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Some multiple Dirichlet series of completely multiplicative arithmetic functions

Nabil Tahmi[1] and Abdallah Derbal[2]

[1] Department of Mathematics, ENS of Laghouat and EDPNLHM Laboratory,
ENS of Kouba, Algiers, Algeria
e-mail: nabil.tahmi@ens-lagh.dz

[2] Department of Mathematics, EDPNLHM Laboratory,
ENS of Kouba, Algiers, Algeria
e-mail: abderbal@yahoo.fr

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Abstract: Let $f_r : \mathbb{N}^r \longrightarrow \mathbb{C}$ be an arithmetic function of r variables, where $r \geq 2$. We study multiple Dirichlet series defined by

$$D(f_r, s_1, \ldots, s_r) = \sum_{\substack{n_1,\ldots,n_r=1 \\ (n_1,\ldots,n_r)=1}}^{+\infty} \frac{f_r(n_1, \ldots, n_r)}{n_1^{s_1} \cdots n_r^{s_r}},$$

where $f_r(n_1, \ldots, n_r) = f(n_1) \cdots f(n_r)$ and f is a completely multiplicative or a specially multiplicative arithmetic function of a single variable. We obtain formulas for these series expressed by infinite products over the primes. We also consider the cases of certain particular completely multiplicative and specially multiplicative functions.

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1 Introduction

A function $f : \mathbb{N} \longrightarrow \mathbb{C}$ is called specially multiplicative if f is the Dirichlet convolution of two completely multiplicative functions. For example, the function $\sigma_k = \mathbf{1} * id_k$, in particular, the divisor function $\tau = \mathbf{1} * \mathbf{1}$ and the sum of divisors function $\sigma = \mathbf{1} * id$ are specially multiplicative. Another example is the alternating sum of divisors function $\beta = \lambda * id$.

We define multiple Dirichlet series by

$$D(f_r, s_1, \dots, s_r) = \sum_{n_1,\dots,n_r=1}^{+\infty} \frac{f_r(n_1, \dots, n_r)}{n_1^{s_1} \cdots n_r^{s_r}},$$

where $f_r : \mathbb{N}^r \longrightarrow \mathbb{C}$ and $(s_1, ..., s_r)$ is in the region of absolute convergence for $D(f_r, s_1, ..., s_r)$. Werner Georg Nowak and L. Tòth [3], L. Tòth ([4–6]), L. Tòth and Wenguang Zhai [8] gave a representation of the multiple Dirichlet series of some special multiplicative arithmetic functions of r variables.

In the present paper, we establish formulas for the multiple Dirichlet series defined by

$$D(f_r, s_1, \dots, s_r) = \sum_{\substack{n_1,\dots,n_r=1 \\ (n_1,\dots,n_r)=1}}^{+\infty} \frac{f_r(n_1, \dots, n_r)}{n_1^{s_1} \cdots n_r^{s_r}},$$

where $f_r(n_1, \dots, n_r) = f(n_1) \cdots f(n_r)$ and f is a completely multiplicative or a specially multiplicative arithmetic function of a single variable. We also consider certain particular completely multiplicative and specially multiplicative functions. Our main results and their proofs are presented in the following sections.

2 Notations

- $\mathbb{N} := \{1, 2, \cdots\}$,
- $(n_1, \dots, n_r)$ is the Greatest Common Divisor of $n_1, \cdots, n_r \in \mathbb{N}$,
- $(f * g)(n) = \sum_{d|n} f(d)g(n/d)$ is the Dirichlet convolution of the functions $f, g : \mathbb{N} \longrightarrow \mathbb{C}$,
- $id(n) = n$, $id_k(n) = n^k (k \in \mathbb{R} \vee \mathbb{C}, n \in \mathbb{N})$,
- $\Omega(n)$ stands for the number of prime power divisors of n,
- $\sigma_k(n) = \sum_{d|n} d^k, \;\; k \in \mathbb{R}$,
- $\sigma(n) = \sigma_1(n)$ is the sum of divisors of n,
- $\tau(n) = \sigma_0(n)$ is the number of divisors of n,
- $\lambda(n) = (-1)^{\Omega(n)}$ is the Liouville function,
- ζ is the Riemann zeta-function,
- $L(s, \chi)$ is the Dirichlet L-function given by $L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$, where χ is the Dirichlet character,
- $D(f, s) = \sum_{n=1}^{+\infty} \frac{f(n)}{n^s}$ is the Dirichlet series,

- $\beta(n)$ is the number of integers x such that $1 \leq x \leq n$ and (x,n) is a square, $\beta(n)$ given by $\beta(n) = \sum\limits_{d|n} d\lambda(n/d)$.

3 Main results

Theorem 3.1. *Let f be a completely multiplicative arithmetic function and let $a \in \mathbb{R}$ be the abscissa of absolute convergence for the Dirichlet series of f. For every $s_1, s_2, \ldots, s_r$ in $D_a = \{s \in \mathbb{C} \ / \ \Re(s) > a\}$ with $r > 1$ such that $\sum\limits_{n=1}^{+\infty} \left|\frac{f(n)}{n^{s_i}}\right| < \infty,\ (i \in \{1, \ldots, r\})$, we have*

$$\sum_{\substack{n_1,\ldots,n_r=1\\(n_1,\ldots,n_r)=1}}^{+\infty} \frac{f(n_1)\cdots f(n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \prod_{i=1}^{r} D(f,s_i) \prod_p \left(1 - \frac{(f(p))^r}{p^{s_1+s_2+\cdots+s_r}}\right).$$

Applying this theorem on a completely multiplicative arithmetic functions, we obtain the following results:

Corollary 3.1. *Let $s_1, s_2, \ldots, s_r \in D_1$ $(r \geq 2)$ where $D_1 = \{s \in \mathbb{C} \ / \ \Re(s) > 1\}$. We have*

$$\sum_{\substack{n_1,\ldots,n_r=1\\(n_1,\ldots,n_r)=1}}^{+\infty} \frac{1}{n_1^{s_1}\cdots n_r^{s_r}} = \frac{\zeta(s_1)\zeta(s_2)\cdots\zeta(s_r)}{\zeta(s_1+s_2+\cdots+s_r)}, \tag{1}$$

where identity (1) is representing the Dirichlet series of the characteristic function of the set of points in $\mathbb{N}^r$, which are visible from the origin (cf.T.M.Apostol [1], page 248, Ex.15), (see formula (17) in [7]).

Corollary 3.2. *Let $s_1, s_2, \ldots, s_r \in D_1$. For any integer number $k \geq 1$ and χ a character modulo k, we have*

$$\sum_{\substack{n_1,\ldots,n_r=1\\(n_1,\ldots,n_r)=1}}^{+\infty} \frac{\chi(n_1)\cdots\chi(n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \frac{L(s_1,\chi)L(s_2,\chi)\cdots L(s_r,\chi)}{L(s_1+s_2+\cdots+s_r,\chi^r)}.$$

Corollary 3.3. *Let $s_1, s_2, \ldots, s_r \in D_1$. For $\chi = \chi_0$ the principal character to modulus k, we have*

$$\sum_{\substack{n_1,\ldots,n_r=1\\(n_1,\ldots,n_r)=1}}^{+\infty} \frac{\chi_0(n_1)\cdots\chi_0(n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \frac{\prod\limits_{m=1}^{r}\zeta(s_m)\prod\limits_{m=1}^{r}\prod\limits_{p|k}\left(1-\frac{1}{p^{s_m}}\right)}{\zeta(s_1+s_2+\cdots+s_r)\prod\limits_{p|k}\left(1-\frac{1}{p^{s_1+s_2+\cdots+s_r}}\right)}.$$

Corollary 3.4. *Let $s_1, s_2, \ldots, s_r \in D_{k+1} = \{s \in \mathbb{C} \ / \ \Re(s) > k+1\}$ and $k \in \mathbb{R} - \{0\}$. We have*

$$\sum_{\substack{n_1,\ldots,n_r=1\\(n_1,\ldots,n_r)=1}}^{+\infty} \frac{id_k(n_1)\cdots id_k(n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \frac{\zeta(s_1-k)\zeta(s_2-k)\cdots\zeta(s_r-k)}{\zeta(s_1+s_2+\cdots+s_r-kr)}.$$

Corollary 3.5. *Let $s_1, s_2, \ldots, s_r \in D_1$ and λ the Liouville lambda function.*

- *If r is even, we have*

$$\sum_{\substack{n_1,\ldots,n_r=1\\(n_1,\ldots,n_r)=1}}^{+\infty} \frac{\lambda(n_1)\cdots\lambda(n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \prod_{i=1}^{r} \frac{\zeta(2s_i)}{\zeta(s_i)} \times \frac{1}{\zeta(s_1+s_2+\cdots+s_r)}.$$

- *If r is odd, we have*

$$\sum_{\substack{n_1,\ldots,n_r=1\\(n_1,\ldots,n_r)=1}}^{+\infty} \frac{\lambda(n_1)\cdots\lambda(n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \prod_{i=1}^{r} \frac{\zeta(2s_i)}{\zeta(s_i)} \times \frac{\zeta(s_1+s_2+\cdots+s_r)}{\zeta(2(s_1+s_2+\cdots+s_r))}.$$

Corollary 3.6. *Given a finite or infinite set of primes $\mathbb{P}$, let $f_{\mathbb{P}}$ denote the characteristic function of the positive integers that do not have a prime divisor belonging to the set $\mathbb{P}$. Thus $f_{\mathbb{P}}$ is the completely multiplicative function defined by $f_{\mathbb{P}} = 1$ if $p \notin \mathbb{P}$ and $f_{\mathbb{P}} = 0$ if $p \in \mathbb{P}$. For all $s_1, s_2, \cdots, s_r \in D_1$, we have*

$$\sum_{\substack{n_1,\ldots,n_r=1\\(n_1,\ldots,n_r)=1}}^{+\infty} \frac{f_{\mathbb{P}}(n_1)\cdots f_{\mathbb{P}}(n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \frac{\zeta(s_1)\zeta(s_2)\ldots\zeta(s_r)}{\zeta(s_1+s_2+\cdots+s_r)} \prod_{i=1}^{r}\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^{s_i}}\right)\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^{s_1+s_2+\cdots+s_r}}\right)^{-1}.$$

Theorem 3.2. *Let g and h be two completely multiplicative functions and let $f = g * h$. Let a and b be the abscissae of absolute convergence for the Dirichlet series of g and h respectively. For every $s_1, s_2, \ldots, s_r$ in $D_{a,b} = \{s \in \mathbb{C} \ / \ \Re(s) > \max(a,b)\}$, $(a, b \in \mathbb{R})$ with $r \geq 2$ such that $\sum_{n=1}^{+\infty}\left|\frac{g(n)}{n^{s_i}}\right| < \infty$ and $\sum_{n=1}^{+\infty}\left|\frac{h(n)}{n^{s_i}}\right| < \infty,\ (i \in \{1,\ldots,r\})$, we have*

$$\sum_{\substack{n_1,\ldots,n_r=1\\(n_1,\ldots,n_r)=1}}^{+\infty} \frac{f(n_1)\cdots f(n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \prod_{i=1}^{r} D(g,s_i)D(h,s_i)\prod_{p} M_r(p),$$

and

$$M_r(p) = \sum_{k=1}^{r}(-1)^{k-1}\sum_{0\leq i_1<i_2<\cdots<i_k\leq r-1} K_{s_{r-i_1}}(p)K_{s_{r-i_2}}(p)\cdots K_{s_{r-i_k}}(p),$$

$$K_{s_{r_i}}(p) = \left(1-\frac{g(p)}{p^{s_{r_i}}}\right)\left(1-\frac{h(p)}{p^{s_{r_i}}}\right).$$

In particular, gives for $r = 2$,

$$\begin{aligned}\sum_{\substack{n_1,n_2=1\\(n_1,n_2)=1}}^{+\infty} \frac{f(n_1)f(n_2)}{n_1^{s_1}n_2^{s_2}} &= \prod_{i=1}^{2} D(g,s_i)D(h,s_i)\prod_{p} M_2(p)\\ &= \prod_{i=1}^{2} D(g,s_i)D(h,s_i)\prod_{p}\left(K_{s_2}(p)+K_{s_1}(p)-K_{s_2}(p)K_{s_1}(p)\right).\end{aligned} \tag{2}$$

In the case $r = 3$, we have

$$\sum_{\substack{n_1,n_2,n_3=1\\(n_1,n_2,n_3)=1}}^{+\infty} \frac{f(n_1)f(n_2)f(n_3)}{n_1^{s_1}n_2^{s_2}n_3^{s_3}} = \prod_{i=1}^{3} D(g,s_i)D(h,s_i)\prod_{p} M_3(p),$$

where

$$\prod_p M_3(p) = \prod_p (K_{s_3}(p) + K_{s_2}(p) + K_{s_1}(p) - K_{s_3}(p)K_{s_2}(p)$$
$$- K_{s_3}(p)K_{s_1}(p) - K_{s_2}(p)K_{s_1}(p) + K_{s_3}(p)K_{s_2}(p)K_{s_1}(p)).$$

Applying this theorem on a specially multiplicative arithmetic functions, we obtain the following results:

Corollary 3.7. *For every s_1, s_2 in $D_1 = \{s \in \mathbb{C} \ /Re\,(s) > 1\}$, we have*

$$\sum_{\substack{n_1,n_2=1\\(n_1,n_2)=1}}^{+\infty} \frac{\tau(n_1)\tau(n_2)}{n_1^{s_1}n_2^{s_2}} = \zeta^2(s_1)\zeta^2(s_2)\prod_p \left(1 - \frac{4}{p^{s_1+s_2}} + \frac{2}{p^{2s_1+s_2}} + \frac{2}{p^{s_1+2s_2}} - \frac{1}{p^{2s_1+2s_2}}\right).$$

Proof. Applying formula (2) for $f = \tau, g = h = \mathbf{1}$, gives

$$\sum_{\substack{n_1,n_2=1\\(n_1,n_2)=1}}^{+\infty} \frac{\tau(n_1)\tau(n_2)}{n_1^{s_1}n_2^{s_2}} = \prod_{i=1}^{2}\zeta^2(s_i)\prod_p \left(\left(1-\frac{1}{p^{s_2}}\right)^2 + \left(1-\frac{1}{p^{s_1}}\right)^2 - \left(1-\frac{1}{p^{s_2}}\right)^2\left(1-\frac{1}{p^{s_1}}\right)^2\right)$$
$$= \zeta^2(s_1)\zeta^2(s_2)\prod_p \left(1 - \frac{4}{p^{s_1+s_2}} + \frac{2}{p^{2s_1+s_2}} + \frac{2}{p^{s_1+2s_2}} - \frac{1}{p^{2s_1+2s_2}}\right). \qquad \square$$

Corollary 3.8. *For every s_1, s_2 in $D_{k+1} = \{s \in \mathbb{C} \ / \Re\,(s) > k+1\}$ $(k \geq 1)$, we have*

$$\sum_{\substack{n_1,n_2=1\\(n_1,n_2)=1}}^{+\infty} \frac{\sigma_k(n_1)\sigma_k(n_2)}{n_1^{s_1}n_2^{s_2}} = \prod_{i=1}^{2}\zeta(s_i)\zeta(s_i-k)$$
$$\prod_p \left(\frac{p^{2s_1+2s_2-2k} - p^{s_1+s_2-2k} + 2p^{s_1+s_2-k} - p^{s_1+s_2} + p^{s_2-k} + p^{s_1-k} + p^{s_2} + p^{s_1} - 1}{p^{2s_1+2s_2-2k}}\right).$$

In the case $k = 1$, the identity in corollary 3.8 reduces to

$$\sum_{\substack{n_1,n_2=1\\(n_1,n_2)=1}}^{+\infty} \frac{\sigma(n_1)\sigma(n_2)}{n_1^{s_1}n_2^{s_2}} = \prod_{i=1}^{2}\zeta(s_i)\zeta(s_i-1)$$
$$\prod_p \left(1 - \frac{1}{p^{s_1+s_2}} - \frac{2}{p^{s_1+s_2-1}} - \frac{1}{p^{s_1+s_2-2}} + \frac{1}{p^{2s_1+s_2-1}} + \frac{1}{p^{s_1+2s_2-1}} + \frac{1}{p^{2s_1+s_2-2}} + \frac{1}{p^{s_1+2s_2-2}} - \frac{1}{p^{2s_1+2s_2-2}}\right),$$

for $\mathcal{R}s_1, \mathcal{R}s_2 > 2$.

Corollary 3.9. *For every s_1, s_2 in $D_2 = \{s \in \mathbb{C} \ / \Re\,(s) > 2\}$, we have*

$$\sum_{\substack{n_1,n_2=1\\(n_1,n_2)=1}}^{+\infty} \frac{\beta(n_1)\beta(n_2)}{n_1^{s_1}n_2^{s_2}} = \prod_{i=1}^{2}\frac{\zeta(2s_i)\zeta(s_i-1)}{\zeta(s_i)}$$
$$\prod_p \left(1 - \frac{1}{p^{s_1+s_2}} + \frac{2}{p^{s_1+s_2-1}} - \frac{1}{p^{s_1+s_2-2}} + \frac{1}{p^{2s_1+s_2-1}} + \frac{1}{p^{s_1+2s_2-1}} - \frac{1}{p^{2s_1+s_2-2}} - \frac{1}{p^{s_1+2s_2-2}} - \frac{1}{p^{2s_1+2s_2-2}}\right).$$

4 Proof of Theorem 3.1

4.1 Preparing lemmas

Lemma 4.1. *For any completely multiplicative arithmetic function f such that $\sum_{n=1}^{+\infty} \left|\frac{f(n)}{n^s}\right| < \infty$ in a domain $D_a \subset \mathbb{C}$ and for all $s \in D_a$ and any $m \in \mathbb{N}$, we have*

$$\sum_{\substack{n=1 \\ (n,m)=1}}^{+\infty} \frac{f(n)}{n^s} = D(f,s) \prod_{p|m} \left(1 - \frac{f(p)}{p^s}\right).$$

Proof. Consider the set $\mathrm{A} = \{n \in \mathbb{N} \mid (n,m) = 1\}$, It follows that $p \in \mathrm{A}$ if, and only if, p does not divide m. Hence, we have

$$\sum_{\substack{n=1 \\ (n,m)=1}}^{+\infty} \frac{f(n)}{n^s} = \sum_{n \in \mathrm{A}}^{+\infty} \frac{f(n)}{n^s} = \prod_{p} \left(1 - \frac{f(p)}{p^s}\right)^{-1} \prod_{p|m} \left(1 - \frac{f(p)}{p^s}\right),$$

from which we obtain

$$\sum_{\substack{n=1 \\ (n,m)=1}}^{+\infty} \frac{f(n)}{n^s} = D(f,s) \prod_{p|m} \left(1 - \frac{f(p)}{p^s}\right).$$

□

Lemma 4.2. *Let f be a completely multiplicative arithmetic function and $m \in \mathbb{N}$.*

1. *For any prime number p and $u, v \in \mathbb{C}$, we have the following formula*

$$\left(1 - \frac{f(p)}{p^u}\right)\left(1 + \sum_{\alpha=1}^{+\infty} \frac{(f(p))^\alpha \left(1 - \frac{(f(p))^m}{p^v}\right)}{p^{\alpha u}}\right) = \left(1 - \frac{(f(p))^{m+1}}{p^{u+v}}\right).$$

$$1 + \sum_{\alpha=1}^{+\infty} \frac{(f(p))^\alpha \left(1 - \frac{(f(p))^m}{p^v}\right)}{p^{\alpha u}} = \left(1 - \frac{f(p)}{p^u}\right)^{-1} \left(1 - \frac{(f(p))^{m+1}}{p^{u+v}}\right)$$

2. *We suppose that $\sum_{n=1}^{+\infty} \left|\frac{f(n)}{n^s}\right| < \infty$ in a domain D for $\mathbb{C}$. then for every pair (u,v) in D_a^2, the following formula makes sense*

$$\sum_{n=1}^{+\infty} \frac{f(n)}{n^u} \prod_{p|n} \left(1 - \frac{(f(p))^m}{p^v}\right) = D(f,u) \prod_{p} \left(1 - \frac{(f(p))^{m+1}}{p^{u+v}}\right).$$

Proof. 1. First we write in the domain D_a

$$1 + \sum_{\alpha=1}^{+\infty} \frac{(f(p))^\alpha \left(1 - \frac{(f(p))^m}{p^v}\right)}{p^{\alpha u}} = 1 + \sum_{\alpha=1}^{+\infty} \frac{(f(p))^\alpha}{p^{\alpha u}} - \sum_{\alpha=1}^{+\infty} \frac{(f(p))^{\alpha+m}}{p^{\alpha u+v}}.$$

Then, taking into account the complete multiplicativity of the function f, the product development

$$\left(1 - \frac{f(p)}{p^u}\right)\left(1 + \sum_{\alpha=1}^{+\infty} \frac{(f(p))^\alpha}{p^{\alpha u}} - \sum_{\alpha=1}^{+\infty} \frac{(f(p))^{\alpha+m}}{p^{\alpha u+v}}\right),$$

gives the first formula from which we deduce the second.

2. The arithmetic function

$$n \mapsto h(n) = f\left(n\right) \prod_{p|n} \left(1 - \frac{\left(f\left(p\right)\right)^m}{p^v}\right) \ (n \in \mathbb{N})$$

is multiplicative. For all prime number p and any $\alpha \in \mathbb{N}$, we have $h\left(p^\alpha\right) = \left(f\left(p\right)\right)^\alpha \left(1 - \frac{(f(p))^m}{p^v}\right)$. One has

$$\sum_{n=1}^{+\infty} \frac{h\left(n\right)}{n^u} = \prod_p \left(1 + \sum_{\alpha=1}^{+\infty} \frac{\left(f\left(p\right)\right)^\alpha \left(1 - \frac{(f(p))^m}{p^v}\right)}{p^{\alpha u}}\right), .$$

According to the first assertion of the lemma, we have

$$1 + \sum_{\alpha=1}^{+\infty} \frac{\left(f\left(p\right)\right)^\alpha \left(1 - \frac{(f(p))^m}{p^v}\right)}{p^{\alpha u}} = \left(1 - \frac{f(p)}{p^u}\right)^{-1} \left(1 - \frac{\left(f\left(p\right)\right)^{m+1}}{p^{u+v}}\right),$$

then

$$\sum_{n=1}^{+\infty} \frac{h\left(n\right)}{n^u} = \prod_p \left(1 - \frac{f(p)}{p^u}\right)^{-1} \prod_p \left(1 - \frac{\left(f\left(p\right)\right)^{m+1}}{p^{u+v}}\right),$$

from which we infer that

$$\sum_{n=1}^{+\infty} \frac{f\left(n\right)}{n^u} \prod_{p|n} \left(1 - \frac{\left(f\left(p\right)\right)^m}{p^v}\right) = D\left(f, u\right) \prod_p \left(1 - \frac{\left(f\left(p\right)\right)^{m+1}}{p^{u+v}}\right). \qquad \square$$

Lemma 4.3. *For any completely multiplicative arithmetic function f and any $m, k \in \mathbb{N}$, we have*

1. *For all $s \in \mathbb{C}$, the arithmetic function $h\left(n\right)$ defined by*

$$n \mapsto h\left(n\right) = f(n) \prod_{p|(n,k)} \left(1 - \frac{\left(f\left(p\right)\right)^m}{p^s}\right) \ \left(n \in \mathbb{N}\right),$$

is multiplicative where

$$h\left(p^\alpha\right) = \begin{cases} f(p^\alpha)\left(1 - \frac{(f(p))^m}{p^s}\right) \text{ if } p \mid k \\ f\left(p^\alpha\right) \text{ if } p \nmid k \end{cases} \quad \left(p \text{ a prime number and } \alpha \in \mathbb{N}\right).$$

2. *We suppose that $\sum_{n=1}^{+\infty} \left|\frac{f(n)}{n^s}\right| < \infty$ in a domain D_a in $\mathbb{C}$. Then, for any pair $\left(u, v\right)$ in D_a^2, we have*

$$\sum_{n=1}^{+\infty} \frac{f\left(n\right)}{n^u} \prod_{p|(n,k)} \left(1 - \frac{\left(f\left(p\right)\right)^m}{p^v}\right) = D\left(f, u\right) \prod_{p|k} \left(1 - \frac{\left(f\left(p\right)\right)^{m+1}}{p^{u+v}}\right).$$

Proof. 1. Let α and β in $\mathbb{N}$ such that $\left(\alpha, \beta\right) = 1$, we have

$$p \mid \left(\alpha \times \beta, k\right) \Leftrightarrow p \mid \left(\alpha, k\right) \text{ or } p \mid \left(\beta, k\right).$$

It follows that $h\left(\alpha \times \beta\right) = h\left(\alpha\right) \times h\left(\beta\right)$. Let p a prime number and $\alpha \in \mathbb{N}$, the expression of $h\left(p^\alpha\right)$ gives the following relation

$$\prod_{p|(p^\alpha,k)} \left(1 - \frac{\left(f\left(p\right)\right)^m}{p^s}\right) = \begin{cases} \left(1 - \frac{(f(p))^m}{p^s}\right) \text{ if } p \mid k \\ 1 \text{ if } p \nmid k \end{cases}$$

2. We consider the multiplicative arithmetic function $h(n)$ of the first assertion

$$h\left(n\right)=f\left(n\right)\prod_{p|(n,k)}\left(1-\frac{\left(f\left(p\right)\right)^{m}}{p^{v}}\right)\ \left(n\in\mathbb{N}\right).$$

We have

$$\sum_{n=1}^{+\infty}\frac{h\left(n\right)}{n^{u}}=\prod_{p}\left(1+\sum_{\alpha=1}^{+\infty}\frac{h\left(p^{\alpha}\right)}{p^{\alpha u}}\right)=\mathrm{P}_{1}\times\mathrm{P}_{2},$$

or

$$\mathrm{P}_{1}=\prod_{p\nmid k}\left(1+\sum_{\alpha=1}^{+\infty}\frac{h\left(p^{\alpha}\right)}{p^{\alpha u}}\right)\text{ and }\mathrm{P}_{2}=\prod_{p|k}\left(1+\sum_{\alpha=1}^{+\infty}\frac{h\left(p^{\alpha}\right)}{p^{\alpha u}}\right).$$

Hence, we can thus write

$$\mathrm{P}_{1}=\prod_{p}\left(1+\sum_{\alpha=1}^{+\infty}\frac{f\left(p^{\alpha}\right)}{p^{\alpha u}}\right)\prod_{p|k}\left(1+\sum_{\alpha=1}^{+\infty}\frac{f\left(p^{\alpha}\right)}{p^{\alpha u}}\right)^{-1}=D\left(f,u\right)\prod_{p|k}\left(1-\frac{f\left(u\right)}{p^{u}}\right),$$

and

$$\mathrm{P}_{2}=\prod_{p|k}\left(1+\sum_{\alpha=1}^{+\infty}\frac{h\left(p^{\alpha}\right)}{p^{\alpha u}}\right)=\prod_{p|k}\left(1+\sum_{\alpha=1}^{+\infty}\frac{f(p^{\alpha})}{p^{\alpha u}}\left(1-\frac{\left(f\left(p\right)\right)^{m}}{p^{v}}\right)\right).$$

The first assertion of Lemma 4.2 implies that

$$\mathrm{P}_{2}=\prod_{p|k}\left(1-\frac{f(p)}{p^{u}}\right)^{-1}\left(1-\frac{\left(f\left(p\right)\right)^{m+1}}{p^{u+v}}\right),$$

we obtain

$$\sum_{n=1}^{+\infty}\frac{h\left(n\right)}{n^{u}}=\mathrm{P}_{1}\times\mathrm{P}_{2}=D\left(f,u\right)\prod_{p|k}\left(1-\frac{\left(f\left(p\right)\right)^{m+1}}{p^{u+v}}\right).\qquad\square$$

4.2 Proof of Theorem 3.1.

For $r=2$, we have

$$\Sigma\left(s_{1},s_{2}\right)=\sum_{\substack{n_{1}=1\\(n_{1},n_{2})=1}}^{+\infty}\sum_{n_{2}=1}^{+\infty}\frac{f\left(n_{1}\right)f\left(n_{2}\right)}{n_{1}^{s_{1}}n_{2}^{s_{2}}}=\sum_{n_{1}=1}^{+\infty}\frac{f\left(n_{1}\right)}{n_{1}^{s_{1}}}\sum_{\substack{n_{2}=1\\(n_{1},n_{2})=1}}^{+\infty}\frac{f\left(n_{2}\right)}{n_{2}^{s_{2}}}.$$

By Lemma 4.1, we obtain

$$\sum_{\substack{n_{2}=1\\(n_{1},n_{2})=1}}^{+\infty}\frac{f\left(n_{2}\right)}{n_{2}^{s_{2}}}=D\left(f,s_{2}\right)\prod_{p|n_{1}}\left(1-\frac{f(p)}{p^{s_{2}}}\right),$$

then

$$\Sigma\left(s_{1},s_{2}\right)=D\left(f,s_{2}\right)\sum_{n_{1}=1}^{+\infty}\frac{f\left(n_{1}\right)}{n_{1}^{s_{1}}}\prod_{p|n_{1}}\left(1-\frac{f(p)}{p^{s_{2}}}\right).$$

The second assertion of Lemma 4.2 implies

$$\sum_{n_1=1}^{+\infty} \frac{f(n_1)}{n_1^{s_1}} \prod_{p|n_1} \left(1 - \frac{f(p)}{p^{s_2}}\right) = D(f, s_1) \prod_p \left(1 - \frac{(f(p))^2}{p^{s_1+s_2}}\right).$$

It follows that

$$\Sigma(s_1, s_2) = D(f, s_1) D(f, s_2) \prod_p \left(1 - \frac{(f(p))^2}{p^{s_1+s_2}}\right).$$

Assuming $r \geq 3$ and writing that

$$\Sigma_r = \Sigma(s_1, s_2, \ldots, s_r) = \sum_{\substack{n_1=1 \\ (n_1, n_2, \ldots, n_r)=1}}^{+\infty} \sum_{n_2=1}^{+\infty} \cdots \sum_{n_r=1}^{+\infty} \frac{f(n_1) f(n_2) \cdots f(n_r)}{n_1^{s_1} n_2^{s_2} \cdots n_r^{s_r}}.$$

We have

$$\Sigma_r = \sum_{n_1=1}^{+\infty} \frac{f(n_1)}{n_1^{s_1}} \sum_{n_2=1}^{+\infty} \frac{f(n_2)}{n_2^{s_2}} \cdots \sum_{n_{r-1}=1}^{+\infty} \frac{f(n_{r-1})}{n_{r-1}^{s_{r-1}}} \sum_{\substack{n_r=1 \\ (n_r, (n_1, n_2, \ldots, n_{r-1}))=1}}^{+\infty} \frac{f(n_r)}{n_r^{s_r}}.$$

The application of Lemma 4.1 with ($n = n_r$ and $m = (n_1, n_2, \ldots, n_{r-1})$) leads to

$$\sum_{\substack{n_r=1 \\ (n_r, (n_1, n_2, \ldots, n_{r-1}))=1}}^{+\infty} \frac{f(n_r)}{n_r^{s_r}} = D(f, s_r) \prod_{p|(n_1, n_2, \ldots, n_{r-1})} \left(1 - \frac{f(p)}{p^{s_r}}\right),$$

we can easily get

$$\Sigma_r = D(f, s_r) \sum_{n_1=1}^{+\infty} \frac{f(n_1)}{n_1^{s_1}} \sum_{n_2=1}^{+\infty} \frac{f(n_2)}{n_2^{s_2}} \cdots \sum_{n_{r-2}=1}^{+\infty} \frac{f(n_{r-2})}{n_{r-2}^{s_{r-2}}} \sum_{n_{r-1}=1}^{+\infty} \frac{h(n_{r-1})}{n_{r-1}^{s_{r-1}}},$$

or

$$\sum_{n_{r-1}=1}^{+\infty} \frac{h(n_{r-1})}{n_{r-1}^{s_{r-1}}} = \sum_{n_{r-1}=1}^{+\infty} \frac{f(n_{r-1})}{n_{r-1}^{s_{r-1}}} \prod_{p|(n_1, n_2, \ldots, n_{r-1})} \left(1 - \frac{f(p)}{p^{s_r}}\right).$$

After $r - 1$ applications of the second assertion of Lemma 4.3 to sums

$$\sum_{n_{r-k}=1}^{+\infty} \frac{h(n_{r-k})}{n_{r-k}^{s_{r-k}}} = \sum_{n_{r-k}=1}^{+\infty} \frac{f(n_{r-k})}{n_{r-k}^{s_{r-k}}} \prod_{p|(n_{r-k}, (n_1, n_2, \ldots, n_{r-k-1}))} \left(1 - \frac{(f(p))^k}{p^{s_{r-k+1}+\cdots+s_r}}\right),$$

$(k = 1, \ldots, r-1)$, which appear in the expression of Σ_r, we obtain the formula

$$\Sigma_r = D(f, s_r) D(f, s_{r-1}) \cdots D(f, s_2) \sum_{n_1=1}^{+\infty} \frac{f(n_1)}{n_1^{s_1}} \prod_{p|n_1} \left(1 - \frac{(f(p))^{r-1}}{p^{s_2+\cdots+s_r}}\right).$$

The second assertion of Lemma 4.2 implies

$$\sum_{n_1=1}^{+\infty} \frac{f(n_1)}{n_1^{s_1}} \prod_{p|n_1} \left(1 - \frac{(f(p))^{r-1}}{p^{s_2+\cdots+s_r}}\right) = D(f, s_1) \prod_p \left(1 - \frac{(f(p))^r}{p^{s_1+s_2+\cdots+s_r}}\right).$$

This completes the proof of the Theorem 3.1. □

5 Proof of Theorem 3.2

We need the following result.

5.1 Preparing lemmas

Lemma 5.1. [2] *Let r be an integer. If $\mathcal{F}(s) = \sum\limits_{\substack{n=1 \\ (n,r)=1}}^{+\infty} \frac{f(n)}{n^s}$ and $\mathcal{G}(s) = \sum\limits_{\substack{n=1 \\ (n,r)=1}}^{+\infty} \frac{g(n)}{n^s}$, where the series converge absolutely for $\mathcal{R}(s) > a$, then*

$$\mathcal{F}(s)\mathcal{G}(s) = \sum_{\substack{n=1 \\ (n,r)=1}}^{+\infty} \frac{(f * g)(n)}{n^s}, \text{ for } \mathcal{R}(s) > a.$$

Proof. For any s for which both series converge absolutely we have

$$\mathcal{F}(s)\mathcal{G}(s) = \sum_{\substack{n=1 \\ (n,r)=1}}^{+\infty} \frac{f(n)}{n^s} \sum_{\substack{m=1 \\ (m,r)=1}}^{+\infty} \frac{g(m)}{m^s} = \sum_{\substack{n,m=1 \\ (nm,r)=1}}^{+\infty} \frac{f(n)g(m)}{(nm)^s},$$

Because of absolute convergence we can multiply these series together and rearrange the terms is any way we please without altering the sum, hence

$$\mathcal{F}(s)\mathcal{G}(s) = \sum_{\substack{k=1 \\ (k,r)=1}}^{+\infty} \frac{1}{k^s} \sum_{mn=k} f(n)g(m) = \sum_{\substack{k=1 \\ (k,r)=1}}^{+\infty} \frac{h(k)}{k^s},$$

where $h(k) = \sum\limits_{mn=k} f(n)g(m) = (f * g)(k)$.
This completes the proof of Lemma 5.1. □

Lemma 5.2. *Let g and h be two completely multiplicative functions and let $f = g * h$ such that $\sum\limits_{n=1}^{+\infty} \left|\frac{g(n)}{n^s}\right| < \infty$ and $\sum\limits_{n=1}^{+\infty} \left|\frac{h(n)}{n^s}\right| < \infty$ for every s in a domain $D_a \subset \mathbb{C}$. For all $s \in D_a$ and any $m \in \mathbb{N}$, we have*

$$\begin{aligned}\sum_{\substack{n=1 \\ (n,m)=1}}^{+\infty} \frac{f(n)}{n^s} &= D(g,s)\, D(h,s) \prod_{p|m} \left(1 - \frac{g(p)}{p^s}\right)\left(1 - \frac{h(p)}{p^s}\right) \\ &= D(f,s)\, K_s(p).\end{aligned}$$

Proof. This is a direct consequence of Lemma 4.1 and Lemma 5.1. □

Lemma 5.3. *Let g and h be two completely multiplicative functions and let $f = g * h$ such that $\sum\limits_{n=1}^{+\infty} \left|\frac{g(n)}{n^s}\right| < \infty$ and $\sum\limits_{n=1}^{+\infty} \left|\frac{h(n)}{n^s}\right| < \infty$ for every s in a domain $D_a \subset \mathbb{C}$.*

1. *For all $s_{r-j} \in D_a$, $(j = 1, \ldots, r)$ and any $r \in \mathbb{N}$, we have the following formula*

$$\sum_{n=1}^{+\infty} \frac{f(n)}{n^{s_{r-j}}} \prod_{p|n} M_j(p) = D(f, s_{r-j}) \prod_{p} M_{j+1}(p), \quad j = 1, \ldots, r$$

2. *For all* $s_{r-j}\in D_a$, $(j=1,\ldots,r)$ *and any* $r,m\in\mathbb{N}$, *we have*

$$\sum_{n=1}^{+\infty}\frac{f(n)}{n^{s_{r-j}}}\prod_{p|(n,m)}M_j(p)=D(f,s_{r-j})\prod_{p|m}M_{j+1}(p),\qquad j=1,\ldots,r,$$

where $M_j(p)=\sum_{k=1}^{j}(-1)^{k-1}\sigma_{(k,j)}(p)$, *and*

$$\sigma_{(k,j)}(p)=\sum_{0\le i_1<i_2<\cdots<i_k\le j-1}K_{s_{r-i_1}}(p)K_{s_{r-i_2}}(p)\cdots K_{s_{r-i_k}}(p),$$

for all $j=1,\ldots,r$.

Proof. 1. The arithmetic function

$$n\mapsto g(n)=f(n)\prod_{p|n}M_j(p),\ (n\in\mathbb{N})$$

is multiplicative. For all prime number p and any $\alpha\in\mathbb{N}$, we have $g(p^\alpha)=(f(p))^\alpha M_j(p)$. One has

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{g(n)}{n^{s_{r-j}}}&=\prod_p\left(1+\sum_{\alpha=1}^{+\infty}\frac{(f(p))^\alpha M_j(p)}{p^{\alpha s_{r-j}}}\right)\\&=\prod_p\left(1+\sum_{k=1}^{j}(-1)^{k-1}\sigma_{(k,j)}(p)\sum_{\alpha=1}^{+\infty}\frac{(f(p))^\alpha}{p^{\alpha s_{r-j}}}\right).\end{aligned}$$

where

$$\sum_{\alpha=1}^{+\infty}\frac{(f(p))^\alpha}{p^{\alpha s_{r-j}}}=\frac{1}{K_{s_{r-j}}(p)}-1.$$

This gives

$$\begin{aligned}&1+\sum_{k=1}^{j}(-1)^{k-1}\sigma_{(k,j)}(p)\sum_{\alpha=1}^{+\infty}\frac{(f(p))^\alpha}{p^{\alpha s_{r-j}}}\\&=1+\sigma_{(1,j)}(p)\left(\frac{1}{K_{s_{r-j}}(p)}-1\right)-\sigma_{(2,j)}(p)\left(\frac{1}{K_{s_{r-j}}(p)}-1\right)\\&\quad+\cdots+(-1)^{j-1}\sigma_{(j,j)}(p)\left(\frac{1}{K_{s_{r-j}}(p)}-1\right),\end{aligned}$$

then

$$\begin{aligned}&K_{s_{r-j}}(p)\left(1+\sum_{k=1}^{j}(-1)^{k-1}\sigma_{(k,j)}(p)\sum_{\alpha=1}^{+\infty}\frac{(f(p))^\alpha}{p^{\alpha s_{r-j}}}\right)\\&=K_{s_{r-j}}(p)+\sigma_{(1,j)}(p)\left(1-K_{s_{r-j}}(p)\right)-\sigma_{(2,j)}(p)\left(1-K_{s_{r-j}}(p)\right)\\&\quad+\cdots+(-1)^{j-1}\sigma_{(j,j)}(p)\left(1-K_{s_{r-j}}(p)\right),\end{aligned}$$

from which we obtain

$$
\begin{aligned}
&K_{s_{r-j}}(p)\left(1+\sum_{k=1}^{j}(-1)^{k-1}\sigma_{(k,j)}(p)\sum_{\alpha=1}^{+\infty}\frac{(f\left(p\right))^{\alpha}}{p^{\alpha s_{r-j}}}\right)\\
&=\left(K_{s_{r-j}}(p)+\sigma_{(1,j)}(p)\right)-\left(\sigma_{(1,j)}(p)K_{s_{r-j}}(p)-\sigma_{(2,j)}(p)\right)\\
&\quad+\cdots+(-)^{j-1}\left(\sigma_{(j-1,j)}(p)K_{s_{r-j}}(p)+\sigma_{(j,j)}(p)\right)+(-1)^{j}\sigma_{(j,j)}(p)K_{s_{r-j}}(p).
\end{aligned}
$$

We can thus write

$$
\begin{aligned}
&K_{s_{r-j}}(p)\left(1+\sum_{k=1}^{j}(-1)^{k-1}\sigma_{(k,j)}(p)\sum_{\alpha=1}^{+\infty}\frac{(f\left(p\right))^{\alpha}}{p^{\alpha s_{r-j}}}\right)\\
&=\sigma_{(1,j+1)}(p)-\sigma_{(2,j+1)}(p)+\cdots+(-)^{j-1}\left(\sigma_{(j,j+1)}(p)\right)+(-1)^{j}\sigma_{(j+1,j+1)}(p)=M_{j+1}(p).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\sum_{n=1}^{+\infty}\frac{f\left(n\right)}{n^{s_{r-j}}}\prod_{p|n}M_{j}(p)&=\sum_{n=1}^{+\infty}\frac{g\left(n\right)}{n^{s_{r-j}}}\\
&=\prod_{p}\left(1+\sum_{k=1}^{j}(-1)^{k-1}\sigma_{(k,j)}(p)\sum_{\alpha=1}^{+\infty}\frac{(f\left(p\right))^{\alpha}}{p^{\alpha s_{r-j}}}\right)\\
&=\prod_{p}\left(K_{s_{r-j}}^{-1}(p)M_{j+1}(p)\right)\\
&=D(f,s_{r-j})\prod_{p}M_{j+1}(p).
\end{aligned}
$$

2. According to the first assertion of the Lemma 4.3, the arithmetic function

$$n\mapsto h(n)=f\left(n\right)\prod_{p|(n,m)}M_{j}(p),\ (n\in\mathbb{N})$$

is multiplicative. For all prime number p and any $\alpha\in\mathbb{N}$, we have

$$\sum_{n=1}^{+\infty}\frac{h\left(n\right)}{n^{n^{s_{r-j}}}}=\prod_{p}\left(1+\sum_{\alpha=1}^{+\infty}\frac{h\left(p^{\alpha}\right)}{p^{\alpha s_{r-j}}}\right)=\mathrm{P}_1\times\mathrm{P}_2,$$

or

$$\mathrm{P}_1=\prod_{p\nmid m}\left(1+\sum_{\alpha=1}^{+\infty}\frac{h\left(p^{\alpha}\right)}{p^{\alpha s_{r-j}}}\right)\ \text{and}\ \mathrm{P}_2=\prod_{p|m}\left(1+\sum_{\alpha=1}^{+\infty}\frac{h\left(p^{\alpha}\right)}{p^{\alpha s_{r-j}}}\right).$$

Hence, we can thus write

$$\mathrm{P}_1=\prod_{p}\left(1+\sum_{\alpha=1}^{+\infty}\frac{f\left(p^{\alpha}\right)}{p^{\alpha s_{r-j}}}\right)\prod_{p|m}\left(1+\sum_{\alpha=1}^{+\infty}\frac{f\left(p^{\alpha}\right)}{p^{\alpha s_{r-j}}}\right)^{-1}=D\left(f,s_{r-j}\right)\prod_{p|m}K_{s_{r-j}}(p),$$

and

$$\mathrm{P}_2=\prod_{p|m}\left(1+\sum_{\alpha=1}^{+\infty}\frac{h\left(p^{\alpha}\right)}{p^{\alpha s_{r-j}}}\right)=\prod_{p|m}\left(1+M_{j}(p)\sum_{\alpha=1}^{+\infty}\frac{f(p^{\alpha})}{p^{\alpha s_{r-j}}}\right).$$

The first assertion of the lemma implies that

$$\mathrm{P}_2 = \prod_{p|m} \left(K_{s_{r-j}}^{-1}(p) M_{j+1}(p) \right),$$

we obtain

$$\sum_{n=1}^{+\infty} \frac{h(n)}{n^{s_{r-j}}} = \mathrm{P}_1 \times \mathrm{P}_2 = D(f, s_{r-j}) \prod_{p|m} M_{j+1}(p).$$

□

5.2 Proof of Theorem 3.2

For $r = 2$, we have

$$\Sigma(s_1, s_2) = \sum_{\substack{n_1=1 \\ (n_1,n_2)=1}}^{+\infty} \sum_{n_2=1}^{+\infty} \frac{f(n_1) f(n_2)}{n_1^{s_1} n_2^{s_2}} = \sum_{n_1=1}^{+\infty} \frac{f(n_1)}{n_1^{s_1}} \sum_{\substack{n_2=1 \\ (n_1,n_2)=1}}^{+\infty} \frac{f(n_2)}{n_2^{s_2}}.$$

According to Lemma 5.2, we have

$$\sum_{\substack{n_2=1 \\ (n_1,n_2)=1}}^{+\infty} \frac{f(n_2)}{n_2^{s_2}} = D(f, s_2) \prod_{p|n_1} M_1(p).$$

where

$$M_1(p) = K_{s_2}(p) = \left(1 - \frac{g(p)}{p^{s_2}}\right)\left(1 - \frac{h(p)}{p^{s_2}}\right),$$

The first assertion of Lemma 5.3 implies

$$\sum_{n_1=1}^{+\infty} \frac{f(n_1)}{n_1^{s_1}} \prod_{p|n_1} M_1(p) = D(f, s_1) \prod_{p} M_2(p),$$

It follows that

$$\begin{aligned} \Sigma(s_1, s_2) &= D(f, s_1) D(f, s_2) \prod_{p} M_2(p) \\ &= D(f, s_1) D(f, s_2) \prod_{p} \left(K_{s_2}(p) + K_{s_1}(p) - K_{s_2}(p) K_{s_1}(p)\right). \end{aligned}$$

Assuming $r \geq 3$ and writing that

$$\Sigma_r = \Sigma(s_1, s_2, \ldots, s_r) = \sum_{\substack{n_1=1 \\ (n_1,n_2,\ldots,n_r)=1}}^{+\infty} \sum_{n_2=1}^{+\infty} \cdots \sum_{n_r=1}^{+\infty} \frac{f(n_1) f(n_2) \cdots f(n_r)}{n_1^{s_1} n_2^{s_2} \cdots n_r^{s_r}}.$$

We have

$$\Sigma_r = \sum_{n_1=1}^{+\infty} \frac{f(n_1)}{n_1^{s_1}} \sum_{n_2=1}^{+\infty} \frac{f(n_2)}{n_2^{s_2}} \cdots \sum_{n_{r-1}=1}^{+\infty} \frac{f(n_{r-1})}{n_{r-1}^{s_{r-1}}} \sum_{\substack{n_r=1 \\ (n_r,(n_1,n_2,\ldots,n_{r-1}))=1}}^{+\infty} \frac{f(n_r)}{n_r^{s_r}}.$$

The application of Lemma 5.2 with ($n = n_r$ and $m = (n_1, n_2, \ldots, n_{r-1})$) we obtain

$$\sum_{\substack{n_r=1 \\ (n_r,(n_1,n_2,\ldots,n_{r-1}))=1}}^{+\infty} \frac{f(n_r)}{n_r^{s_r}} = D(f, s_r) \prod_{p|(n_1,n_2,\ldots,n_{r-1})} M_1(p)$$

we can easily get

$$\Sigma_r = D(f, s_r) \sum_{n_1=1}^{+\infty} \frac{f(n_1)}{n_1^{s_1}} \sum_{n_2=1}^{+\infty} \frac{f(n_2)}{n_2^{s_2}} \cdots \sum_{n_{r-2}=1}^{+\infty} \frac{f(n_{r-2})}{n_{r-2}^{s_{r-2}}} \sum_{n_{r-1}=1}^{+\infty} \frac{h(n_{r-1})}{n_{r-1}^{s_{r-1}}}$$

or

$$\sum_{n_{r-1}=1}^{+\infty} \frac{h(n_{r-1})}{n_{r-1}^{s_{r-1}}} = \sum_{n_{r-1}=1}^{+\infty} \frac{f(n_{r-1})}{n_{r-1}^{s_{r-1}}} \prod_{p|(n_1,n_2,\ldots,n_{r-1})} M_1(p),$$

After $r-1$ applications of the second assertion of Lemma 5.3 to sums

$$\sum_{n_{r-k}=1}^{+\infty} \frac{h(n_{r-k})}{n_{r-k}^{s_{r-k}}} = \sum_{n_{r-k}=1}^{+\infty} \frac{f(n_{r-k})}{n_{r-k}^{s_{r-k}}} \prod_{p|(n_{r-k},(n_1,n_2,\ldots,n_{r-k-1}))} M_k(p),$$

which appear in the expression of Σ_r, we obtain the formula

$$\Sigma_r = D(f, s_r)\, D(f, s_{r-1}) \cdots D(f, s_2) \sum_{n_1=1}^{+\infty} \frac{f(n_1)}{n_1^{s_1}} \prod_{p|n_1} M_{r-1}(p).$$

The first assertion of Lemma 5.3 implies

$$\sum_{n_1=1}^{+\infty} \frac{f(n_1)}{n_1^{s_1}} \prod_{p|n_1} M_{r-1}(p) = D(f, s_1) \prod_{p} M_r(p).$$

This completes the proof of the Theorem 3.2.

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