)$, the set of all k-th powers ($k \geq 2$) under the A-convolution. Then $\mu_{S,A} = \lambda_{A,k}$, where $\lambda_{A,k}$ is the multiplicative function given as

$$\lambda_{A,k}(p^a) = \lambda_{A,k}(p^{st}) = \begin{cases} 1, & \text{if } s \equiv 0 \pmod{k}, \\ -1, & \text{if } s \equiv 1 \pmod{k}, \\ 0, & \text{otherwise}, \end{cases}$$

where $a \geq 1$. We denote $\lambda_{D,k} = \lambda_k$ $(k \geq 1)$. Then $\lambda_2 = \lambda$, the famous Liouville function, and $\lambda_1 = \delta$. Further, $\lambda_{U,k} = \mu^{\oplus}$ for $k > 1$ and $\lambda_{U,1} = \delta$. More generally, $\lambda_{A,1} = \delta$.

Theorem 4.1. *Let P, Q be any conjugate pair of sets under the A-convolution. Then*

1) $\chi_P *_A \chi_Q = \zeta$,

2) $\chi_P^{-1} = \mu_{Q,A}$, $\chi_Q^{-1} = \mu_{P,A}$,

3) $\mu_{P,A} *_A \mu_{Q,A} = \mu_A$,

where the inverses are under the A-convolution.

Proof. 1) Since every $n \in \mathbb{Z}^+$ can be written uniquely as $n = ab$, where $a \in P$, $b \in Q$, a, $b \in A(n)$, we have

$$(\chi_P *_A \chi_Q)(n) = \sum_{d \in A(n)} \chi_P(d)\chi_Q(n/d) = \chi_P(a)\chi_Q(b) = 1.$$

2) Applying Equation (2) and Part 1 we obtain

$$\chi_P *_A \mu_{Q,A} = \chi_P *_A \chi_Q *_A \mu_A = \zeta *_A \mu_A = \delta :$$

hence $\chi_P^{-1} = \mu_{Q,A}$. The proof of the second result is similar.

3) Applying Parts 1 and 2 we obtain

$$\mu_{P,A} *_A \mu_{Q,A} = \chi_Q^{-1} *_A \chi_P^{-1} = (\chi_P *_A \chi_Q)^{-1} = \zeta^{-1} = \mu_A.$$

This completes the proof. □

Remark 1. For the pair $P_k(A)$, $Q_k(A)$ of the k-free integers and the k-th power integers under the A-convolution Parts 1 and 2 of Theorem 4.1 become

$$\chi_{P_k(A)}^{-1} = \lambda_{A,k}, \quad \chi_{Q_k(A)}^{-1} = \mu_{A,k}, \quad \mu_{A,k} *_A \lambda_{A,k} = \mu_A. \tag{3}$$

5 An inversion formula

Let S be any subset of the set of positive integers with $1 \in S$. Then

$$f = \chi_S *_A g \Leftrightarrow g = \chi_S^{-1} *_A f. \tag{4}$$

Noting that the inverse of χ_S under the A-convolution exists if and only if $1 \in S$ we obtain the equivalency (4) multiplying the two Equations in (4) with χ_S^{-1} and χ_S, respectively.

If P, Q is a conjugate pair of sets under the A-convolution, then (4) can be written as

$$f = \chi_P *_A g \Leftrightarrow g = \mu_{Q,A} *_A f. \tag{5}$$

If $A = D$ and P is the set of squarefree integers, then (5) is Sivaramakrishnan's [10] inversion formula given as

$$f(n) = \sum_{\substack{d|n \\ d \text{ squarefree}}} g(n/d) \Leftrightarrow g(n) = \sum_{d|n} \lambda(d) f(n/d).$$

Our main interest in the inversion formula (4) is to show that it puts under the same roof the following inversion formulas of Suryanarayana [12]: For any positive integer k:

$$f(n) = \sum_{d^k e=n} g(e) \Leftrightarrow g(n) = \sum_{d^k e=n} \mu(d) f(e), \tag{6}$$

$$f(n) = \sum_{\substack{d^k e=n \\ (d,e)=1}} g(e) \Leftrightarrow g(n) = \sum_{\substack{d^k e=n \\ (d,e)=1}} \mu^{\oplus}(d) f(e). \tag{7}$$

Example 5.1. If S is the set of k-th powers under the A-convolution, that is, if $S = Q_k(A)$, then $\chi_S^{-1} = \mu_{A,k}$. In particular, $\mu_{D,k}(n) = \mu(n^{1/k})$ if n is a k-th power (in the usual sense), and $= 0$ otherwise. Thus, if $S = Q_k(D)$ and $A = D$, then (4) reduces to (6). With $k = 1$ this is the classical number-theoretic Möbius inversion formula. Note that $\mu_{U,k} = \delta$ $(k > 1)$ and thus (4) with $S = Q_k(U)$ and $A = U$ is the trivial equivalency $f = g \Leftrightarrow f = g$. The special case (7) is shown in the next example.

Example 5.2. Let $S = Q_k(D)$ and $A = U$ in (4). Then $\chi_S^{-1}(n) = \mu^{\oplus}(n^{1/k})$ if n is a k-th power (in the usual sense), and $= 0$ otherwise. Thus (4) reduces to (7). With $k = 1$ this is the unitary analogue of the Möbius inversion formula [2].

Remark 2. Equations (6) and (7) are put under the same roof using the same method in [11]. Unfortunately, however, [11] contains some misprints. For example, $\chi_{Q_k(D)}^{-1} \neq \mu_{P_k(D),U}$, unlike stated in [11], where the inverse is under the unitary convolution (cf. Example 5.2). Note that we have $\chi_{Q_k(D)}^{-1} = \mu_{P_k(D),D}$ and $\chi_{Q_k(U)}^{-1} = \mu_{P_k(U),U}$, where the inverses are under the Dirichlet convolution and the unitary convolution, respectively, by virtue of the property of a conjugate pair of sets given in Part 2 of Theorem 4.1.

6 Totient functions

Let S be any subset of the set of positive integers. We define the generalized totient function $\phi_{S,A}(n)$ associated with S as the number of integers $x \pmod{n}$ such that $(x,n)_A \in S$. It is known [3] that

$$\phi_{S,A}(n) = \sum_{d \in A(n)} d\mu_{S,A}(n/d) = (N *_A \mu_{S,A})(n), \tag{8}$$

where $N(n) = n$ for all n. If $S = \{1\}$, then $\phi_{S,A} = \phi_A$, the A-analogue of the Euler totient function. The functions ϕ_D and ϕ_U are the Euler totient function and its unitary analogue [2], respectively. For $S = P_k(D)$ and $A = D$ and for $S = Q_2(D)$ and $A = D$ the function $\phi_{S,A}$ reduces to Klee's totient function (see [5]) and Sivaramakrishnan's totient function [10], respectively.

Theorem 6.1. *Let P, Q be a conjugate pair of sets. Then*

$$n = \sum_{\substack{d \in A(n) \\ d \in Q}} \phi_{P,A}(n/d) \tag{9}$$

and

$$\phi_{P,A}(n) = \sum_{\substack{d \in A(n) \\ d \in P}} \phi_A(n/d). \tag{10}$$

Proof. By Part 2 of Theorem 4.1 and Equation (8) we obtain Equation (9) as follows

$$N = \phi_{P,A} *_A (\mu_{P,A})^{-1} = \phi_{P,A} *_A \chi_Q.$$

Further, by Equations (8) and (2), we have

$$\phi_{P,A} = N *_A \mu_{P,A} = N *_A \mu_A *_A \chi_P = \phi_A *_A \chi_P.$$

This proves (10). □

Theorem 6.2. *For any subset S of the set of positive integers*

$$\sum_{d\in A(n)} \phi_{S,A}(d) = n \sum_{\substack{d\in A(n)\\ d\in S}} d^{-1}. \tag{11}$$

Proof. We have

$$\begin{aligned} \phi_{S,A} *_A \zeta &= N *_A \mu_{S,A} *_A \zeta = N *_A \chi_S *_A \mu_A *_A \zeta \\ &= N *_A \chi_S *_A \delta = N *_A \chi_S. \end{aligned}$$

This proves (11). □

The functions $\phi_{P_k(A),A}(n)$ and $\phi_{Q_k(A),A}(n)$ count the number of integers $x \pmod n$ such that $(x,n)_A$ is k-free and k-th power, respectively, under the A-convolution. We next apply Equations (8), (9), (10) and (11) in the context the functions $\phi_{P_k(A),A}$ and $\phi_{Q_k(A),A}$. For this purpose we let $(m,n)_A^{(k)}$ denote the greatest common k-th power divisor of m and n under the A-convolution. Note that $(m,n)_A^{(k)} \neq (m,n)_{A,k}$ and also $(m,n)_A^{(1)} \neq (m,n)_{A,1}$.

Now application of Equations (8), (9), (10) and (11) to the functions $\phi_{P_k(A),A}$ and $\phi_{Q_k(A),A}$ gives the equations

$$\begin{aligned} \phi_{P_k(A),A}(n) &= \sum_{d\in A(n)} \mu_{A,k}(d)(n/d) = \sum_{\substack{d\in A(n)\\ d^k\in A(n)}} \mu_A(d)(n/d^k), && (12)\\ n &= \sum_{\substack{d\in A(n)\\ d^k\in A(n)}} \phi_{P_k(A),A}(n/d^k), && (13)\\ \phi_{P_k(A),A}(n) &= \sum_{\substack{d\in A(n)\\ (d,n)_A^{(k)}=1}} \phi_A(n/d), && (14)\\ \sum_{d\in A(n)} \phi_{P_k(A),A}(d) &= n \sum_{\substack{d\in A(n)\\ (d,n)_A^{(k)}=1}} d^{-1} && (15) \end{aligned}$$

and

$$\begin{aligned} \phi_{Q_k(A),A}(n) &= \sum_{d\in A(n)} \lambda_{A,k}(d)(n/d), && (16)\\ n &= \sum_{\substack{d\in A(n)\\ (d,n)_A^{(k)}=1}} \phi_{Q_k(A),A}(n/d), && (17)\\ \phi_{Q_k(A),A}(n) &= \sum_{\substack{d\in A(n)\\ d^k\in A(n)}} \phi_A(n/d^k), && (18)\\ \sum_{d\in A(n)} \phi_{Q_k(A),A}(d) &= n \sum_{\substack{d\in A(n)\\ d^k\in A(n)}} d^{-k}. && (19) \end{aligned}$$

The function $\phi_{A,k}(n)$ is defined as the number of integers $x \pmod{n^k}$ such that $(x, n^k)_{A,k} = 1$. It is known [9] that

$$\phi_{A,k}(n) = \sum_{d \in A_k(n)} \mu_{A_k}(d)(n/d)^k.$$

In general, $\phi_{A,k}(n) \neq \phi_{P_k(A),A}(n^k)$. However, for the Dirichlet convolution, we have

$$\phi_{D,k}(n) = \phi_{P_k(D),D}(n^k).$$

The function $\phi_{D,k}(n)$ is Cohen's totient function $\phi_k(n)$ defined as the number of integers $x \pmod{n^k}$ such that $(x, n^k)_k = 1$, and as noted above $\phi_{P_k(D),D}(n)$ is Klee's totient function $\psi_k(n)$, which is defined as the number of integers $x \pmod{n}$ such that (x, n) is k-free, that is, $(x, n)_k = 1$ (for Cohen's and Klee's totient functions, see e.g. [5, Chapter 1]). The set (12)–(15) of equations gives identities for Klee's totient as follows:

$$\psi_k(n) = \sum_{d^k | n} \mu(d)(n/d^k), \tag{20}$$

$$n = \sum_{d^k | n} \psi_k(n/d^k), \tag{21}$$

$$\psi_k(n) = \sum_{\substack{d|n \\ (d,n)_k = 1}} \phi(n/d), \tag{22}$$

$$\sum_{d|n} \psi_k(d) = n \sum_{\substack{d|n \\ (d,n)_k = 1}} d^{-1}. \tag{23}$$

Equations (20) and (22) give expressions for Cohen's totient ϕ_k as follows

$$\phi_k(n) = \sum_{d|n} \mu(d)(n/d)^k = \sum_{\substack{d|n^k \\ (d,n^k)_k = 1}} \phi(n^k/d).$$

Sivaramakrishnan's totient function $b(n)$ is defined as the number of integers $x \pmod{n}$ such that (x, n) is a square. As noted above $b(n) = \phi_{Q_2(D),D}(n)$. The set (16)–(19) of equations gives thus identities for $b(n)$ as follows:

$$b(n) = \sum_{d|n} \lambda(d)(n/d), \tag{24}$$

$$n = \sum_{\substack{d|n \\ (d,n)_2 = 1}} b(n/d) = \sum_{d|n} \mu^2(d) b(n/d), \tag{25}$$

$$b(n) = \sum_{d^2|n} \phi(n/d^2), \tag{26}$$

$$\sum_{d|n} b(d) = n \sum_{d^2|n} d^{-2}. \tag{27}$$

Identities (20)–(27) are known identities ([5,10]). The approach adopted in this paper, however, puts these identities into a new framework.

The function $\phi_{P_1(U),U}$ and the function $\phi_{Q_k(U),U}$ $(k > 1)$ is the unitary analogue $\phi^{\oplus}$ of Euler's totient function ϕ, where $\phi^{\oplus}(n)$ counts the number of integers $x \pmod n$ such that $(x, n)^{\oplus} = 1$. Further, $\phi_{P_k(U),U}(n) = \phi_{Q_1(U),U}(n) = n$ $(k > 1)$. Equations (12)–(19) for these functions give only trivial identities and the two well-known identities

$$\begin{aligned} \phi^{\oplus}(n) &= \sum_{d\|n} \mu^{\oplus}(d)(n/d), \\ n &= \sum_{d\|n} \phi^{\oplus}(n/d). \end{aligned}$$

It appears that for any conjugate pair of sets P, Q there is a connection between the functions $\phi_{P,A}$ and $\phi_{Q,A}$ and a generalization of Pillai's function β. The function $\beta(n)$ is defined as

$$\beta(n) = \sum_{x \pmod n} (x, n).$$

We define the function $\beta_A(n)$ as

$$\beta_A(n) = \sum_{x \pmod n} (x, n)_A.$$

In particular, $\beta_D(n) = \beta(n)$ and $\beta_U(n)$ is the unitary analogue of Pillai's function [13]. It is known [4] that

$$\beta_A(n) = (N *_A N *_A \mu_A)(n). \tag{28}$$

Theorem 6.3. *For any conjugate pair of sets P, Q*

$$\phi_{P,A} *_A \phi_{Q,A} = \beta_A.$$

Proof. By Equation (8),

$$\phi_{P,A} = N *_A \mu_{P,A}, \quad \phi_{Q,A} = N *_A \mu_{Q,A}.$$

With the aid of Part 3 of Theorem 4.1 we have

$$\phi_{P,A} *_A \phi_{Q,A} = N *_A N *_A \mu_A.$$

Thus, by (28), we obtain Theorem 6.3. □

Example 6.1. Let $b_k = \phi_{Q_k(D),D}$. In particular, $b_2 = b$. Then

$$\sum_{d|n} \psi_k(d) b_k(n/d) = \beta(n)$$

and, in particular,

$$\sum_{d|n} \psi_2(d) b(n/d) = \beta(n).$$

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