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Asymptotic formula of a "hyperbolic" summation related to the Piltz divisor function

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Abstract: In this paper, we obtain asymptotic formula on the "hyperbolic" summation

$$\sum_{mn\leq x} D_k\left(\gcd\left(m,n\right)\right) \quad \left(k\in\mathbb{Z}_{\geq 2}\right),$$

such that $D_k\left(n\right)=\frac{\tau_k\left(n\right)}{\tau_k^*\left(n\right)}$, where $\tau_k\left(n\right)=\sum\limits_{n_1 n_2 \ldots n_k=n} 1$ denotes the Piltz divisor function, and $\tau_k^*\left(n\right)$ is the unitary analogue function of $\tau_k\left(n\right)$.

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1 Introduction

For all integers m, $n\geq 1$, we denote by $\gcd(m,n)=(m,n)$ the greatest common divisor of the integers m and n. The details of the function $D_k\left(n\right)$ is given in [4] and for many properties of the classical functions $\tau_k\left(n\right)$, $\tau_k^*\left(n\right)$ see, e.g. [5, 6]. Let f, g be two arithmetic functions. The estimation of the $\sum_{mn\leq x}\frac{f\left((m,n)\right)}{g\left((m,n)\right)}$ is a research topic of many researchers, see, for e.g., [1] and [3].

In [3], the authors give an important result of the sum $\sum_{mn\leq x} f\left((m,n)\right)$ in the case where $f(n)$ satisfies the condition

$$f\left(n\right) \ll n^{\beta}\left(\log n\right)^{\delta} \tag{1}$$

with β, $\delta \in \mathbb{R}$ and $\beta < 1$. In this case, they gave an error term as follows

$$R_f\left(x\right) \ll \begin{cases} x^{(\beta+1)/2}\left(\log x\right)^{\delta+1}, & \text{if } 0<\beta<1 \text{ or } \beta=0,\ \delta\neq -1, \\ x^{1/2}\log\log x, & \text{if } \beta=0,\ \delta=-1, \\ x^{1/2}\lambda\left(x\right), & \text{if } \beta<0, \end{cases}$$

where

$$\lambda\left(x\right) := e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}},$$

with some constant $c > 0$.

The function $D_k\left(n\right)$ checks the estimate (1), where $0 < \beta < 1$ and $\delta = 0$. So, the application of Theorem 2.2 in [3], gives the following error term

$$R_{D_k}\left(x\right) \ll x^{(\beta+1)/2}\log x.$$

Note that, if $0 < \beta < 1$, then $R_{D_k}\left(x\right) = O\left(x^{\theta+\varepsilon}\right)$ where $1/2 < \theta < 1$ and $\varepsilon > 0$ is a constant.

The aim of this paper is to give an estimate for the sum $\sum_{mn\leq x} D_k\left((m,n)\right)$ with an optimized error term. We use elementary methods to prove $R_{D_k}\left(x\right) = O\left(x^{\theta+\varepsilon}\left(\log x\right)^{k-1}\right)$, where $\theta = 517/1648 \simeq 0.3113\ldots$. More precisely, we will prove the following result:

Theorem 1.1. *For $x \geq e^2$, $\varepsilon > 0$ and for any fixed integer $k \geq 2$, we have*

$$S\left(x\right) = B_k x\left(\log x + M_k\right) + O\left(x^{\theta+\varepsilon}\left(\log x\right)^{k-1}\right),$$

where $\theta = \dfrac{517}{1648} \simeq 0.3113\ldots,$

$$B_k = \zeta^{-1}\left(2\right)\prod_p\left(1-\frac{1}{k}+\frac{1}{k}\left(1-\frac{1}{p^2}\right)^{-k}\right)$$

and

$$M_k = 2\gamma - 1 - 2\sum_{n=1}^{\infty}\frac{\left(D_k * \mu\right)\left(n\right)\log n}{n^2}.$$

2 Proof of Theorem 1.1

In order to prove the above result, we first establish some auxiliaries lemmas.

Lemma 2.1. *For each integer $k \geq 2$, and for each real number $x \geq 1$, we have*

$$\begin{aligned} S\left(x\right) &= \sum_{s\leq x}\sum_{t^2\ell=s}\left(D_k*\mu\right)\left(t\right)\tau\left(\ell\right) \\ &= \sum_{t\leq x^{1/2}} g_k\left(t\right)\sum_{\ell\leq x/t^2}\tau\left(\ell\right), \end{aligned} \tag{2}$$

where $g_k = D_k * \mu$.

Proof. We put $m = hm'$ and $n = hn'$, with $(m', n') = 1$, one has, for any integer $s \geq 2$,

$$L(s) = \sum_{mn=s} D_k((m,n)) = \sum_{\substack{h^2 m' n' = s \\ (m',n')=1}} D_k(h).$$

By using the property of the Möbius μ function,

$$\sum_{\delta \mid (m',n')} \mu(\delta) = \begin{cases} 1, & \text{if } (m',n') = 1 \\ 0, & \text{if } (m',n') > 1, \end{cases}$$

we get

$$\begin{aligned} L(s) &= \sum_{h^2 m' n' = s} D_k(h) \sum_{\delta \mid (m',n')} \mu(\delta) \\ &= \sum_{h^2 \delta^2 m'' n'' = s} D_k(h) \mu(\delta) = \sum_{h^2 \delta^2 \ell = s} D_k(h) \mu(\delta) \sum_{m'' n'' = \ell} 1 \\ &= \sum_{h^2 \delta^2 \ell = s} D_k(h) \mu(\delta) \tau(\ell), \end{aligned}$$

where $\tau(n)$ is the number of divisors of n. Therefore, by the Dirichlet convolution product we have

$$\begin{aligned} L(s) &= \sum_{t^2 \ell = s} \tau(\ell) \sum_{h\delta = t} D_k(h) \mu(\delta) \\ &= \sum_{t^2 \ell = s} (D_k * \mu)(t) \tau(\ell). \end{aligned}$$

So, by this last result, one has

$$\begin{aligned} S(x) &= \sum_{s \leq x} \sum_{t^2 \ell = s} (D_k * \mu)(t) \tau(\ell) \\ &= \sum_{t^2 \leq x} (D_k * \mu)(t) \sum_{\ell \leq x/t^2} \tau(\ell) \\ &= \sum_{t \leq x^{1/2}} g_k(t) \sum_{\ell \leq x/t^2} \tau(\ell). \end{aligned}$$ □

Lemma 2.2. *Let $n \in \mathbb{Z}_{\geq 0}$ and $\alpha > 1$. For any $z \geq e$, we have*

$$\int_1^z t^{-\alpha} (\log t)^n \, dt \leq \frac{n!}{(\alpha - 1)^{n+1}}. \tag{3}$$

$$\int_z^{+\infty} t^{-\alpha} (\log t)^n \, dt \leq \frac{n! (\log z)^n}{(\alpha - 1) z^{\alpha - 1}} \left(\frac{\alpha}{\alpha - 1} \right)^n. \tag{4}$$

Proof. We put

$$J_n = \int_1^z t^{-\alpha} (\log t)^n \, dt,$$

and the integration by parts gives

$$J_n = -\frac{1}{(\alpha - 1) z^{\alpha - 1}} (\log z)^n + \frac{n}{(\alpha - 1)} J_{n-1}$$

and by recurrence we obtain

$$J_n = -\frac{1}{(\alpha-1)\,z^{\alpha-1}}\left(\sum_{k=0}^{n-1} k!\begin{pmatrix} n \\ k \end{pmatrix}\frac{(\log z)^{n-k}}{(\alpha-1)^k} + \frac{n!}{(\alpha-1)^n}\right) + \frac{n!}{(\alpha-1)^{n+1}},$$

from which

$$J_n \leq \frac{n!}{(\alpha-1)^{n+1}}.$$

In the same way, we have

$$\begin{aligned} I_n &= \int_z^{+\infty} t^{-\alpha}(\log t)^n\,dt \\ &= \frac{(\log z)^n}{(\alpha-1)\,z^{\alpha-1}} + \frac{n}{\alpha-1}I_{n-1} \\ &= \frac{1}{(\alpha-1)\,z^{\alpha-1}}\sum_{k=0}^{n} k!\begin{pmatrix} n \\ k \end{pmatrix}\frac{(\log z)^{n-k}}{(\alpha-1)^k} \\ &\leq \frac{n!\,(\log z)^n}{(\alpha-1)\,z^{\alpha-1}}\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\frac{1}{(\alpha-1)^k} \\ &= \frac{n!\,(\log z)^n}{(\alpha-1)\,z^{\alpha-1}}\left(\frac{\alpha}{\alpha-1}\right)^n. \end{aligned}$$

□

Lemma 2.3. *Let $k \geq 2$ be a fixed integer. For any real number $x \geq 1$, we have*

$$\sum_{t \leq x^{1/2}} \frac{g_k(t)}{t^2} = B_k + O\left(x^{-3/4}(\log x)^{k-2}\right), \tag{5}$$

$$\sum_{t \leq x^{1/2}} \frac{g_k(t)\log t}{t^2} = C_k + O\left(x^{-3/4}(\log x)^{k-1}\right), \tag{6}$$

$$\sum_{t \leq x^{1/2}} \frac{g_k(t)}{t^{2\theta+2\varepsilon}} = O(1) \text{ for every } \varepsilon > 0, \tag{7}$$

such that $\theta = \frac{517}{1648}$, $C_k = \sum_{t=1}^{\infty} \frac{g_k(t)\log t}{t^2}$ *and*

$$B_k = \zeta^{-1}(2)\prod_p \left(1 - \frac{1}{k} + \frac{1}{k}\left(1 - \frac{1}{p^2}\right)^{-k}\right).$$

Proof. Firstly, we write

$$\sum_{t \leq x^{1/2}} \frac{g_k(t)}{t^2} = \sum_{t=1}^{\infty} \frac{g_k(t)}{t^2} - \sum_{t > x^{1/2}} \frac{g_k(t)}{t^2}.$$

For the first series we have

$$\begin{aligned} \sum_{t=1}^{\infty} \frac{g_k(t)}{t^2} &= \prod_p \left(1 + \sum_{\alpha=1}^{\infty} \frac{g_k(p^\alpha)}{p^{2\alpha}}\right) \\ &= \prod_p \left(1 + \sum_{\alpha=1}^{\infty} \frac{(D_k * \mu)(p^\alpha)}{p^{2\alpha}}\right), \end{aligned}$$

and by Identity 10 of Lemma 4 in [4], we have

$$(D_k * \mu)(p^\alpha) = \frac{k-1}{k}(s_2 \times D_{k-1})(p^\alpha),$$

where s_2 is the characteristic function of the square-full number, so

$$s_2(p^\alpha) = 0 \text{ if } \alpha = 1 \text{ and } s_2(p^\alpha) = 1 \text{ if } \alpha \geq 2.$$

Then

$$\begin{aligned} \sum_{t=1}^{\infty} \frac{g_k(t)}{t^2} &= \prod_p \left(1 + \frac{k-1}{k} \sum_{\alpha=1}^{\infty} \frac{(s_2 \times D_{k-1})(p^\alpha)}{p^{2\alpha}}\right) \\ &= \prod_p \left(1 + \frac{k-1}{k} \sum_{\alpha=2}^{\infty} \frac{D_{k-1}(p^\alpha)}{p^{2\alpha}}\right). \end{aligned}$$

In addition, by (2) and (3) in [4] we get

$$D_{k-1}(p^\alpha) = \frac{\tau_{k-1}(p^\alpha)}{\tau^*_{k-1}(p^\alpha)} = \frac{1}{k-1}\begin{pmatrix} k+\alpha-2 \\ \alpha \end{pmatrix},$$

then

$$\begin{aligned} \sum_{t=1}^{\infty} \frac{g_k(t)}{t^2} &= \prod_p \left(1 + \frac{1}{k} \sum_{\alpha=2}^{\infty} \frac{1}{p^{2\alpha}} \begin{pmatrix} k+\alpha-2 \\ \alpha \end{pmatrix}\right) \\ &= \prod_p \left(1 + \frac{1}{kp^2} \sum_{\alpha=1}^{\infty} \frac{1}{p^{2\alpha}} \begin{pmatrix} k+\alpha-1 \\ \alpha+1 \end{pmatrix}\right) \\ &= \prod_p \left(1 + \frac{k-1}{kp^2} \sum_{\alpha=1}^{\infty} \frac{1}{(\alpha+1)p^{2\alpha}} \begin{pmatrix} k+\alpha-1 \\ \alpha \end{pmatrix}\right). \end{aligned}$$

On the other hand, by Formula 9 of Lemma 3 in [4] we get

$$\frac{k-1}{p^2} \sum_{\alpha=0}^{\infty} \frac{1}{(\alpha+1)p^{2\alpha}} \begin{pmatrix} k+\alpha-1 \\ \alpha \end{pmatrix} = \left(1 - \frac{1}{p^2}\right)^{1-k} - 1$$

from which we deduce

$$\frac{k-1}{kp^2} \sum_{\alpha=1}^{\infty} \frac{1}{(\alpha+1)p^{2\alpha}} \begin{pmatrix} k+\alpha-1 \\ \alpha \end{pmatrix} = \frac{1}{k}\left(1 - \frac{1}{p^2}\right)^{1-k} - \frac{1}{k} - \frac{k-1}{kp^2}.$$

Finally,

$$\begin{aligned} \sum_{t=1}^{\infty} \frac{g_k(t)}{t^2} &= \prod_p \left(1 - \frac{1}{k} - \frac{k-1}{kp^2} + \frac{1}{k}\left(1 - \frac{1}{p^2}\right)^{1-k}\right) \\ &= \prod_p \left(\left(1 - \frac{1}{k}\right)\left(1 - \frac{1}{p^2}\right) + \frac{1}{k}\left(1 - \frac{1}{p^2}\right)^{1-k}\right) \\ &= \prod_p \left(1 - \frac{1}{p^2}\right)\left(1 - \frac{1}{k} + \frac{1}{k}\left(1 - \frac{1}{p^2}\right)^{-k}\right). \end{aligned}$$

Note that the series $\sum_{t=1}^{\infty}\frac{g_k(t)}{t^2}$ is convergent if and only if $\prod_p\left(1-\frac{1}{k}+\frac{1}{k}\left(1-\frac{1}{p^2}\right)^{-k}\right)$ is convergent. Indeed,

$$\begin{aligned}\sum_p\left(-\frac{1}{k}+\frac{1}{k}\left(1-\frac{1}{p^2}\right)^{-k}\right) &= \frac{1}{k}\sum_p\left(-1+\left(1-\frac{1}{p^2}\right)^{-k}\right)\\ &= \frac{1}{k}\sum_p\frac{-\left(p^2-1\right)^k+p^{2k}}{\left(p^2-1\right)^k}\\ &= \frac{1}{k}\sum_p\frac{-\sum_{m=0}^{k}(-1)^m\binom{k}{m}p^{2k-2m}+p^{2k}}{\left(p^2-1\right)^k}\\ &= \frac{1}{k}\sum_p\frac{\sum_{m=1}^{k}(-1)^{m+1}\binom{k}{m}p^{2k-2m}}{\left(p^2-1\right)^k}<\infty.\end{aligned}$$

Hence,

$$\sum_{t=1}^{\infty}\frac{g_k(t)}{t^2}=\zeta^{-1}(2)\prod_p\left(1-\frac{1}{k}+\frac{1}{k}\left(1-\frac{1}{p^2}\right)^{-k}\right)=B_k.$$

Then, by partial summation [2, p. 15], we have

$$\sum_{t>x^{1/2}}\frac{g_k(t)}{t^2}=-\frac{1}{x}\sum_{n\leq x^{1/2}}g_k(n)+2\int_{x^{1/2}}^{+\infty}\left(\sum_{n\leq t}g_k(n)\right)\frac{dt}{t^3}.$$

Hence, by (4) and the estimate [4, p. 6]

$$\sum_{n\leq t}g_k(n)\ll t^{1/2}(\log t)^{k-2}, \tag{8}$$

one obtains

$$\sum_{t>x^{1/2}}\frac{g_k(t)}{t^2}\ll x^{-3/4}(\log x)^{k-2}.$$

From this result, we get (5).

Secondly, we note that by the same method used in the previous paragraph we can show that the series $\sum_{t=1}^{\infty}\frac{g_k(t)}{t}$ is convergent. So, because $\frac{\log t}{t}\leq 1$, then

$$\sum_{t=1}^{\infty}\frac{g_k(t)\log t}{t^2}\leq\sum_{t=1}^{\infty}\frac{g_k(t)}{t}<\infty.$$

On the other hand, we have

$$\sum_{t\leq x^{1/2}}\frac{g_k(t)\log t}{t^2}=\sum_{t=1}^{\infty}\frac{g_k(t)\log t}{t^2}-\sum_{t>x^{1/2}}\frac{g_k(t)\log t}{t^2}$$

and by partial summation [2, p. 15], we obtain

$$\sum_{t>x^{1/2}} \frac{g_k(t)\log t}{t^2} = \frac{\log x}{2x} \sum_{n\le x^{1/2}} g_k(n) - \int\limits_{x^{1/2}}^{+\infty} \left(\frac{2\log t - 1}{t^3} \sum_{n\le t} g_k(n) \right) dt.$$

Using formulas (4) and (8) we get

$$\begin{aligned}
\sum_{t\le x^{1/2}} \frac{g_k(t)\log t}{t^2} &= C_k + O\left(x^{-3/4}(\log x)^{k-1}\right) + O\left(\int_{x^{1/2}}^{+\infty} t^{-5/2}(\log t)^{k-1}\right) dt \\
&= C_k + O\left(x^{-3/4}(\log x)^{k-1}\right).
\end{aligned}$$

Finally, by (3) and (8)

$$\begin{aligned}
\sum_{t\le x^{1/2}} \frac{g_k(t)}{t^{2\theta+2\varepsilon}} &= \frac{1}{x^{\theta+\varepsilon}} \sum_{n\le x^{1/2}} g_k(n) + (2\theta+2\varepsilon) \int_1^{x^{1/2}} \frac{1}{t^{2\theta+2\varepsilon+1}} \left(\sum_{n\le t} g_k(n) \right) dt \\
&= O\left(x^{1/4-(\theta+\varepsilon)}(\log x)^{k-2}\right) + O\left(\int_1^{x^{1/2}} t^{-(2\theta+2\varepsilon+1/2)}(\log t)^{k-2}\right) dt \\
&= O(1).
\end{aligned}$$

□

2.1 Proof of Theorem 1.1

We have the following estimate [2, p. 472]

$$\sum_{n\le n} \tau(n) = x(\log x + C) + O\left(x^{\theta+\varepsilon}\right),$$

where $C = 2\gamma - 1$, $\theta = \frac{517}{1648}$, and so, by (2), we get

$$\begin{aligned}
S(x) &= \sum_{t\le x^{1/2}} g_k(t) \left(\frac{x}{t^2}\left(\log\frac{x}{t^2} + C\right) + O\left(\left(\frac{x}{t^2}\right)^{\theta+\varepsilon}\right) \right) \\
&= x(\log x + C) \sum_{t\le x^{1/2}} \frac{g_k(t)}{t^2} - 2x \sum_{t\le x^{1/2}} \frac{g_k(t)\log t}{t^2} + O\left(x^{\theta+\varepsilon} \sum_{t\le x^{1/2}} \frac{g_k(t)}{t^{2\theta+2\varepsilon}} \right).
\end{aligned}$$

Substituting (5), (6) and (7) in this last relation we find

$$\begin{aligned}
S(x) &= B_k x(\log x + C) + O\left(x^{1/4}(\log x)^{k-1}\right) \\
&\quad -2x\left(C_k + O\left(x^{-3/4}(\log x)^{k-1}\right)\right) + O\left(x^{\theta+\varepsilon}\right) \\
&= B_k x(\log x + C) - 2C_k x + O\left(x^{1/4}(\log x)^{k-1}\right) + O\left(x^{\theta+\varepsilon}\right) \\
&= B_k x(\log x + C - 2C_k) + O\left(x^{\theta+\varepsilon}(\log x)^{k-1}\right).
\end{aligned}$$

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