Hook type tableaux and partition identities

Koustav Banerjee[1] and Manosij Ghosh Dastidar[2]

[1] Research Institute for Symbolic Computation, Johannes Kepler University
Altenberger Straße 69, A-4040 Linz, Austria
e-mail: `Koustav.Banerjee@risc.uni-linz.ac.at`

[2] Technische Universität Wien
Wiedner Hauptstraße 8–10/104, 1040 Wien, Austria
e-mail: `gdmanosij@gmail.com`

Abstract: In this paper we exhibit the box-stacking principle (BSP) in conjunction with Young diagrams to prove generalizations of Stanley's and Elder's theorems without even the use of partition statistics in general. We primarily focus on to study Stanley's theorem in color partition context.
Keywords: Hook type, Partitions, Young tableaux, Stanley's theorem.
2020 Mathematics Subject Classification: 05A19, 11P81, 11P84.

1 Introduction

In 1959, Fine [6] did a comprehensive study on counting number of parts in partitions and its association with parts imposed by restriction in partitions. For more details about the history of the literature, we refer to [8]. The hook lengths of partitions are widely studied in the paradigm of integer partitions, algebraic combinatorics and representation theory of groups. Work in the direction of partitions has been done by Bessenrodt [3], Bacher and Manivel [2] and Han [9], to name a few.

We have endeavoured to show how the combinatorial framework BSP, demonstrated subsequently in Section 2, can be used to provide proofs of Theorem 1.2, 1.3, 1.4 and 1.5 in elementary and elegant manner. Following the same ethos, we continued further to generalize the strategy in context of color partitions to prove Theorem 1.6.

A partition of $n \geq 0$ is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of positive integers whose sum is n, denoted by $\lambda \vdash n$. $p(n)$ denotes the number of partitions of n and $P(n)$ is the set of all partitions of n. Define $\ell = \ell(\lambda)$ to be the number of parts in λ, $a(\lambda)$ to be the largest part of λ and $\text{mult}(\lambda_i)$ to be the multiplicity of the part λ_i. We also use $\lambda = (\lambda_1^{m_1} \ldots \lambda_\ell^{m_\ell})$ as an alternative notation for partition. For partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ define the sum $\lambda + \mu$ to be the partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$. Similarly, define the union $\lambda \cup \mu$ to be the partition with parts $\{\lambda_i, \mu_j\}$, arranged in non-increasing order.

Due to Euler, the generating function of $p(n)$ is

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}. \tag{1}$$

To each partition $\lambda \vdash n$ we associate Y_λ, the celebrated graphical representation called the Young diagram of λ.

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$, we may define a new partition $\lambda' = (\lambda_1', \lambda_2', \ldots, \lambda_m') \vdash n$ (where m is the largest part of λ) by choosing λ_i' as the number of parts of λ that are $\geq i$. The partition λ' is called the conjugate of λ. Notice that the graphical representation of the conjugate is obtained by reflecting the Young diagram (which we will define in few lines) in the main diagonal. For example, if $\lambda = (6, 3, 3, 2, 1)$, then conjugate of λ is $\lambda' = (5, 4, 3, 1, 1, 1)$.

For $\ell \in \mathbb{Z}_{\geq 2}$, we define the **$\ell$-color partitions** of n to be $P^{(\ell)}(n)$ is the set of all partitions of n where multiples of ℓ can occur with 2 colors and $P^{(\ell)}(0) := \{()\}$; moreover, $p^{(\ell)}(n) := |P^{(\ell)}(n)|$. Let λ be a ℓ-color partition of n, then we tag parts of λ, those are not multiple of ℓ (resp. those are) by indexing with 1 (resp. with 1 and 2). Consequently in context of Young diagram, those parts which are indexed by 1 (resp. parts are indexed by both 1 and 2) interpreted as being white (resp. green) in color.

For example, there are 9 partitions enumerated by $p^{(2)}(4)$ are 4_2, 4_1, $3_1 + 1_1$, $2_2 + 2_2$, $2_1 + 2_2$, $2_1 + 2_1$, $2_2 + 1_1 + 1_1$, $2_1 + 1_1 + 1_1$, $1_1 + 1_1 + 1_1 + 1_1$ and for $\lambda = (2_1, 2_2) \in P^{(2)}(4)$, the associated colored Young diagram Y_λ is
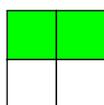

Figure 1. Y_λ for the colored partition $\lambda = (2_1, 2_2)$.

Similar to (1), the generating function of $p^{(\ell)}(n)$ is

$$\sum_{n=0}^{\infty} p^{(\ell)}(n)q^n = \prod_{j=1}^{\infty} \frac{1}{(1 - q^j)(1 - q^{\ell j})}. \tag{2}$$

Plugging in $\ell = 2$ into (2), it is immediate that $p^{(2)}(n)$ is the number of cubic partitions of n, usually denoted by $a(n)$ (cf. [4]) originated from the study of Ramanujan's cubic continued fraction.

For each box v in Y_λ, define the hook length of v, denoted by $h_v(\lambda)$, to be the number of boxes u such that $u = v$ or u lies in the same column as v and above v or in the same row as v and to the right of v. The hook length multiset of λ, denoted by $\mathcal{H}_\lambda$, is the multiset of all hook lengths of λ.

Each hook length h can be split into $h = a + l + 1$, where a is the arm length (the no. of boxes to the right in the same row) and l the leg length (the no. of boxes above in the same column). The ordered pair (a, l) is called hook type of the chosen box in the Young tableau. A hook length tableau (resp. hook type tableau) is obtained by filling in the boxes of the Young diagram with hook length (resp. hook type) of each box. The boxes will be colored according to the index of the parts in the color partition considered.

For $\lambda = (6, 3, 3, 2) \vdash 14$, the hook length tableau is

2	1				
4	3	1			
5	4	2			
9	8	6	3	2	1

Figure 2. Hook length tableau for the partition $\lambda = (6, 3, 3, 2)$

and the hook type tableau of Y_λ is

$(1,0)$	$(0,0)$				
$(2,1)$	$(1,1)$	$(0,0)$			
$(2,2)$	$(1,2)$	$(0,1)$			
$(5,3)$	$(4,3)$	$(3,2)$	$(2,0)$	$(1,0)$	$(0,0)$

Figure 3. Hook type tableau for the partition $\lambda = (6, 3, 3, 2)$.

In color context, the hook type tableau for $\lambda = (6_2, 3_1, 3_2, 2_1) \in P^{(3)}(14)$ is

$(1,0)$	$(0,0)$				
$(2,1)$	$(1,1)$	$(0,0)$			
$(2,2)$	$(1,2)$	$(0,1)$			
$(5,3)$	$(4,3)$	$(3,2)$	$(2,0)$	$(1,0)$	$(0,0)$

Figure 4. Hook type tableau for the colored partition $\lambda = (6_2, 3_1, 3_2, 2_1)$.

Throughout this paper, we shall follow the notations given below. For positive integers n, k and for $\ell \in \mathbb{Z}_{\geq 2}$, we define

$Q_k(n) :=$ Number of occurences of part k in $P(n)$ and $Q_k(n) = 0$ if $k > n$,

$V_k(n) :=$ The number of parts occurring k or more times in the partitions of n,

$$S(n) := \sum_{\lambda \vdash n} \mathrm{dist}(\lambda), \text{ where } \mathrm{dist}(\lambda) \text{ denotes the number of distinct parts in } \lambda,$$

$$Q_k^{(\ell)}(n) := \begin{cases} \text{number of occurences of parts } k_1 \text{ and } k_2 \text{ in } P^{(\ell)}(n), & \text{if } k \equiv 0 (\mathrm{mod}\ \ell), \\ \text{number of occurences of the part } k_1 \text{ in } P^{(\ell)}(n), & \text{otherwise.} \end{cases}$$

In short we will say, $Q_k^{(\ell)}(n)$ is the number of occurences of part k.

In continuation of Fine's discovery [6], Stanley proposed a beautiful theorem (cf. Theorem 1.2) in 1972, finally published with his original bijective proof in 1986. Elder generalizes Stanley's theorem that extends to enumeration of total number of occurrences of a positive integer in partitions (cf. Theorem 1.3). In [5], Dastidar and Sengupta carried out a recursive argument to prove Theorem 1.4 and 1.5. We organize the rest of this section first by providing a brief sketch of proofs for both Theorem 1.2 and 1.3 by exploiting Theorem 1.1 in a point-wise sense. The argument of proof is then subsumed into the BSP, our second objective. Finally, following the BSP, we will prove Theorem 1.4 and 1.5 which allows us to observe in general how these recursions can be obtained from the combinatorial construction primarily based on hook type enumeration which we will describe in Lemma 3.1.

In [2, Theorem 1.1 and Theorem 1.2], Bacher and Manivel studied the association between enumeration of hook type in Young diagram of a partition and enumeration of statistics based on parts of the considered partition. Proof of both Theorem 1.1 and Theorem 1.2 was given by argument based on q-binomial theorem but additionally, a bijective proof for Theorem 1.2 was given due to C. Bessenrodt.

Theorem 1.1. [2, Theorem 1.1 and Theorem 1.2] *Let $1 \le k \le n$ be two integers. Then, for every positive $0 \le j < k$, the total number of occurrences of the part k among all partitions of n is equal to the number of boxes whose hook type is $(j, k - j - 1)$.*

Theorem 1.2. [10, page 6] *The total number of 1's in all partitions of a positive integer n is equal to the sum of the numbers of distinct parts of those partitions of n.*

Proof. It is enough to show that number of distinct parts of a partition $\lambda \vdash n$ is equal to the number of boxes in Y_λ with hook-type $(0, 0)$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ with respective multiplicities $m_1, m_2, \ldots, m_r$. Note that, the boxes with hook-type $(0, 0)$ appear exactly once in Y_λ corresponding to the part λ_i subject to the condition that the immediate next part λ_j with $i \ne j$. Therefore, it is clear that the number of boxes with hook-type $(0, 0)$ equals the number of distinct parts of λ. Now, summing over all $\lambda \vdash n$ we get the Stanley's theorem. $\square$

Theorem 1.3. [10, page 8] *The total number of occurences of an integer k among all partitions of n is equal to the number of occasions that a part occurs greater or equal k times in $P(n)$.*

Remark 1.1. *Restricting $k = 1$, Theorem 1.3 reduces to the Theorem 1.2.*

Proof of Theorem 1.3. We need only to show that the number of boxes with hook type $(0, k - 1)$, $k > 1$, in a partition $\lambda \vdash n$ is equal to the number of parts that occur k or more times in λ. Now, a box with hook-type $(0, k - 1)$ in Y_λ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ precisely describes that there are $k - 1$ boxes on the above of it but having no box to the right. Therefore, corresponding to

each box with hook-type $(0, k-1)$ there exists a part that occurs at least k times. Now, summing over all partitions of n we have Elder's theorem. $\square$

Theorem 1.4. [5, Theorem 1] *For positive integers n and k,*

$$S(n) = Q_k(n) + Q_k(n+1) + Q_k(n+2) + \cdots + Q_k(n+k-1) = \sum_{j=0}^{k-1} Q_k(n+j).$$

Remark 1.2. *For $k = 1$, Theorem 1.4 reduces to the Theorem 1.2.*

Theorem 1.5. [5, Theorem 2] *For positive integers n, r and k,*

$$V_k(n) = Q_{rk}(n) + Q_{rk}(n+k) + Q_{rk}(n+2k) + \cdots + Q_{rk}(n+(r-1)k) = \sum_{\ell=0}^{r-1} Q_{rk}(n+\ell k).$$

Remark 1.3. *For $k = 1$ and $r = 1$, Theorem 1.5 reduces to the Theorem 1.3. We also note that for $k = 1$, Theorem 1.5 corresponds to the Theorem 1.4.*

We shall further extends the idea of BSP, described in Section 2, in order to generalize the Stanley's theorem in the context of color partitions which is analogous to Theorem 1.4, stated in the following theorem.

Theorem 1.6. *Let $n \in \mathbb{N}$. Now, for positive integers k and ℓ, with $\ell \in \mathbb{Z}_{\geq 2}$,*

$$Q_1^{(\ell)}(n) = \begin{cases} (Q_k^{(\ell)}(n) + Q_k^{(\ell)}(n+1) + \cdots + Q_k^{(\ell)}(n+k-1))/2, & \text{if } \ell \mid k, \\ Q_k^{(\ell)}(n) + Q_k^{(\ell)}(n+1) + \cdots + Q_k^{(\ell)}(n+k-1), & \text{otherwise.} \end{cases}$$

Remark 1.4. *In order to prove the Theorem 1.6, it is enough to prove the following recursions: Let $n \in \mathbb{N}$. Now, for positive integers k and l, with $l \in \mathbb{Z}_{\geq 2}$, it follows that*

$$\begin{aligned} \frac{Q_k^{(\ell)}(n+k)}{2} &= p^{(\ell)}(n) + \frac{Q_k^{(\ell)}(n)}{2} \ \text{if } \ell \mid k, \\ Q_k^{(\ell)}(n+k) &= p^{(\ell)}(n) + Q_k^{(\ell)}(n) \ \text{otherwise.} \end{aligned} \tag{3}$$

2 Box Stacking Principle

2.1 BSP for Partitions

In this subsection, we shall introduce a specific type of combinatorial construction which we call the "Box Stacking Principle" (BSP). The BSP consists of a set of rules to produce from all partitions of n a new set of partitions of $n + k$ where k is a positive integer. Given a partition $\lambda \vdash n$, the new partitions are produced by adding k boxes as follows:

Case 2.1. *For $k = 1$: We add one box to all permissible places in Y_λ. One can trivially add one box in two ways: (i) Add to the bottom row of Y_λ. (ii) Stack the box on the above of the top row of Y_λ. Also, one can add one box to a row in Y_λ if and only if the difference between the number of boxes in the chosen row and its immediate next is at least 1. In other words, for $\lambda := (\lambda_1, \ldots, \lambda_r) \vdash n$, following rule (i) the trivial addition of one box corresponds to $\mu := ((\lambda_1 + 1), \ldots, \lambda_r) \vdash n + 1$ whereas by rule (ii) we have $\mu := (\lambda_1, \ldots, \lambda_r, 1) \vdash n + 1$. Nontrivial addition of one box can be done if and only if for any two consecutive part say, λ_i and λ_j ($\lambda_i \geq \lambda_j$), we have $\lambda_i - \lambda_j \geq 1$.*

For example, the total number of newly generated partitions obtained by applying the stacking principle for adding one box to Young diagrams of partitions of 4 is given in Figures 5–9.
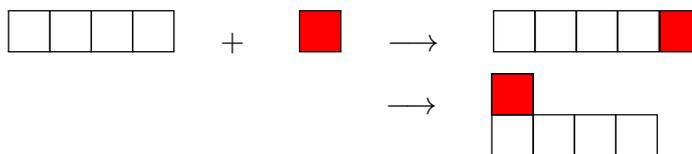
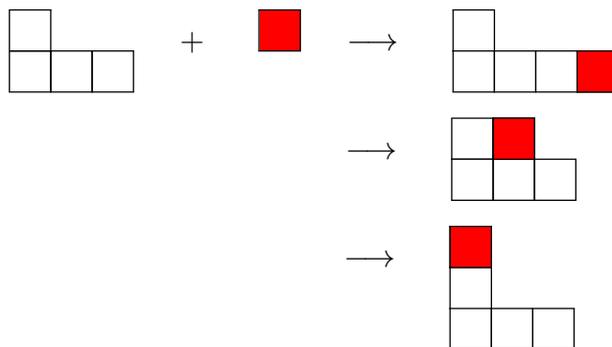

Figure 5. BSP for the partition 4.
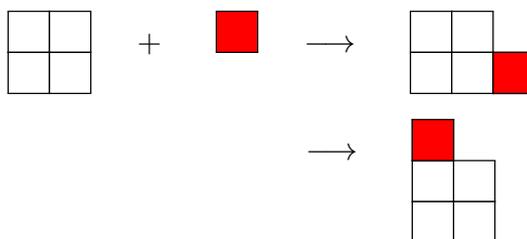

Figure 6. BSP for the partition $3 + 1$.
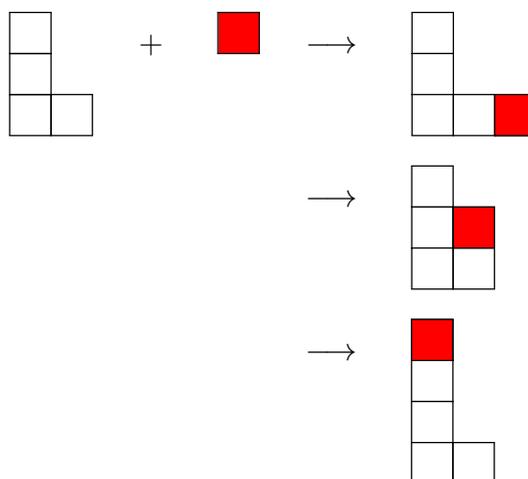

Figure 7. BSP for the partition $2 + 2$.
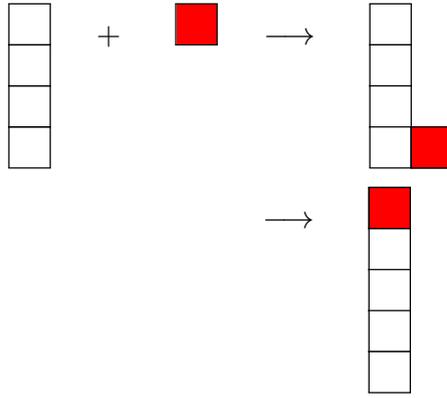

Figure 8. BSP for the partition $2 + 1 + 1$.

Figure 9. BSP for the partition $1 + 1 + 1 + 1$.

Case 2.2. $k > 1$: *Here we consider the addition of k boxes as a 'packet of k boxes', instead of adding 'k single boxes'. Again one can trivially add a 'packet of k boxes' to the bottom row of Y_λ with $\lambda := (\lambda_1, \ldots, \lambda_r)$. In this context, by adding 'packet of k boxes', we mean that adding k to λ_1 so that the resulting partition $\mu := ((\lambda_1 + k), \ldots, \lambda_r) \vdash n + k$. Now a nontrivial addition of a packet of k boxes to Y_λ can be done if and only if for any two consecutive part say, λ_i and λ_j ($\lambda_i \geq \lambda_j$), we have $\lambda_i - \lambda_j \geq k$. We do not consider the addition of 'k single boxes' which means that we do not allow the cases $\mu_1 := (\lambda_1, \ldots, \lambda_r, 1, \ldots, 1) \vdash n + k$ and $\mu_2 := (\lambda_1, \ldots, (\lambda_{j_1} + 1), \ldots, (\lambda_{j_2} + 1), \ldots, (\lambda_{j_k} + 1), \ldots, \lambda_r) \vdash n + k$.*

Next the specific example, considered below, will show what we do not allow. For stacking of $k = 2$ boxes with $\lambda = (3, 1) \vdash 4$, those situations that will be regarded as violating our rules depicted in Figure 10. Whereas, the correct addition of 'packet of 2 boxes' following BSP is given in Figure 11.
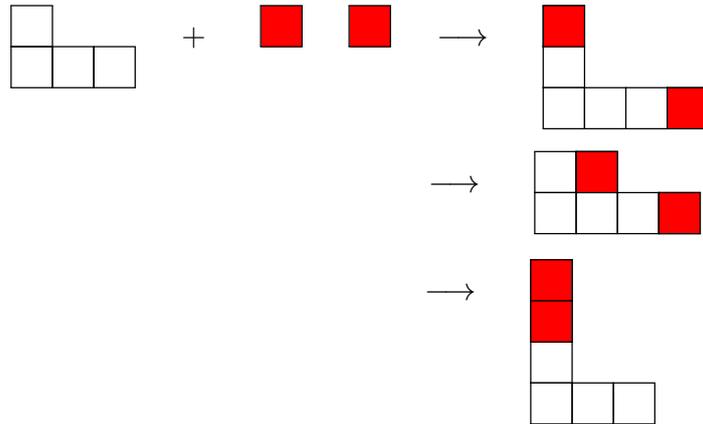

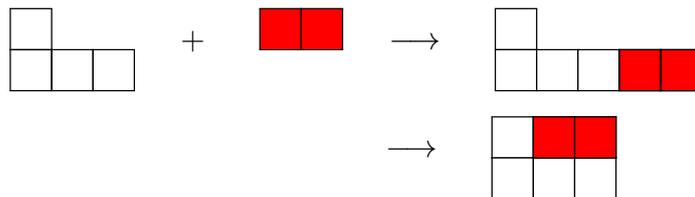Figure 10. Exclusion for the partition $\lambda = (3, 1)$.

Figure 11. Addition by BSP for the partition $\lambda = (3, 1)$.

641

2.2 BSP for Color Partitions

The BSP in color partitions context consists of a set of rules to produce from all color partitions λ of n a new set of color partitions of $n + k$. This is done by adding k boxes in a particular way. Here 'packet of k boxes' has the same meaning as in the case of the BSP for the classical partition function. But in the color context we have to take care about the color of a 'packet of k boxes' (because following the notations in Section 1, one can observe that for a $\lambda \in P^{(l)}(n)$ if k is multiple of l then k appears twice k_1, k_2 in λ). If k is not a multiple of l, without loss of generality, we always add a 'packet of k boxes' prescribed by, white color (one may also add a 'packet of k boxes' prescribed by green color). The set of rules are described in the following paragraphs.

Whenever, we say adding a 'packet of k boxes' it will mean that 'packet of k boxes' is colored by white color. Let $\lambda := (\lambda_{1_{i_1}}, \lambda_{2_{i_2}}, \ldots, \lambda_{r_{i_r}}) \in P^{(l)}(n)$ with $i_k \in \{1, 2\}$ and $1 \leq k \leq r$, $k \in \mathbb{N}$. So when we say $\lambda_{1_{i_1}}$ is the largest part of λ, it means that $\lambda_1 \geq \cdots \geq \lambda_r$ (independent of the indices). First, we will look at the index of the largest part $\lambda_{1_{i_1}}$. If $i_1 = 1$, then trivially we add the packet of k boxes to $\lambda_{1_{i_1}}$; i.e., to bottom row of Y_λ so that the resulting partition $\mu := ((\lambda_{1_{i_1}} + k_1), \ldots, \lambda_{r_{i_r}}) \in P^{(l)}(n + k)$. If $i_1 = 2$, then two cases (cf. 2.3 and 2.4) will arise.

Case 2.3. *If $\lambda_1 \geq k$, then we consider following two cases: (i) If there exist any two consecutive parts say $\lambda_{s_{i_s}}$ and $\lambda_{t_{i_t}}$ ($\lambda_s \geq \lambda_t$, where $t = s + 1$) with $i_t = 1$ and $\lambda_s - \lambda_t \geq k$, then we add a packet of k boxes to the row corresponding to the part $\lambda_{t_{i_t}}$ in Y_λ. (ii) If there does not exists any two consecutive parts with the condition given in (i), then we simply insert the packet of k-boxes as a new row into Y_λ (cf. Figure 19).*

Case 2.4. *If $\lambda_1 < k$, then we adjoin the packet of k boxes below the bottom row of Y_λ so that resulting partition is $\mu := (k_1, \lambda_{1_{i_1}}, \ldots, \lambda_{r_{i_r}}) \in P^{(l)}(n + k)$ (cf. Figure 18).*

Case 2.5 (Exclusion). *We have already stated all the rules of adding a packet of k boxes. Now, we state an exclusion rule; i.e, a case in which we will not allow for addition of k boxes. Here, index of parts in the partition $\lambda \in P^{(l)}(n)$ is important. For any part, say $\lambda_{m_{i_m}}$ with $i_m = 2$, we do not allow the addition of a packet of k boxes to the row corresponding to the part $\lambda_{m_{i_m}}$ in Y_λ. In short, if the color of the row corresponding to the part with index 2 is green, we do not allow the addition of a packet of k boxes to it (cf. Figure 20).*

As an example to illustrate these rules, let us consider all 3 color partitions of 4 and applying the color BSP for adding a packet of 2 boxes to Young diagrams gives the newly generated partitions which is given by
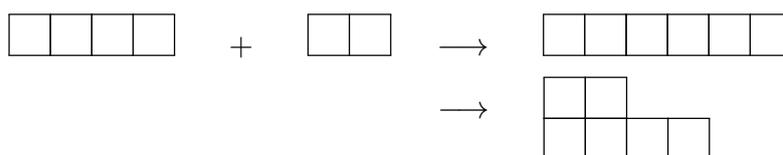

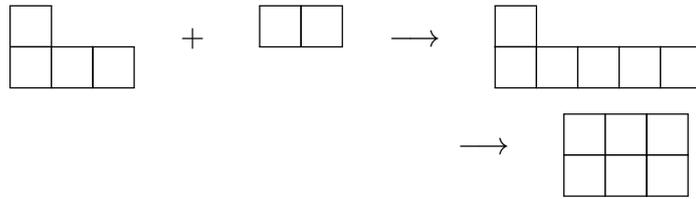Figure 12. Color BSP for the partition 4_1.

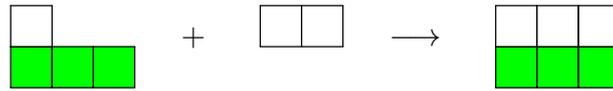Figure 13. Color BSP for the partition $3_1 + 1_1$.

Figure 14. Color BSP for the partition $3_2 + 1_1$.
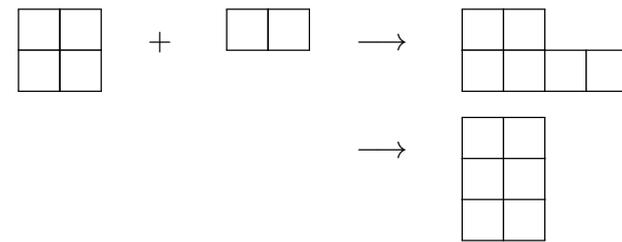

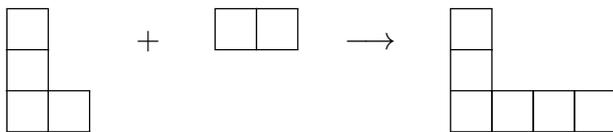Figure 15. Color BSP for the partition $2_1 + 2_1$.

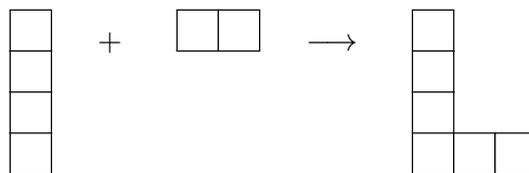Figure 16. Color BSP for the partition $2_1 + 1_1 + 1_1$.

Figure 17. Color BSP for the partition $1_1 + 1_1 + 1_1 + 1_1$.

Note 2.1. *We added the packet of two boxes on the top of Y_λ (with $\lambda = (2_1, 2_1)$) to get Y_{μ_2} (with $\mu_2 = (2_1, 2_1, 2_1)$) given in Figure 15.*

Now let us consider the addition of a packet of 5 boxes and 3 boxes, respectively, to the partition $(3_2, 1_1)$. Then following rules in Cases 2.4 and 2.3(ii), the newly generated Young diagrams given in Figures 18 and 19, respectively.

Next, we will give an example for the exclusion in context of color BSP. For instance, for $n = 11$, $l = 3$, $k = 2$ and $\lambda = (6_2, 3_2, 2_1) \in p^{(3)}(11)$, following rule in the Case 2.5. In Figure 20, one can observe that we did not allow the addition of a packet of 2 white boxes to the parts which is colored with green color; i.e., by exclusion we mean that we excluded all possible addition of a packet of 2 boxes being white colored to the parts colored with.
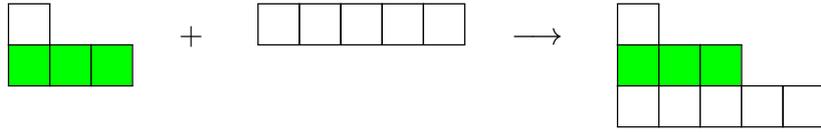
Figure 18. Adjoining for the partition $\lambda = (3_2, 1_1)$ (by Case 2.4).
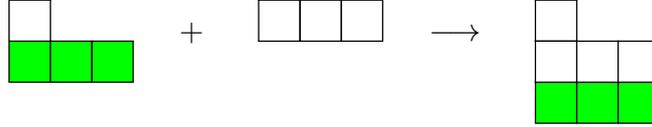

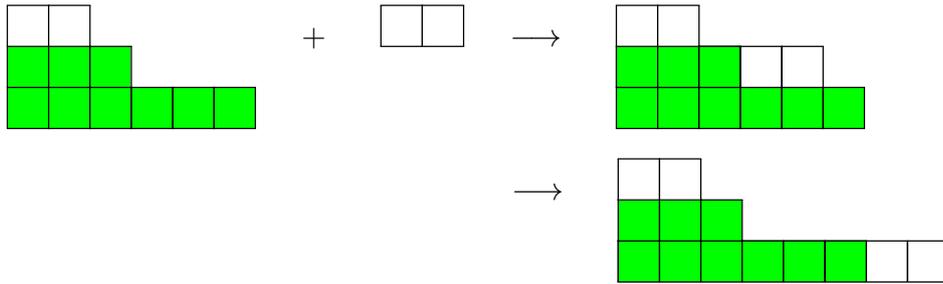Figure 19. Adjoining for the partition $\lambda = (3_2, 1_1)$ (by Case 2.3(ii)).

Figure 20. Exclusion for the partition $\lambda = (6_2, 3_2, 2_1)$ (by Case 2.5).

3   Lemmas and Proof of Theorems 1.4, 1.5 and 1.6

Lemma 3.1. *Stacking k boxes to the Young diagrams corresponding to all partitions of n following the BSP generates as many new partitions as there are occurences of k in all partitions of $n + k$.*

Proof. The proof is divided into two cases as follows:

I. (The Trivial Stacking): We can obviously add a packet of k boxes to the largest part of a partition $\lambda \vdash n$ (as discussed in the principle of construction) and immediately observe that the total number of generated new partition is $p(n)$.

II. (Non-trivial Stacking): Adding k-boxes to a Young diagram Y_λ following BSP is possible if and only if there exists a box in Y_λ with hook-type $(k - 1, 0)$. This is because having a box with hook-type $(k - 1, 0)$ implies that above this box there are k-consecutive empty places where we can place the packet of k boxes and on the other hand, to place a packet of k boxes in the diagram without violating the BSP and structure of Y_λ there must exist k-consecutive empty places; i.e., a box with hook-type $(k - 1, 0)$. This explicitly shows the one to one correspondence between the number of permissible ways of non-trivial addition of packet of k boxes and the number of boxes with hook-type $(k-1, 0)$ in Y_λ. Summing over all partitions of n gives the number of occurrences of part k in $P(n)$. Altogether I and II give that the total newly generated partition is $p(n) + Q_k(n)$. On the other hand, enumeration of total number of occurences of the part k in $P(n + k)$ can be split into two cases. For $k > n$, number of occurences of part k enumerated by counting partitions $\lambda = \lambda_1 \cup \lambda_2$ with $\lambda \vdash n + k$, $\lambda_2 \vdash n$ and $\lambda_1 = (k)$, which is $P(n)$. Whereas for $k \leq n$, number of occurences of part k is enumerated by $Q_k(n)$. Therefore, $p(n) + Q_k(n) = Q_k(n + k)$. □

Proof. Theorems 1.4 and 1.5. By Lemma 3.1, we have

$$Q_k(n) = P(n-k) + Q_k(n-k) = P(n-k) + P(n-2k) + Q_k(n-k)$$
$$= P(n-k) + P(n-2k) + \dots .$$

(4)

Hence

$$\sum_{m=0}^{k-1} Q_k(n+m) = \sum_{m=0}^{k-1} (P(n+m-k) + P(n+m-2k) + \dots) \quad \text{(by (4))}$$
$$= \sum_{m=0}^{n-1} P(m) = Q_1(n) = S(n) \quad \text{(by Theorem 1.2).}$$

Similarly we obtain

$$\sum_{m=0}^{r-1} Q_{rk}(n+mk) = \sum_{m=0}^{r-1} (P(n+mk-rk) + P(n+mk-2rk) + \dots) \quad \text{(by (4))}$$
$$= P(n-k) + P(n-2k) + \dots = Q_k(n) = V_k(n) \quad \text{(by Theorem 1.3).} \qquad \square$$

Now we prove the recursion (3) given in the Remark 1.4 by rephrasing recursions using color BSP presented before.

Lemma 3.2. *Adding a packet of k boxes to the Young diagrams of $\lambda \in P^{(\ell)}(n)$ following the color BSP generates as many new color partitions as there are occurences of a part k in $P^{(\ell)}(n+k)$ subject to the condition that k is not a multiple of ℓ. But if k is a multiple of ℓ, then adding a packet of k boxes generates as many new color partitions which equals to half of the total number of occurences of the part k in $P^{(\ell)}(n+k)$.*

Proof. Following the rules defined in the Subsection 2.2, one can immediately observe that the trivial addition of a packet of k boxes generates the number of color partitions which equals to $p^{(\ell)}(n)$.

Now, if $\ell \nmid k$, then following rule in Case 2.3(ii), we conclude that the number of nontrivial addition of a packet of k boxes to Young diagrams is $Q_k^{(\ell)}(n)$. Therefore, total number of new generated color partitions is $p^{(\ell)}(n) + Q_k^{(\ell)}(n)$ and $p^{(\ell)}(n) + Q_k^{(\ell)}(n) = Q_k^{(\ell)}(n+k)$. On the other hand, for $\ell \mid k$, the part k in $\lambda \in P^{(\ell)}(n)$ appears with two colors. Now, adding a packet of k boxes to Young Diagrams enumerate half of the total number of occurrences of k in $P^{(\ell)}(n)$ because we add only a white colored packet of k boxes. In short, we have chosen only one representative of k_1 and k_2 in terms of adding only a white colored packet of k boxes. Hence one can observe that the total number of generated color partition is $p^{(\ell)}(n) + \frac{Q_k^{(\ell)}(n)}{2}$ and by similar argument given in Lemma 3.1, we have $p^{(\ell)}(n) + \frac{Q_k^{(\ell)}(n)}{2} = \frac{Q_k^{(\ell)}(n+k)}{2}$. $\qquad \square$

Proof. Theorem 1.6. For $\ell \nmid k$, by Lemma 3.2, we have

$$Q_k^{(\ell)}(n+k) = p^{(\ell)}(n) + Q_k^{(\ell)}(n).$$

Following the same line of argument given in Proof of Theorems 1.4 and 1.5, we have

$$Q_1^{(\ell)}(n) = \sum_{m=0}^{k-1} Q_k^{(\ell)}(n+m).$$

Now, for $\ell \mid k$, we split the recursion

$$\frac{Q_k^{(\ell)}(n+k)}{2} = p^{(\ell)}(n) + \frac{Q_k^{(\ell)}(n)}{2}$$

into following two recursions

$$\begin{aligned} Q_{k_1}^{(\ell)}(n+k) &= p^{(\ell)}(n) + Q_{k_1}^{(\ell)}(n), \\ Q_{k_2}^{(\ell)}(n+k) &= p^{(\ell)}(n) + Q_{k_2}^{(\ell)}(n). \end{aligned} \tag{5}$$

Following the Proof of Theorems 1.4 and 1.5 for each of the recursions given in (5), we have

$$Q_1^{(\ell)}(n) = \sum_{m=0}^{k-1} Q_{k_1}^{(\ell)}(n+m) \ \text{ and } \ Q_1^{(\ell)}(n) = \sum_{m=0}^{k-1} Q_{k_2}^{(\ell)}(n+m).$$

We conclude the proof by

$$2Q_1^{(\ell)}(n) = \sum_{m=0}^{k-1} \sum_{i=1}^{2} Q_{k_i}^{(\ell)}(n+m) = \sum_{m=0}^{k-1} Q_k^{(\ell)}(n+m). \qquad \square$$

4 Conclusion

It may be worthwhile to study Theorem 1.4, 1.5 and 1.6 in more generalize setup by BSP. By generalizing we mean to say that for positive integers k, ℓ and non-negative integers α, β if one consider the sequence $C_{k,\ell}^{(\alpha,\beta)}(n)$, defined by

$$\sum_{n=0}^{\infty} C_{k,\ell}^{(\alpha,\beta)}(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^{kn})^{\alpha}(1-q^{\ell n})^{\beta}}.$$

Acknowledgements

References

[1] Andrews, G. E. (1998). *The Theory of Partitions*. Addison-Wesley Pub. Co., NY, 300 pp. (1976). Reissued, Cambridge University Press, New York.

[2] Bacher, R. & Manivel, L. (2002). Hooks and powers of parts in partitions. *Séminaire Lotharingien de Combinatoire*, 47, Article B47d.

[3] Bessenrodt, C. (1998). On hooks of Young diagrams. *Annals of Combinatorics*, 2, 103–110.

[4] Chan, H. C. (2010). Ramanujan's cubic continued fraction and an analogue of his most beautiful identity. *International Journal of Number Theory*, 06, 673–680.

[5] Dastidar, M. G., & Sengupta, S. (2013). Generalization of a few results in integer partitions. *Notes on Number Theory and Discrete Mathematics*, 19(2), 69–76.

[6] Fine, N. J. (1959). Sums over partitions. *Report of the Institute of the Theory of Numbers*, Colorado Boulder, 86–94.

[7] Fine, N. J. (1988). *Basic Hypergeometric Series and Applications*. Mathematical Surveys and Monograph Series, No. 27, American Mathematical Society.

[8] Gilbert, R. A. (2015). A fine rediscovery. *American Mathematical Monthly*, 4(122), 322–331.

[9] Han, G. N. (2010). Hook lengths and shifted parts of partitions. *Ramanujan Journal*, 23, 127–135.

[10] Honsberger, R. (1985). *Mathematical Gems III*, The Mathematical Association of America, Washington DC.