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Asymptotics of sums of divisor functions over sequences with restricted factorization structure

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Abstract: We compute asymptotics of the sums of general divisor functions over h-free numbers, h-full numbers and other arithmetically interesting sets and conditions. The main tool for obtaining these results is Perron's formula.

Keywords: Divisor function, Sum of divisors function, h-free number, h-full number, Perfect powers, h-free part, h-full part.
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1 Introduction

Let $f(n)$ be an arithmetic function defined over the positive integers, and let S be a specific subset of $\mathbb{Z}_{>0}$. In this note, we are interested in sums of the form

$$\sum_{\substack{n\leq x\\ n\in S}} f(n),$$

where the sum is taken over all the elements in S that are smaller than x.

The sums where S is the set of positive integers have been largely studied for many arithmetic functions such as the divisor function $d(n)$, the sum of divisors $\sigma(n)$, Euler's totient function $\varphi(n)$, the prime divisor counting functions $\omega(n)$ and $\Omega(n)$, among several others. For example, for the divisor function $d(n) := \sum_{d|n} 1$, Dirichlet proved that [1, Theorem 320]

$$\sum_{n\le x} d(n) = x\log x + (2\gamma - 1)x + O\left(x^{\frac{1}{2}}\right),$$

where $\gamma = 0.57721\ldots$ is the Euler–Mascheroni constant. The error term has been improved by various authors, and the best known result, due to Huxley [2], is $O\left(x^{\alpha+\varepsilon}\right)$, with $\alpha \le \frac{131}{416}$.

For the sum of divisors function $\sigma(n)$ given by $\sigma(n) := \sum_{d|n} d$, we have that [1, Theorem 324]

$$\sum_{n\le x} \sigma(n) = \frac{\zeta(2)}{2}x^2 + O(x\log x),$$

and this error term has been improved to $O\left(x(\log x)^{\frac{2}{3}}\right)$ by Walfisz [5, Chapter III.2].

The sum of the inverses of divisors function $\sigma_{-1}(n)$ satisfies

$$\sum_{n\le x} \sigma_{-1}(n) = \zeta(2)x + O\left(\log x\right).$$

In this note, we propose the study of some analogous sums, where the n are restricted to certain special sets of natural numbers S such as the h-free, h-full numbers, and h-powers. We also consider going over the h-free and h-full parts of n. The main tool in this note is Perron's formula, that yields an estimate for such sums by the computation of a residue related to the generating function. For example, we recall that the generating functions for the arithmetic functions described above are given by

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta(s)^2, \qquad \mathrm{Re}(s) > 1,$$

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1), \qquad \mathrm{Re}(s) > 2,$$

$$\sum_{n=1}^{\infty} \frac{\sigma_{-1}(n)}{n^s} = \zeta(s)\zeta(s+1), \qquad \mathrm{Re}(s) > 1.$$

We will work more broadly with the general divisor function defined by $\sigma_j(n) = \sum_{d|n} d^j$, where j is a real number. We can think of $d(n)$ as the case $j = 0$.

Our results are typically given by asymptotics involving coefficients that are expressed as Euler products. When we write $\prod_p$, we mean that the product is taken over all the natural primes p.

Let h be an integer greater than 1. Recall that a positive integer n is said to be *h-free* if all the primes in its factorization have exponent strictly less than h. In other words, the prime factorization is given by $n = q_1^{s_1}\cdots q_r^{s_r}$ with $s_i \le h-1$ for $i = 1, \ldots, r$. Let F_h denote the set of h-free numbers. Then we have the following result.

Theorem 1.1. *Let $h > 1$ be an integer and $j > 0$. Then for any $\varepsilon > 0$, we have*

$$\sum_{\substack{n \le x \\ n \in F_h}} \sigma_{-j}(n) = \zeta(j+1)\mathcal{F}_{-j,h}(1)x + O\left(x^{\frac{1}{2}+\frac{1}{2h^2}+\varepsilon}\right), \tag{1}$$

where

$$\mathcal{F}_{-j,h}(1) = \prod_p \left(1 - \frac{1}{p^h} - \frac{\left(1-p^{-1}\right)\left(1-p^{-jh}\right)}{p^{h+j}\left(1-p^{-j}\right)}\right).$$

Recall that the above product is taken over all the natural primes.

Similarly, for any $\varepsilon > 0$, we have

$$\sum_{\substack{n \le x \\ n \in F_h}} \sigma_j(n) = \frac{\zeta(j+1)}{j+1}\mathcal{F}_{-j,h}(1)x^{j+1} + O\left(x^{j+\frac{1}{2}+\frac{1}{2h^2}+\varepsilon}\right). \tag{2}$$

For any $\varepsilon > 0$,

$$\sum_{\substack{n \le x \\ n \in F_h}} d(n) = \mathcal{F}_{0,h}(1)x\log x + \left[(2\gamma-1)\mathcal{F}_{0,h}(1) + \mathcal{F}'_{0,h}(1)\right]x + O\left(x^{\frac{3}{4}+\varepsilon}\right), \tag{3}$$

where

$$\mathcal{F}_{0,h}(1) = \prod_p \left(1 - \frac{h+1}{p^h} + \frac{h}{p^{h+1}}\right),$$

$$\frac{\mathcal{F}'_{0,h}(1)}{\mathcal{F}_{0,h}(1)} = h(h+1)\sum_p \frac{\log p\left(p^{-h} - p^{-(h+1)}\right)}{1-(h+1)p^{-h} + hp^{-(h+1)}},$$

and γ is the Euler–Mascheroni constant.

Recall that $\omega(n)$ denotes the number of distinct divisors of n. Then $2^{\omega(n)}$ denotes the number of square-free divisors of n. For $h \ge 2$ a positive integer we consider $h^{\omega(n)}$. We have the following.

Theorem 1.2. *Let $h > 1$ be an integer. For any $\varepsilon > 0$, we have*

$$\sum_{n \le x} h^{\omega(n)} = \frac{\mathcal{H}_h(1)}{(h-1)!}x\log^{h-1} x + c_{h-2,h}x\log^{h-2} x + \cdots + c_{0,h}x + O\left(x^{1-\frac{1}{2h}+\varepsilon}\right),$$

where $c_{h-2,h}, \ldots, c_{0,h}$ are certain constants depending on h and

$$\mathcal{H}_h(1) = \prod_p \left(1-\frac{1}{p}\right)^h \left(1+\frac{h}{p-1}\right).$$

Let P_h denote the set of h-powers. The constants from Theorem 1.2 appear also when considering the sum of the divisor function over elements of P_h.

Theorem 1.3. *Let $h \ge 1$ be an integer. For any $\varepsilon > 0$, we have*

$$\sum_{\substack{n \le x \\ n \in P_h}} d(n) = \frac{\mathcal{H}_h(1)}{h!h^h}x^{\frac{1}{h}}\log^h x + \frac{c_{h-1,h+1}}{h^{h-1}}x^{\frac{1}{h}}\log^{h-1} x + \cdots + c_{0,h+1}x^{\frac{1}{h}} + O\left(x^{\frac{1}{h}-\frac{1}{2(h+1)h}+\varepsilon}\right),$$

where the constants are defined as in Theorem 1.2.

We consider the sum of divisors $\sigma_j(n)$ over the h-powers as well.

Theorem 1.4. *Let $h \geq 1$ be an integer and $0 < j < 1$. Then for any $\varepsilon > 0$ we have*

$$\sum_{\substack{n \leq x \\ n \in P_h}} \sigma_{-j}(n) = \mathcal{P}_{-j,h}\left(\frac{1}{h}\right) x^{\frac{1}{h}} + O\left(x^{\frac{1}{2h}+\frac{(1-j)^2}{2h^3}+\varepsilon}\right), \tag{4}$$

where

$$\mathcal{P}_{-j,h}\left(\frac{1}{h}\right) = \prod_p \left(1 + \frac{1-p^{-hj}}{(1-p^{-j})(p^{1+j}-p^{-(h-1)j})}\right).$$

For $j \geq 1$, we get the same formula with the error term replaced by $O\left(x^{\frac{1}{2h}+\varepsilon}\right)$.

Similarly, for $0 < j < 1$ and any $\varepsilon > 0$, we have

$$\sum_{\substack{n \leq x \\ n \in P_h}} \sigma_{j}(n) = \mathcal{P}_{-j,h}\left(\frac{1}{h}\right) x^{j+\frac{1}{h}} + O\left(x^{j+\frac{1}{2h}+\frac{(1-j)^2}{2h^3}+\varepsilon}\right). \tag{5}$$

For $j \geq 1$, we get the same formula with the error term replaced by $O\left(x^{j+\frac{1}{2h}+\varepsilon}\right)$.

Recall that a positive integer n is said to be h-*full* if all the primes in its factorization have exponent larger or equal than h. In other words, the prime factorization is given by $n = q_1^{s_1} \cdots q_r^{s_r}$ with $s_i \geq h$ for $i = 1, \ldots, r$. Let G_h denote the set of h-full numbers. Then we have the following results.

Theorem 1.5. *Let $h \geq 1$ be an integer and $j > \frac{1}{h+1}$. Then for any $\varepsilon > 0$ we have*

$$\sum_{\substack{n \leq x \\ n \in G_h}} \sigma_{-j}(n) = \mathcal{G}_{-j,h}\left(\frac{1}{h}\right) x^{\frac{1}{h}} + O\left(x^{\frac{1}{2h}+\frac{1}{2h(h+1)^2}+\varepsilon}\right), \tag{6}$$

where

$$\mathcal{G}_{-j,h}\left(\frac{1}{h}\right) = \prod_p \left(1 + \frac{p^{-j}+p^{-\frac{1}{h}}-p^{-j-\frac{1}{h}}}{p\left(1-p^{-j}\right)\left(1-p^{-\frac{1}{h}}\right)} - \frac{p^{-(h+1)j}}{p\left(1-p^{-j}\right)\left(1-p^{-j-\frac{1}{h}}\right)} + \frac{1}{p^2\left(1-p^{-j}\right)}\left(\frac{p^{-(h+1)j}}{1-p^{-j-\frac{1}{h}}} - \frac{1}{1-p^{-\frac{1}{h}}}\right)\right).$$

For $0 < j \leq \frac{1}{h+1}$, we get the same formula with the error term replaced by $O\left(x^{\frac{1}{2h}+\frac{(1-j)^2}{2h^3}+\varepsilon}\right)$.

Similarly, for $j > \frac{1}{h+1}$ and for any $\varepsilon > 0$ we have

$$\sum_{\substack{n \leq x \\ n \in G_h}} \sigma_{j}(n) = \frac{\mathcal{G}_{-j,h}\left(\frac{1}{h}\right)}{hj+1} x^{j+\frac{1}{h}} + O\left(x^{j+\frac{1}{2h}+\frac{1}{2h(h+1)^2}+\varepsilon}\right). \tag{7}$$

For $0 < j \leq \frac{1}{h+1}$, we get the same formula with the error term replaced by $O\left(x^{j+\frac{1}{2h}+\frac{(1-j)^2}{2h^3}+\varepsilon}\right)$.

Finally, for any $\varepsilon > 0$,

$$\sum_{\substack{n \leq x \\ n \in G_h}} d(n) = \frac{\mathcal{G}_{0,h}\left(\frac{1}{h}\right)}{h!h^h} x^{\frac{1}{h}} \log^h x + \frac{d_{h-1,h+1}}{h^{h-1}} x^{\frac{1}{h}} \log^{h-1} x + \cdots + d_{0,h+1} x^{\frac{1}{h}} + O\left(x^{\frac{1}{h}-\frac{1}{2h(h+1)}+\varepsilon}\right), \tag{8}$$

where $d_{h-1,h+1},\ldots,d_{0,h+1}$ are certain constants depending on h and

$$\mathcal{G}_{0,h}\left(\frac{1}{h}\right)=\prod_p\left(1-\frac{1}{p}\right)^{h+1}\left(1+\frac{hp^{-1}}{1-p^{-\frac{1}{h}}}+\frac{p^{-1}}{\left(1-p^{-\frac{1}{h}}\right)^2}\right).$$

We consider the h-free and h-full parts. Let $n=q_1^{s_1}\cdots q_r^{s_r}$ be the prime factorization of n as usual. Let

$$L_h(n)=\prod_{\substack{1\le j\le r\\ s_j<h}}q_j^{s_j}\qquad\text{and}\qquad U_h(n)=\prod_{\substack{1\le j\le r\\ h\le s_j}}q_j^{s_j}.$$

We say that $L_h(n)$ is the *h-free part* of n and that $U_h(n)$ is the *h-full part* of n. By convention, $L_1(n)=1$, while naturally $U_1(n)=n$. Similarly, when $h>\max_j s_j$, we have $L_h(n)=n$ and $U_h(n)=1$. It is natural to ask similar questions to the ones addressed in Theorems 1.1 and 1.5 replacing the conditions $n\in F_h$ or $n\in G_h$ by $L_h(n)$ and $U_h(n)$, respectively. This is what we do next.

Theorem 1.6. *Let $h>1$ be an integer and let $0<j<1$. For any $\varepsilon>0$, we have*

$$\sum_{n\le x}^{\infty}\sigma_{-j}(L_h(n))=\zeta(j+1)\mathcal{L}_{-j,h}(1)x+O\left(x^{\frac{1}{2}+\frac{(1-j)^2}{2h^2}+\varepsilon}\right),\tag{9}$$

and

$$\sum_{n\le x}^{\infty}\sigma_{j}(L_h(n))=\frac{\zeta(j+1)}{j+1}\mathcal{L}_{-j,h}(1)x^{j+1}+O\left(x^{j+\frac{1}{2}+\frac{(1-j)^2}{2h^2}+\varepsilon}\right),\tag{10}$$

where

$$\mathcal{L}_{-j,h}(1)=\prod_p\left(1+\frac{p^{-h-(h+1)j}-p^{-h-j}-p^{-(h+1)(j+1)}+p^{-h-1-2j}}{1-p^{-j}}\right).$$

For $j\ge1$, we get the same formulas with the error terms replaced by $O\left(x^{\frac{1}{2}+\varepsilon}\right)$ and $O\left(x^{j+\frac{1}{2}+\varepsilon}\right)$ respectively.

For $0<j<1$, we also have

$$\sum_{n\le x}^{\infty}\sigma_{-j}(U_h(n))=\zeta(j+1)\mathcal{U}_{-j,h}(1)x+O\left(x^{\frac{1}{2}+\frac{(1-j)^2}{2h^2}+\varepsilon}\right),\tag{11}$$

and

$$\sum_{n\le x}^{\infty}\sigma_{j}(U_h(n))=\frac{\zeta(j+1)}{j+1}\mathcal{U}_{-j,h}(1)x^{j+1}+O\left(x^{j+\frac{1}{2}+\frac{(1-j)^2}{2h^2}+\varepsilon}\right),\tag{12}$$

where

$$\mathcal{U}_{-j,h}(1)=\prod_p\left(1-\frac{1}{p^{j+1}}+\frac{-p^{-h-(h+1)j}+p^{-h-j}+p^{-(h+1)(j+1)}-p^{-h-1-2j}}{1-p^{-j}}\right).$$

For $j\ge1$, we get the same formulas with the error terms replaced by $O\left(x^{\frac{1}{2}+\varepsilon}\right)$ and $O\left(x^{j+\frac{1}{2}+\varepsilon}\right)$ respectively.

Finally, we have

$$\sum_{n\le x} d(L_h(n)) = \mathcal{L}_{0,h}(1)x\log x + \left[(2\gamma-1)\mathcal{L}_{0,h}(1) + \mathcal{L}'_{0,h}(1)\right]x + O\left(x^{\frac{3}{4}+\varepsilon}\right), \tag{13}$$

where

$$\mathcal{L}_{0,h}(1) = \prod_p \left(1 - \frac{h}{p^h} + \frac{h-1}{p^{h+1}}\right)$$

and

$$\frac{\mathcal{L}'_{0,h}(1)}{\mathcal{L}_{0,h}(1)} = \sum_p \log p \frac{h^2p^{-h} - (h^2-1)p^{-(h+1)}}{1 - hp^{-h} + (h-1)p^{-(h+1)}}.$$

Similarly, we have

$$\sum_{n\le x} d(U_h(n)) = \mathcal{U}_{0,h}(1)x + O\left(x^{\frac{1}{2}+\frac{1}{2h^2}+\varepsilon}\right), \tag{14}$$

where

$$\mathcal{U}_{0,h}(1) = \prod_p \left(1 + (h-1)p^{-h} + \frac{p^{-h}}{1-p^{-1}}\right).$$

The main tool for proving our results is Perron's formula, which we review in Section 2. The results involving the divisor functions for h-free, h-powers, and h-full numbers are proven in Sections 3, 4 and 5, respectively. We treat the h-free and h-full parts in Section 6.

2 Some background on the zeta function and Perron's formula

In this section we recall some bounds for the Riemann zeta function and prove some versions of Perron's formula that will be specially useful for the results of this article.

First we recall some bounds for the Riemann zeta function.

Theorem 2.1. ([3, Theorem 1.9]) *For* $t \ge t_0 > 0$ *uniformly in* σ,

$$\zeta(\sigma+it) \ll \begin{cases} 1 & \textit{for} \quad 2 \le \sigma, \\ \log t & \textit{for} \quad 1 \le \sigma \le 2, \\ t^{\frac{1-\sigma}{2}}\log t & \textit{for} \quad 0 \le \sigma \le 1, \\ t^{\frac{1}{2}-\sigma}\log t & \textit{for} \quad \sigma \le 0. \end{cases}$$

We will work with the following version of Perron's formula.

Theorem 2.2. (Perron's formula [4, 4.4.15]) *Suppose that the Dirichlet series*

$$\mathcal{G}(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

is absolutely convergent for $\mathrm{Re}(s) > \sigma_0$. *Let* $x > 0$ *that is not an integer and let* $\sigma > \max\{0, \sigma_0\}$. *Then, we have*

$$\sum_{n\le x} a_n = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \mathcal{G}(s)\frac{x^s}{s}ds + O\left(x^\sigma \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} \min\left\{1, \frac{1}{T|\log\frac{x}{n}|}\right\}\right).$$

Corollary 2.1. ([4, 4.4.16]) *Suppose that $a_n = O(n^\varepsilon)$ with $\varepsilon > 0$. Then if x is not an integer,*

$$\sum_{n\le x} a_n = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \mathcal{G}(s)\frac{x^s}{s}ds + O\left(\frac{x^{\sigma+\varepsilon}}{T}\right).$$

Corollary 2.2. ([4, 4.4.17], modified version) *Suppose that $a_n = O(n^\varepsilon)$ with $\varepsilon > 0$ and*

$$\mathcal{G}(s) = \zeta(s)^\ell \mathcal{G}_1(s),$$

where k is a positive integer and $\mathcal{G}_1(s)$ is a Dirichlet series absolutely convergent in $\mathrm{Re}(s) > 1-\delta$ for some $\frac{1}{\ell} \le \delta \le 1$. Then

$$\sum_{n\le x} a_n = \frac{d^{\ell-1}}{ds^{\ell-1}}\left((s-1)^\ell\zeta(s)^\ell\mathcal{G}_1(s)\frac{x^s}{s}\right)\Bigg|_{s=1} + O\left(x^{1-\frac{1}{2\ell}+\varepsilon}\right).$$

If $0 < \delta < \frac{1}{\ell}$, then the error term should be replaced by

$$O\left(x^{1-\delta+\frac{\delta^2\ell}{2}+\varepsilon}\right).$$

Proof. We start by fixing $\sigma_1 > 1$. By Corollary 2.1, we have

$$\sum_{n\le x} a_n = \frac{1}{2\pi i}\int_{\sigma_1-iT}^{\sigma_1+iT} \mathcal{G}(s)\frac{x^s}{s}ds + O\left(\frac{x^{\sigma_1+\varepsilon}}{T}\right).$$

Now fix $1 > \sigma_2 > 1-\delta$, and consider the integration along the rectangle with corners $\sigma_1 - iT, \sigma_1 + iT, \sigma_2 + iT, \sigma_2 - iT$. By Cauchy's integral formula,

$$\frac{1}{2\pi i}\left(\int_{\sigma_1-iT}^{\sigma_1+iT} + \int_{\sigma_1+iT}^{\sigma_2+iT} + \int_{\sigma_2+iT}^{\sigma_2-iT} + \int_{\sigma_2-iT}^{\sigma_1-iT}\right)\mathcal{G}(s)\frac{x^s}{s}ds = \mathrm{Res}_{s=1}\left((s-1)^\ell\zeta(s)^\ell\mathcal{G}_1(s)\frac{x^s}{s}\right).$$

By Theorem 2.1, the contribution from the other vertical integral is bounded by

$$\left|\frac{1}{2\pi i}\int_{\sigma_2+iT}^{\sigma_2-iT}\mathcal{G}(s)\frac{x^s}{s}ds\right| \ll x^{\sigma_2}T^{\frac{(1-\sigma_2)\ell}{2}}\log^\ell T.$$

Again by Theorem 2.1, the contribution from the horizontal integrals is bounded by

$$\left|\frac{1}{2\pi i}\int_{\sigma_1\pm iT}^{\sigma_2\pm iT}\mathcal{G}(s)\frac{x^s}{s}ds\right| \ll \frac{x^{\sigma_1}}{\log x}T^{\frac{(1-\sigma_2)\ell-2}{2}}\log^\ell T.$$

Now take $T = x^{\sigma_1-\sigma_2}$ to make the above error terms comparable. This gives an error term of

$$\ll x^{\sigma_2+\frac{(\sigma_1-\sigma_2)(1-\sigma_2)\ell}{2}+\varepsilon}.$$

Our goal is to optimize this error term by appropriately choosing σ_2 and σ_1. Thinking of the exponent as a quadratic equation on σ_2, this is minimized when $\sigma_2 = \frac{1+\sigma_1}{2} - \frac{1}{\ell}$.

We can take $\sigma_1 = 1+\varepsilon_1$, then $\sigma_2 = 1 + \frac{\varepsilon_1}{2} - \frac{1}{\ell}$. This gives an error term of

$$\ll x^{1-\frac{1}{2\ell}+\frac{\varepsilon_1}{2}-\frac{\ell\varepsilon_1^2}{8}+\varepsilon}.$$

If $\frac{1}{\ell} \le \delta$, we can take ε_1 arbitrarily small, and we get the result.

Otherwise, take $\sigma_1 = 1+\varepsilon_1$, $\sigma_2 = 1-\delta+\varepsilon_2$ and we get an error term of

$$\ll x^{1-\delta+\varepsilon_2+\frac{(\delta+\varepsilon_1-\varepsilon_2)(\delta-\varepsilon_2)\ell}{2}+\varepsilon}.$$

Taking ε_i arbitrarily small, we get the result. □

We end this section by recalling the following useful result.

Theorem 2.3 (Abel's summation formula)**.** *Let a_n be a sequence of complex numbers and $f(x)$ be a continuously differentiable function on $[1, x]$. If $A(x) = \sum_{n \le x} a_n$, then*

$$\sum_{n \le x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$$

3 h-free numbers

Recall that a natural number n with prime factorization $q_1^{s_1} \cdots q_r^{s_r}$ is said to be h-free if $s_i \le h-1$ for $i = 1, \dots, r$. For example, if $h = 2$, we have the square-free numbers, if $h = 3$, we obtain the cube-free numbers, etc. Let F_h denote the set of h-free numbers. In this section we prove Theorem 1.1.

It is well-known that the number of h-free positive integers not exceeding x is given by

$$\sum_{\substack{n \le x \\ n \in F_h}} 1 = \frac{x}{\zeta(h)} + O\left(x^{\frac{1}{h}}\right).$$

That is, these numbers have positive density $\frac{1}{\zeta(h)} \to 1$ if h is large.

Before proceeding to the proofs, we remark that for $j > 0$,

$$\sigma_{-j}(n) \le d(n) = O(n^{\varepsilon}).$$

It follows that the generating Dirichlet series of σ_{-j} is convergent for $\mathrm{Re}(s) > 1$. We also have that

$$\frac{\sigma_j(n)}{n^j} = \sigma_{-j}(n) = O(n^{\varepsilon}),$$

which will be helfpul to relate sums of $\sigma_j(n)$ to sums of $\sigma_{-j}(n)$.

Proof of Theorem 1.1. Assume that $j > 0$. We consider the generating Dirichlet series of σ_{-j} over F_h. Since σ_{-j} is multiplicative, and the structure of F_h is also defined multiplicatively, the generating series can be directly computed by considering the case of n a prime power, and then multiplying together all the factors corresponding to different primes.

$$\begin{aligned}\sum_{n \in F_h} \frac{\sigma_{-j}(n)}{n^s} &= \prod_p \left(1 + \frac{1+p^{-j}}{p^s} + \frac{1+p^{-j}+p^{-2j}}{p^{2s}} + \cdots + \frac{1+p^{-j}+\cdots+p^{-(h-1)j}}{p^{(h-1)s}}\right) \\ &= \prod_p \left(1-p^{-j}\right)^{-1}\left(1-p^{-j} + \frac{1-p^{-2j}}{p^s} + \cdots + \frac{1-p^{-hj}}{p^{(h-1)s}}\right) \\ &= \prod_p \left(1-p^{-j}\right)^{-1}\left(\frac{1-p^{-hs}}{1-p^{-s}} - \frac{p^{-j}(1-p^{-h(j+s)})}{1-p^{-(j+s)}}\right) \\ &= \zeta(s)\zeta(j+s)\mathcal{F}_{-j,h}(s),\end{aligned} \tag{15}$$

where

$$\mathcal{F}_{-j,h}(s) = \prod_p \left(1 - \frac{p^{-hs} - p^{-(h+1)s-j} - p^{-hs-(h+1)j} + p^{-(h+1)(s+j)}}{1-p^{-j}}\right),$$

which converges for $\mathrm{Re}(s) > \frac{1}{h}$ when $j > 0$.

We apply Corollary 2.2, where we have that $\ell = 1$, and $\delta = 1 - \frac{1}{h}$. This gives

$$\sum_{\substack{n \le x \\ n \in F_h}} \sigma_{-j}(n) = \zeta(j+1)\mathcal{F}_{-j,h}(1)x + O\left(x^{\frac{1}{2}+\frac{1}{2h^2}+\varepsilon}\right).$$

This concludes the proof of (1).

To study $\sigma_j(n)$ with $j > 0$, we first notice that $\frac{\sigma_j(n)}{n^j} = \sigma_{-j}(n)$. Therefore we have

$$\sum_{\substack{n \le x \\ n \in F_h}} \frac{\sigma_j(n)}{n^j} = \zeta(j+1)\mathcal{F}_{-j,h}(1)x + O\left(x^{\frac{1}{2}+\frac{1}{2h^2}+\varepsilon}\right).$$

A simple application of Abel's summation gives

$$\begin{aligned}\sum_{\substack{n \le x \\ n \in F_h}} \sigma_j(n) &= \zeta(j+1)\mathcal{F}_{-j,h}(1)x^{j+1} + O\left(x^{j+\frac{1}{2}+\frac{1}{2h^2}+\varepsilon}\right) \\ &\qquad - j\int_0^x \left[\zeta(j+1)\mathcal{F}_{-j,h}(1)t^j + O\left(t^{j-\frac{1}{2}+\frac{1}{2h^2}+\varepsilon}\right)\right] dt \\ &= \frac{\zeta(j+1)}{j+1}\mathcal{F}_{-j,h}(1)x^{j+1} + O\left(x^{j+\frac{1}{2}+\frac{1}{2h^2}+\varepsilon}\right).\end{aligned}$$

This concludes the proof of (2).

Finally, to prove (3), we consider as before the Dirichlet series of $d(n)$ over F_h

$$\begin{aligned}\sum_{n \in F_h} \frac{d(n)}{n^s} &= \prod_p \left(1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + \cdots + \frac{h}{p^{(h-1)s}}\right) \\ &= \prod_p \left(\frac{1-(h+1)p^{-hs} + hp^{-(h+1)s}}{(1-p^{-s})^2}\right) \qquad (16) \\ &= \zeta(s)^2\mathcal{F}_{0,h}(s),\end{aligned}$$

where $\mathcal{F}_{0,h}(s)$ is absolutely convergent in $\mathrm{Re}(s) > \frac{1}{h}$. Also we have

$$\frac{\mathcal{F}'_{0,h}(s)}{\mathcal{F}_{0,h}(s)} = \sum_p \log p \frac{h(h+1)\left(p^{-hs} - p^{(h+1)s}\right)}{1-(h+1)p^{-hs} + hp^{-(h+1)s}}.$$

We apply Corollary 2.2. In this case we have $\ell = 2$, $\delta = 1 - \frac{1}{h}$. Thus, we are in the first case for the error term and we get

$$\sum_{\substack{n \le x \\ n \in F_h}} d(n) = \frac{d}{ds}\left((s-1)^2\zeta(s)^2\mathcal{F}_{0,h}(s)\frac{x^s}{s}\right)\Bigg|_{s=1} + O\left(x^{\frac{3}{4}+\varepsilon}\right). \qquad (17)$$

We proceed to compute

$$\begin{aligned}\frac{d}{ds}\left((s-1)^2\zeta(s)^2\mathcal{F}_{0,h}(s)\frac{x^s}{s}\right)\Bigg|_{s=1} &= \frac{d}{ds}\left((s-1)^2\zeta(s)^2\right)\Big|_{s=1}\mathcal{F}_{0,h}(1)x \\ &\qquad + \lim_{s\to 1}(s-1)^2\zeta(s)^2\left(\mathcal{F}'_{0,h}(1)x + \mathcal{F}_{0,h}(1)(x\log x - x)\right) \\ &= 2\gamma\mathcal{F}_{0,h}(1)x + \mathcal{F}'_{0,h}(1)x + \mathcal{F}_{0,h}(1)(x\log x - x) \\ &= \mathcal{F}_{0,h}(1)x\log x + \left[(2\gamma-1)\mathcal{F}_{0,h}(1) + \mathcal{F}'_{0,h}(1)\right]x. \qquad (18)\end{aligned}$$

Combining (18) with (17), we finally obtain (3). □

Theorem 1.1 takes a particularly elegant form when $j = 1$.

Corollary 3.1. *Let $h > 1$ be an integer. For any $\varepsilon > 0$*

$$\sum_{\substack{n\le x\\ n\in F_h}} \sigma_{-1}(n) = \frac{\zeta(2)}{\zeta(h)\zeta(h+1)}x + O\left(x^{\frac{1}{2}+\frac{1}{2h^2}+\varepsilon}\right),$$

$$\sum_{\substack{n\le x\\ n\in F_h}} \sigma(n) = \frac{\zeta(2)}{2\zeta(h)\zeta(h+1)}x^2 + O\left(x^{\frac{3}{2}+\frac{1}{2h^2}+\varepsilon}\right).$$

4 h-powers

In this section we consider various versions of h-powers and prove Theorems 1.2, 1.3, and 1.4.

We are firstly interested in the function $h^{\omega(n)}$, where $\omega(n)$ denotes the number of distinct prime divisors. Since $2^{\omega(n)}$ denotes the number of square-free divisors of n, we have $2^{\omega(n)} \le d(n) \ll n^{\varepsilon}$ for all $\varepsilon > 0$. This gives $h^{\omega(n)} = \left(2^{\omega(n)}\right)^{\frac{\log h}{\log 2}} \ll n^{\varepsilon}$ also for all $\varepsilon > 0$. Therefore the generating Dirichlet series for $h^{\omega(n)}$ converges for $\mathrm{Re}(s) > 1$. Indeed, it is well-known that

$$\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta(s)^2}{\zeta(2s)}.$$

We will generalize this result in the proof of Theorem 1.2.

Proof of Theorem 1.2. We start by considering the generating Dirichlet series of h^{ω}.

$$\sum_{n=1}^{\infty} \frac{h^{\omega(n)}}{n^s} = \prod_p \left(1 + \frac{h}{p^s} + \frac{h}{p^{2s}} + \cdots\right) = \prod_p \left(1 + \frac{h}{p^s - 1}\right) = \zeta(s)^h \mathcal{H}_h(s),$$

where

$$\begin{aligned}
\mathcal{H}_h(s) &= \prod_p \left(1 - \frac{1}{p^s}\right)^h \left(1 + \frac{h}{p^s - 1}\right) \\
&= \prod_p \left(\sum_{m=0}^{h} \binom{h}{m} \frac{(-1)^m}{p^{ms}}\right) \left(1 + \frac{h}{p^s} + \frac{h}{p^{2s}} + \cdots\right) \\
&= \prod_p \left(1 - \frac{h(h+1)}{2p^{2s}} + \cdots\right).
\end{aligned}$$

Therefore, $\mathcal{H}_h(s)$ converges for $\mathrm{Re}(s) > \frac{1}{2}$.

We apply Corollary 2.2, where we find that $\ell = h$, $\delta = \frac{1}{2}$. We are in the first case and this gives

$$\begin{aligned}
\sum_{n\le x} h^{\omega(n)} &= \frac{1}{(h-1)!} \frac{d^{h-1}}{ds^{h-1}} \left((s-1)^h \zeta(s)^h \mathcal{H}_h(s) \frac{x^s}{s}\right)\Bigg|_{s=1} + O\left(x^{1-\frac{1}{2h}+\varepsilon}\right) \\
&= \frac{\mathcal{H}_h(1)}{(h-1)!} x \log^{h-1} x + c_{h-2,h} x \log^{h-2} x + \cdots + c_{0,h} x + O\left(x^{1-\frac{1}{2h}+\varepsilon}\right),
\end{aligned}$$

where $c_{h-2,h}, \ldots, c_{0,h}$ are certain constants depending on h. $\square$

Corollary 4.1. *For any $\varepsilon > 0$, we have*

$$\sum_{n\leq x} 2^{\omega(n)} = \frac{1}{\zeta(2)} x\log x + \frac{1}{\zeta(2)}\left[(2\gamma-1) + 2\sum_p \frac{\log p}{p^2-1}\right] x + O\left(x^{\frac{3}{4}+\varepsilon}\right).$$

Let P_h denote the set of h-powers. We now consider the sum of the divisor function over elements of P_h. Before proceeding to the proof of Theorem 1.3, we shed some light into why we have very similar asymptotics to Theorem 1.2.

Lemma 4.1. *The following formulas hold*

$$d(n^h) = \sum_{d|n} h^{\omega(d)} \tag{19}$$

and

$$\sum_{\substack{n\leq x\\ n\in P_h}} d(n) = \sum_{1\leq d\leq x^{\frac{1}{h}}} h^{\omega(d)} \left\lfloor \frac{x^{\frac{1}{h}}}{d} \right\rfloor. \tag{20}$$

Proof. Since both sides of equation (19) are multiplicative, it suffices to prove this identity for prime powers p^a. The left-hand side gives $d(p^{ah}) = ah+1$. The right-hand side gives

$$\sum_{k=0}^{a} h^{\omega(p^{ah})} = 1 + \sum_{k=1}^{a} h = 1 + ah,$$

and this proves equation (19).

From this, we have

$$\sum_{n^h\leq x} d\left(n^h\right) = \sum_{1\leq n\leq x^{\frac{1}{h}}} \left(\sum_{d|n} h^{\omega(d)}\right) = \sum_{1\leq d\leq x^{\frac{1}{h}}} h^{\omega(d)} \left\lfloor \frac{x^{\frac{1}{h}}}{d} \right\rfloor,$$

and we obtain equation (20). □

The combination of Theorem 1.2, Abel's summation, and equation (20) gives us the following estimate

$$\sum_{\substack{n\leq x\\ n\in P_h}} d(n) = \sum_{n^h\leq x} d(n^h) \sim \frac{\mathcal{H}_h(1)}{(h-1)!h^{h+1}} x^{\frac{1}{h}} \log^h x = \frac{\mathcal{H}_h(1)}{h!h^h} x^{\frac{1}{h}} \log^h x.$$

We give a more precise formula in Theorem 1.3.

Proof of Theorem 1.3. As usual we consider the generating Dirichlet series

$$\begin{aligned}
\sum_{n\in P_h} \frac{d(n)}{n^s} &= \prod_p \left(1 + \frac{h+1}{p^{hs}} + \frac{2h+1}{p^{2hs}} + \frac{3h+1}{p^{3hs}} + \cdots\right)\\
&= \prod_p \left(\frac{p^{hs}}{p^{hs}-1} + \frac{hp^{hs}}{(p^{hs}-1)^2}\right)\\
&= \prod_p \left(1 - \frac{1}{p^{hs}}\right)^{-1} \left(1 + \frac{h}{p^{hs}-1}\right)\\
&= \zeta(hs)^{h+1}\mathcal{H}_h(hs),
\end{aligned}$$

and we recall that $\mathcal{H}_h(s)$ converges for $\mathrm{Re}(s) > \frac{1}{2}$.

We apply Corollary 2.2. First we make the change of variables $hs = s_1$. In this case we have $\ell = h + 1$ and $\delta = \frac{1}{2}$. We are in the first case and we get

$$\sum_{\substack{n\le x\\ n\in P_h}} d(n) = \frac{1}{h!}\frac{d^h}{ds_1^h}\left((s_1-1)^{h+1}\zeta(s_1)^{h+1}\mathcal{H}_h(s_1)\frac{x^{\frac{s_1}{h}}}{s_1}\right)\Bigg|_{s_1=1} + O\left(x^{\frac{1}{h}-\frac{1}{2(h+1)h}+\varepsilon}\right).$$

The statement of Theorem 1.3 follows by developping the above residue and comparing it with the statement of Theorem 1.2. □

We now consider the results regarding the sums of powers of divisors.

Proof of Theorem 1.4. Assume that $j > 0$. We consider the generating Dirichlet series of σ_{-j} over P_h.

$$\begin{aligned}
\sum_{n\in P_h}\frac{\sigma_{-j}(n)}{n^s} &= \prod_p\left(1+\frac{1+p^{-j}+\cdots+p^{-hj}}{p^{hs}}+\frac{1+p^{-j}+\cdots+p^{-2hj}}{p^{2hs}}+\cdots\right)\\
&= \prod_p\left(1+(1-p^{-j})^{-1}\left(\frac{1-p^{-(h+1)j}}{p^{hs}}+\frac{1-p^{-(2h+1)j}}{p^{2hs}}+\cdots\right)\right)\\
&= \prod_p\left(1+(1-p^{-j})^{-1}\left(\frac{1}{p^{hs}-1}-\frac{p^{-j}}{p^{h(j+s)}-1}\right)\right)\\
&= \zeta(hs)\mathcal{P}_{-j,h}(s),
\end{aligned}$$

where

$$\mathcal{P}_{-j,h}(s) = \prod_p\left(1+\frac{1-p^{-hj}}{(1-p^{-j})(p^{hs+j}-p^{-(h-1)j})}\right),$$

which is convergent for $\mathrm{Re}(s) > \frac{1-j}{h}$.

Now we apply Corollary 2.2. First we make the change of variables $hs = s_1$. Here we have $\ell = 1$, $\delta = 1-\frac{1-j}{h}$ for $0 < j \le 1$ and $\delta = 1$, otherwise. If $0 < j < 1$, we are in the second case for the error term

$$\sum_{\substack{n\le x\\ n\in P_h}} \sigma_{-j}(n) = \mathcal{P}_{-j,h}\left(\frac{1}{h}\right)x^{\frac{1}{h}} + O\left(x^{\frac{1}{2h}+\frac{(1-j)^2}{2h^3}+\varepsilon}\right).$$

If $j \ge 1$, we are in the first case and we get an error term of $O\left(x^{\frac{1}{2h}+\varepsilon}\right)$.

This concludes the proof of (4).

Now we consider σ_j and (5). We use once again that $\frac{\sigma_j(n)}{n^j} = \sigma_{-j}(n)$. Therefore we have

$$\sum_{\substack{n\le x\\ n\in P_h}} \frac{\sigma_j(n)}{n^j} = \mathcal{P}_{-j,h}\left(\frac{1}{h}\right)x^{\frac{1}{h}} + O\left(x^{\frac{1}{2h}+\frac{(1-j)^2}{2h^3}+\varepsilon}\right).$$

By applying Abel's summation to the above equation, we obtain

$$\sum_{\substack{n\le x\\ n\in P_h}} \sigma_j(n) = \mathcal{P}_{-j,h}\left(\frac{1}{h}\right) x^{j+\frac{1}{h}+} + O\left(x^{j+\frac{1}{2h}+\frac{(1-j)^2}{2h^3}+\varepsilon}\right)$$

$$- j\int_0^x \left[\mathcal{P}_{-j,h}\left(\frac{1}{h}\right) t^{j-1+\frac{1}{h}} + O\left(t^{j-1+\frac{1}{2h}+\frac{(1-j)^2}{2h^3}+\varepsilon}\right)\right] dt$$

$$= \frac{\mathcal{P}_{-j,h}\left(\frac{1}{h}\right)}{jh+1} x^{j+\frac{1}{h}} + O\left(x^{j+\frac{1}{2h}+\frac{(1-j)^2}{2h^3}+\varepsilon}\right).$$

For $j \ge 1$, the error term is $O\left(x^{j+\frac{1}{2h}+\varepsilon}\right)$ instead.

This concludes the proof of Theorem 1.4. □

5 *h*-full numbers

Recall that a natural number n with prime factorization $q_1^{s_1}\cdots q_r^{s_r}$ is said to be h-full if $s_i \ge h$ for $i = 1,\dots,r$. For example, if $h = 2$, we have the square-full numbers, if $h = 3$, we obtain the cube-full numbers, etc. Let G_h denote the set of h-full numbers. In this section we prove Theorem 1.5.

Proof of Theorem 1.5. Assume that $j > 0$. We consider the corresponding generating Dirichlet series.

$$\begin{aligned}
\sum_{n\in G_h} \frac{\sigma_{-j}(n)}{n^s} &= \prod_p \left(1 + \frac{1+p^{-j}+\cdots+p^{-hj}}{p^{hs}} + \frac{1+p^{-j}+\cdots+p^{-(h+1)j}}{p^{(h+1)s}} + \cdots\right)\\
&= \prod_p \left(1 + p^{-hs}\left(1-p^{-j}\right)^{-1}\left(\left(1-p^{-(h+1)j}\right) + \frac{1-p^{-(h+2)j}}{p^s} + \cdots\right)\right)\\
&= \prod_p \left(1 + p^{-hs}\left(1-p^{-j}\right)^{-1}\left(\frac{1}{1-p^{-s}} - \frac{p^{-(h+1)j}}{1-p^{-j-s}}\right)\right)\\
&= \zeta(hs)\mathcal{G}_{-j,h}(s),
\end{aligned} \tag{21}$$

where

$$\mathcal{G}_{-j,h}(s) = \prod_p \left(1 + \frac{p^{-j}+p^{-s}-p^{-j-s}}{p^{hs}\left(1-p^{-j}\right)\left(1-p^{-s}\right)} - \frac{p^{-(h+1)j}}{p^{hs}\left(1-p^{-j}\right)\left(1-p^{-j-s}\right)} + \frac{1}{p^{2hs}\left(1-p^{-j}\right)}\left(\frac{p^{-(h+1)j}}{1-p^{-j-s}} - \frac{1}{1-p^{-s}}\right)\right),$$

which converges for $\mathrm{Re}(s) > \frac{1}{h+1}, \frac{1-j}{h}$.

We apply Corollary 2.2. We make the change of variables $hs = s_1$. Here we have $\ell = 1$, $\delta = 1 - \frac{1}{h+1}$ when $j > \frac{1}{h+1}$ and $\delta = 1 - \frac{1-j}{h}$ when $j \le \frac{1}{h+1}$. We are in the second case and we get, for $j > \frac{1}{h+1}$,

$$\sum_{\substack{n\le x\\ n\in G_h}} \sigma_{-j}(n) = \mathcal{G}_{-j,k}\left(\frac{1}{h}\right) x^{\frac{1}{h}} + O\left(x^{\frac{1}{2h}+\frac{1}{2h(h+1)^2}+\varepsilon}\right).$$

When $j \le \frac{1}{h+1}$, the error term is replaced by $O\left(x^{\frac{1}{2h}+\frac{(1-j)^2}{2h^3}+\varepsilon}\right)$.

This proves equation (6).

We now consider (7). The previous result gives, for $j > \frac{1}{h+1}$,

$$\sum_{\substack{n\le x\\ n\in G_h}} \frac{\sigma_j(n)}{n^j} = \mathcal{G}_{-j,k}\left(\frac{1}{h}\right) x^{\frac{1}{h}} + O\left(x^{\frac{1}{2h}+\frac{1}{2h(h+1)^2}+\varepsilon}\right).$$

By Abel's summation,

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in G_h}} \sigma_j(n) &= \mathcal{G}_{-j,k}\left(\frac{1}{h}\right) x^{j+\frac{1}{h}} + O\left(x^{j+\frac{1}{2h}+\frac{1}{2h(h+1)^2}+\varepsilon}\right)\\ &\quad - j\int_0^x \left[\mathcal{G}_{-j,k}\left(\frac{1}{h}\right) t^{j-1+\frac{1}{h}} + O\left(t^{j-1+\frac{1}{2h}+\frac{1}{2h(h+1)^2}+\varepsilon}\right)\right] dt\\ &= \frac{\mathcal{G}_{-j,k}\left(\frac{1}{h}\right)}{hj+1} x^{j+\frac{1}{h}} + O\left(x^{j+\frac{1}{2h}+\frac{1}{2h(h+1)^2}+\varepsilon}\right).\end{aligned}$$

When $j \le \frac{1}{h+1}$, the error term is replaced by $O\left(x^{j+\frac{1}{2h}+\frac{(1-j)^2}{2h^3}+\varepsilon}\right)$.

We now consider the generating Dirichlet series for the divisor function.

$$\begin{aligned}\sum_{n\in G_h} \frac{d(n)}{n^s} &= \prod_p \left(1 + \frac{h+1}{p^{hs}} + \frac{h+2}{p^{(h+1)s}} + \frac{h+3}{p^{(h+2)s}} + \cdots\right)\\ &= \prod_p \left(1 + \frac{hp^{-hs}}{1-p^{-s}} + \frac{p^{-hs}}{(1-p^{-s})^2}\right) \qquad (22)\\ &= \zeta(hs)^{h+1}\mathcal{G}_{0,h}(s),\end{aligned}$$

where

$$\begin{aligned}\mathcal{G}_{0,h}(s) &= \prod_p \left(1 - \frac{1}{p^{hs}}\right)^{h+1} \left(1 + \frac{hp^{-hs}}{1-p^{-s}} + \frac{p^{-hs}}{(1-p^{-s})^2}\right)\\ &= \prod_p \left(\sum_{m=0}^{h+1} \binom{h+1}{m} \frac{(-1)^m}{p^{mhs}}\right) \left(1 + \frac{h+1}{p^{hs}} + \frac{h+2}{p^{(h+1)s}} + \frac{h+3}{p^{(h+2)s}} + \cdots\right)\\ &= \prod_p \left(1 + \frac{h+2}{p^{(h+1)s}} + \cdots\right)\end{aligned}$$

is convergent for $\mathrm{Re}(s) > \frac{1}{h+1}$.

We apply Corollary 2.2. We start by making the change of variables $hs = s_1$. Here we have $\ell = h+1$, $\delta = \frac{1}{h+1}$. We are in the first case and this yields

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in G_h}} d(n) &= \frac{1}{h!}\frac{d^h}{ds_1^h}\left((s_1-1)^{h+1}\zeta(s_1)^{h+1}\mathcal{G}_{0,h}\left(\frac{s_1}{h}\right)\frac{x^{\frac{s_1}{h}}}{s_1}\right)\Bigg|_{s_1=1} + O\left(x^{\frac{1}{h}-\frac{1}{2h(h+1)}+\varepsilon}\right)\\ &= \frac{\mathcal{G}_{0,h}\left(\frac{1}{h}\right)}{h!h^h} x^{\frac{1}{h}}\log^h x + \frac{d_{h-1,h+1}}{h^{h-1}} x^{\frac{1}{h}}\log^{h-1} x + \cdots + d_{0,h+1}x^{\frac{1}{h}} + O\left(x^{\frac{1}{h}-\frac{1}{2h(h+1)}+\varepsilon}\right). \square\end{aligned}$$

6 h-free and h-full parts

Let $n = q_1^{s_1} \cdots q_r^{s_r}$ be the prime factorization of n. Recall that

$$L_h(n) = \prod_{\substack{1\le j\le r\\ s_j<h}} q_j^{s_j} \qquad \text{and} \qquad U_h(n) = \prod_{\substack{1\le j\le r\\ h\le s_j}} q_j^{s_j}$$

are the h-free and h-full parts of n respectively. For a fixed $h > 1$ integer, we can write $n = L_h(n)U_h(n)$ uniquely. For an arithmetic function $f(n)$, we can investigate the sums of $f(L_h(n))$ and $f(U_h(n))$, which correspond to summing over the h-free and h-full parts of the numbers n not exceeding x.

Proof of Theorem 1.6. Suppose that $j > 0$. We start by considering the generating series. Following a similar calculation to the one in (15),

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\sigma_{-j}(L_h(n))}{n^s} &= \prod_p \left(1 + \frac{1+p^{-j}}{p^s} + \cdots + \frac{1+p^{-j}+\cdots+p^{-(h-1)j}}{p^{(h-1)s}} + \frac{1}{p^{hs}} + \cdots\right)\\ &= \prod_p \left(\frac{1-p^{-hs}}{(1-p^{-j})(1-p^{-s})} - \frac{p^{-j}(1-p^{-h(j+s)})}{(1-p^{-j})(1-p^{-(j+s)})} + \frac{p^{-hs}}{1-p^{-s}}\right)\\ &= \zeta(s)\zeta(j+s)\mathcal{L}_{-j,h}(s),\end{aligned}$$

where

$$\mathcal{L}_{-j,h}(s) = \prod_p \left(1 + \frac{p^{-hs-(h+1)j} - p^{-hs-j} - p^{-(h+1)(j+s)} + p^{-(h+1)s-2j}}{1-p^{-j}}\right),$$

which converges for $\mathrm{Re}(s) > \frac{1-j}{h}$.

We apply Corollary 2.2. We have that $\ell = 1$, and $\delta = 1 - \frac{1-j}{h}$ for $0 < j \le 1$ and $\delta = 1$ otherwise. If $0 < j < 1$, we are in the second case. This gives

$$\sum_{n\le x}^{\infty} \sigma_{-j}(L_h(n)) = \zeta(j+1)\mathcal{L}_{-j,h}(1)x + O\left(x^{\frac{1}{2}+\frac{(1-j)^2}{2h^2}+\varepsilon}\right).$$

If $j \ge 1$, we are in the first case and we simply get an error term of $O\left(x^{\frac{1}{2}+\varepsilon}\right)$.

Using that $\frac{\sigma_j(n)}{n^j} = \sigma_{-j}(n)$, we have, for $0 < j < 1$,

$$\sum_{n\le x}^{\infty} \frac{\sigma_j(L_h(n))}{n^j} = \zeta(j+1)\mathcal{L}_{-j,h}(1)x + O\left(x^{\frac{1}{2}+\frac{(1-j)^2}{2h^2}+\varepsilon}\right).$$

By applying Abel's summation to the above equation, we have

$$\begin{aligned}\sum_{n\le x}^{\infty} \sigma_j(L_h(n)) &= \zeta(j+1)\mathcal{L}_{-j,h}(1)x^{j+1} + O\left(x^{j+\frac{1}{2}+\frac{(1-j)^2}{2h^2}+\varepsilon}\right)\\ &\qquad - j\int_0^x \left[\zeta(j+1)\mathcal{L}_{-j,h}(1)t^j + O\left(t^{j-\frac{1}{2}+\frac{(1-j)^2}{2h^2}+\varepsilon}\right)\right] dt\\ &= \frac{\zeta(j+1)}{j+1}\mathcal{L}_{-j,h}(1)x^{j+1} + O\left(x^{j+\frac{1}{2}+\frac{(1-j)^2}{2h^2}+\varepsilon}\right).\end{aligned}$$

For $j \ge 1$, the error term is $O\left(x^{j+\frac{1}{2}+\varepsilon}\right)$ instead.

For the sums over $U_n(n)$, we have, for $j > 0$,

$$\sum_{n=1}^{\infty} \frac{\sigma_{-j}(U_h(n))}{n^s} = \prod_p \left(1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{(h-1)s}} + \frac{1 + p^{-j} + \cdots + p^{-hj}}{p^{hs}} \right.$$
$$\left. + \frac{1 + p^{-j} + \cdots + p^{-(h+1)j}}{p^{(h+1)s}} + \cdots \right)$$
$$= \prod_p \left(\frac{1 - p^{-hs}}{1 - p^{-s}} + \frac{p^{-hs}}{(1 - p^{-j})(1 - p^{-s})} - \frac{p^{-(h+1)j - hs}}{(1 - p^{-j})(1 - p^{-j-s})} \right)$$
$$= \zeta(s)\zeta(j + s)\mathcal{U}_{-j,h}(s),$$

where we have employed the computation from (21), and

$$\mathcal{U}_{-j,h}(s) = \prod_p \left(1 - \frac{1}{p^{j+s}} + \frac{-p^{-hs-(h+1)j} + p^{-hs-j} + p^{-(h+1)(j+s)} - p^{-(h+1)s-2j}}{1 - p^{-j}} \right),$$

which converges for $\mathrm{Re}(s) > \frac{1-j}{h}$.

We apply Corollary 2.2. We have that $\ell = 1$, and $\delta = 1 - \frac{1-j}{h}$ for $0 < j \leq 1$ and $\delta = 1$ otherwise. If $0 < j < 1$, we are in the second case. This gives

$$\sum_{n \leq x}^{\infty} \sigma_{-j}(U_h(n)) = \zeta(j + 1)\mathcal{U}_{-j,h}(1)x + O\left(x^{\frac{1}{2} + \frac{(1-j)^2}{2h^2} + \varepsilon}\right).$$

If $j \geq 1$, we are in the first case and we get an error term of $O\left(x^{\frac{1}{2}+\varepsilon}\right)$.

By using that $\frac{\sigma_j(n)}{n^j} = \sigma_{-j}(n)$, we have, for $0 < j < 1$,

$$\sum_{n \leq x}^{\infty} \frac{\sigma_j(U_h(n))}{n^j} = \zeta(j + 1)\mathcal{U}_{-j,h}(1)x + O\left(x^{\frac{1}{2} + \frac{(1-j)^2}{2h^2} + \varepsilon}\right).$$

By applying Abel's summation to the above equation, we have

$$\sum_{n \leq x}^{\infty} \sigma_j(U_h(n)) = \zeta(j + 1)\mathcal{U}_{-j,h}(1)x^{j+1} + O\left(x^{j + \frac{1}{2} + \frac{(1-j)^2}{2h^2} + \varepsilon}\right)$$
$$- j \int_0^x \left[\zeta(j + 1)\mathcal{U}_{-j,h}(1)t^j + O\left(t^{j - \frac{1}{2} + \frac{(1-j)^2}{2h^2} + \varepsilon}\right)\right] dt$$
$$= \frac{\zeta(j + 1)}{j + 1}\mathcal{U}_{-j,h}(1)x^{j+1} + O\left(x^{j + \frac{1}{2} + \frac{(1-j)^2}{2h^2} + \varepsilon}\right).$$

For $j \geq 1$, the error term is $O\left(x^{j+\frac{1}{2}+\varepsilon}\right)$ instead.

Now we treat the divisor function. We find the corresponding generating function by following a computation similar to that in (16).

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{d(L_h(n))}{n^s} &= \prod_p\left(1+\frac{2}{p^s}+\cdots+\frac{h}{p^{(h-1)s}}+\frac{1}{p^{hs}}+\cdots\right)\\
&= \prod_p\left(\frac{p^{-hs}}{1-p^{-s}}+\frac{1-(h+1)p^{-hs}+hp^{-(h+1)s}}{(1-p^{-s})^2}\right)\\
&= \zeta(s)^2\prod_p\left(1-hp^{-hs}+(h-1)p^{-(h+1)s}\right)\\
&= \zeta(s)^2\mathcal{L}_{0,h}(s),
\end{aligned}
$$

where $\mathcal{L}_{0,h}(s)$ is convergent for $\mathrm{Re}(s)>\frac{1}{h}$. We remark that

$$
\frac{\mathcal{L}'_{0,h}(s)}{\mathcal{L}_{0,h}(s)}=\sum_p \log p\frac{h^2p^{-hs}-(h^2-1)p^{-(h+1)s}}{1-hp^{-hs}+(h-1)p^{-(h+1)s}}.
$$

We now apply Corollary 2.2. We have $\ell=2$, $\delta=1-\frac{1}{h}$. Since $h\geq 2$, we are in the first case and we get

$$
\sum_{n\leq x}d(L_h(n))=\frac{d}{ds}\left((s-1)^2\zeta(s)^2\mathcal{L}_{0,h}(s)\frac{x^s}{s}\right)\bigg|_{s=1}+O\left(x^{\frac{3}{4}+\varepsilon}\right).
$$

By following similar steps to the computation in (18), we obtain

$$
\frac{d}{ds}\left((s-1)^2\zeta(s)^2\mathcal{L}_{0,h}(s)\frac{x^s}{s}\right)\bigg|_{s=1}=\mathcal{L}_{0,h}(1)x\log x+\left[(2\gamma-1)\mathcal{L}_{0,h}(1)+\mathcal{L}'_{0,h}(1)\right]x.
$$

This finishes the proof of equation (13).

For equation (14), we follow a computation similar to (22) and obtain

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{d(U_h(n))}{n^s} &= \prod_p\left(1+\frac{1}{p^s}+\cdots+\frac{1}{p^{(h-1)s}}+\frac{h+1}{p^{hs}}+\frac{h+2}{p^{(h+1)s}}+\cdots\right)\\
&= \prod_p\left(\frac{1-p^{-hs}}{1-p^{-s}}+\frac{hp^{-hs}}{1-p^{-s}}+\frac{p^{-hs}}{(1-p^{-s})^2}\right)\\
&= \zeta(s)\prod_p\left(1+(h-1)p^{-hs}+\frac{p^{-hs}}{1-p^{-s}}\right)\\
&= \zeta(s)\mathcal{U}_{0,h}(s),
\end{aligned}
$$

where $\mathcal{U}_{0,h}(s)$ is convergent for $\mathrm{Re}(s)>\frac{1}{h}$.

Now apply Corollary 2.2, we have $\ell=1$, $\delta=1-\frac{1}{h}$. We are in the second case. Thus, we conclude

$$
\sum_{n\leq x}d(U_h(n))=\mathcal{U}_{0,h}(1)x+O\left(x^{\frac{1}{2}+\frac{1}{2h^2}+\varepsilon}\right). \qquad \square
$$

7 Conclusion

We have obtained asymptotics for the sums of general divisor functions over certain sequences with restricted factorization structure. The techniques exhibited here can be easily adapted to obtain similar results for other multiplicative functions, and to other contexts, such as function fields.

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