

# On a new class of the generalized Gauss $k$ -Pell numbers and their polynomials

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**Abstract:** In this article, we generalize the well-known Gauss Pell numbers and refer to them as generalized Gauss  $k$ -Pell numbers. There are relationships discovered between the class of generalized Gauss  $k$ -Pell numbers and the typical Gauss Pell numbers. Also, we generalize the known Gauss Pell polynomials, and call such polynomials as the generalized Gauss  $k$ -Pell polynomials. We obtain relations between the class of the generalized Gauss  $k$ -Pell polynomials and the typical Gauss Pell polynomials. Furthermore, we provide matrices for the novel generalizations of these numbers and polynomials. After that, we obtain Cassini's identities for these numbers and polynomials.

**Keywords:** Gauss Pell numbers, Gauss Pell polynomials, Gauss Fibonacci numbers, Gauss Fibonacci polynomials, Cassini's identity.

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## 1 Introduction

Fibonacci and Lucas numbers have received much interest in recent years and are studied by a wide range of researchers in a variety of branches of mathematics, including linear algebra, applied mathematics, and calculus. Furthermore, the study of Gaussian numbers is a very

interesting academic field, and various research have been done in this area. In 1963, Horadam [6] introduced complex Fibonacci numbers and named them Gaussian Fibonacci numbers. In 1965, Jordan [7] investigated two complex number sequences and developed some of the characteristics associated with usual Fibonacci sequences. Several identities are given as the relation between the typical Fibonacci and Gaussian Fibonacci sequences [5]. Gaussian Fibonacci numbers and Lucas numbers are defined by  $GF_{n+1} = GF_n + GF_{n-1}$ , where  $GF_0 = i, GF_1 = 1$  and  $GL_{n+1} = GL_n + GL_{n-1}$ , where  $GL_0 = 2 - i, GL_1 = 1 + 2i$ , respectively [7]. Then the closed form of Gaussian Fibonacci numbers are defined by Berzsenyi [3] as Fibonacci Q matrix. Also, different concept of Fibonacci and Lucas numbers and their polynomials are studied by various authors [1–5, 8, 12]. A new class of  $k$ -Gaussian Fibonacci numbers is given by Taş [10] and a new family of Gauss  $k$ -Fibonacci polynomials are defined by Taştan and Özkan [11].

Now, we give the structure of the paper. In Section 2, we introduce several well-known definitions and characteristics. In Section 3.1, we introduce a new class of the generalized Gauss  $k$ -Pell numbers. These numbers are a generalization of the Gauss Pell numbers in [5]. We give relations between the generalized Gauss  $k$ -Pell numbers and the Gauss Pell numbers. Also, we determine the matrix for new generalization of these numbers. Then we demonstrate Cassini's identity for these numbers.

In Section 3.2, we define a new class of the generalized Gauss  $k$ -Pell polynomials. These polynomials are a generalization of the Gauss Pell polynomials in [12]. We give relations between the generalized Gauss  $k$ -Pell polynomials and the Gauss Pell polynomials. Moreover, we determine the matrix for new generalization of these polynomials. Then we prove Cassini's identity for these polynomials. In Section 4, we conclude the paper.

## 2 Materials and methods

In this section, we provide the Gauss Fibonacci numbers  $GF_n$ , the Gauss Fibonacci polynomials  $GF_n(x)$ , Gauss Pell numbers  $GP_n$ , the Gauss Pell polynomials  $GP_n(x)$ , and the Gauss Pell polynomial matrix  $gp_n(x)$  in this section.

**Definition 2.1.** *The Gaussian Fibonacci numbers  $\{GF_n\}_{n=0}^{\infty}$  are determined by:*

$$GF_{n+1} = GF_n + GF_{n-1}, \quad n \geq 1 \quad (1)$$

with initial conditions  $GF_0 = i$  and  $GF_1 = 1$  [3].

The following is the Binet formulas for  $GF_n$ :

$$GF_n = \left( \frac{\alpha^n + \beta^n}{\alpha - \beta} \right) + i \left( \frac{\alpha^{n-1} + \beta^{n-1}}{\alpha - \beta} \right), \quad (2)$$

where  $\alpha = \frac{1+\sqrt{5}}{2}$  and  $\frac{1-\sqrt{5}}{2}$  [6].

The Cassini's identity for the Gauss Fibonacci numbers are given as follows [6]:

$$GF_{n+1}GF_{n-1} - GF_n^2 = (-1)^n (2 - i), \quad n \geq 1. \quad (3)$$

**Definition 2.2.** The Gaussian Fibonacci polynomials  $\{GF_n(x)\}_{n=0}^{\infty}$  are determined by

$$GF_{n+1}(x) = xGF_n(x) + GF_{n-1}(x), n \geq 2 \quad (4)$$

with initial conditions  $GF_1(x) = 1$  and  $GF_2 = x + i$  [9].

The following is the Binet formulas for  $GF_n(x)$ :

$$GF_n(x) = \left( \frac{\alpha^n(x) + \beta^n(x)}{\alpha(x) - \beta(x)} \right) + i \left( \frac{\alpha^{n-1}(x) + \beta^{n-1}(x)}{\alpha(x) - \beta(x)} \right), \quad (5)$$

where  $\alpha(x) = \frac{x + \sqrt{x^2 + 4}}{2}$  and  $\beta(x) = \frac{x - \sqrt{x^2 + 4}}{2}$  [9].

The Cassini's identity [11] for the Gauss Fibonacci polynomials are given as follows:

$$GF_{n+1}(x)GF_{n-1}(x) - GF_n^2(x) = (-1)^n(2 - xi), n \geq 1. \quad (6)$$

In [11], the Gauss Fibonacci polynomial matrix  $gf_n(x)$  is given by

$$gf_n(x) = \begin{bmatrix} GF_{n+1}(x) & GF_n(x) \\ GF_n(x) & GF_{n-1}(x) \end{bmatrix}, n \geq 0.$$

**Definition 2.3.** The Gaussian Pell numbers  $\{GP_n\}_{n=0}^{\infty}$  are determined by

$$GP_{n+1} = 2GP_n + GP_{n-1}, n \geq 1 \quad (7)$$

with initial conditions  $GP_0 = i$  and  $GP_1 = 1$  [5].

The following is the Binet formulas for  $GP_n$ :

$$GP_n = \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right) + i \left( \frac{\alpha\beta^n - \beta\alpha^n}{\alpha - \beta} \right), \quad (8)$$

where  $\alpha = 1 + \sqrt{2}$  and  $\beta = 1 - \sqrt{2}$  [5].

The Cassini's identity [5] for the Gauss Pell numbers are given as follows:

$$GP_{n+1}GP_{n-1} - GP_n^2 = (-1)^n 2(1 - i), n \geq 1. \quad (9)$$

**Definition 2.4.** The Gaussian Pell polynomials  $\{GP_n(x)\}_{n=0}^{\infty}$  are determined by

$$GP_{n+1}(x) = 2xGP_n(x) + GP_{n-1}(x), n \geq 1 \quad (10)$$

with initial conditions  $GP_0(x) = i$  and  $GP_1 = 1$  [4].

The following is the Binet formulas for  $GP_n(x)$ :

$$GP_n(x) = \left( \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \right) + i \left( \frac{\alpha(x)\beta^n(x) - \beta(x)\alpha^n(x)}{\alpha(x) - \beta(x)} \right), \quad (11)$$

where  $\alpha(x) = x + \sqrt{1 + x^2}$  and  $\beta(x) = x - \sqrt{1 + x^2}$  [4].

The Cassini's identity [4] for the Gauss Pell polynomials are given as follows:

$$GP_{n+1}(x)GP_{n-1}(x) - GP_n^2(x) = 2(-1)^n(1 - xi), n \geq 1. \quad (12)$$

In [4], The Gauss Pell polynomial matrix  $gp_n(x)$  is given by

$$gp_n(x) = Q^n(x)P(x) = \begin{bmatrix} GP_{n+2}(x) & GP_{n+1}(x) \\ GP_{n+1}(x) & GP_n(x) \end{bmatrix}, n \geq 1.$$

where  $GP_n(x)$  is the  $n$ th Gaussian Pell polynomial,  $Q(x) = \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}$  and  $P(x) = \begin{bmatrix} 2x+i & 1 \\ 1 & i \end{bmatrix}$  are special matrices.

### 3 Main results

#### 3.1 The generalized Gauss $k$ -Pell numbers

**Definition 3.1.** Let  $m$  and  $r$  be unique numbers such that  $n = mk + r$  and  $0 \leq r < k$ , for  $n \in \mathbb{N}$  and  $m, k \in \mathbb{Z}^+$ . Then generalized Gauss  $k$ -Pell numbers  $GP_n^{(k)}$  is defined by

$$GP_n^{(k)} := \left[ \left( \frac{\alpha^m - \beta^m}{\alpha - \beta} \right) + i \left( \frac{\alpha\beta^m - \beta\alpha^m}{\alpha - \beta} \right) \right]^{k-r} \left[ \left( \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta} \right) + i \left( \frac{\alpha\beta^{m+1} - \beta\alpha^{m+1}}{\alpha - \beta} \right) \right]^r, n = mk + r,$$

where  $\alpha = 1 + \sqrt{2}$  and  $\beta = 1 - \sqrt{2}$ .

Furthermore, using the matrix methods, we can derive the generalized Gauss  $k$ -Pell number. Clearly, it can be said that

$$GP_n^{k-1} gp_n = \begin{bmatrix} GP_{kn+1}^{(k)} & GP_{kn}^{(k)} \\ GP_{kn}^{(k)} & GP_{kn-1}^{(k)} \end{bmatrix},$$

where  $n > 0$  and

$$gp_n = \begin{bmatrix} GP_{n+1} & GP_n \\ GP_n & GP_{n-1} \end{bmatrix}.$$

From Eq. (8) and Definition 3.1, we get the following relationship between the generalized Gauss  $k$ -Pell numbers and the Gauss Pell numbers:

$$GP_n^{(k)} := (GP_m)^{k-r} (GP_{m+1})^r, n = mk + r. \quad (13)$$

If we take  $k = 1$  in Eq. (13), then we get that  $m = n$  and  $r = 0$ , so  $GP_n^{(1)} = GP_n$ .

For  $k = 2, 3, 4, \dots, n$ , we get the following relations between the generalized Gauss  $k$ -Pell numbers and the Gauss Pell numbers by Eq. (7) and Eq. (13):

1.  $GP_{2n}^{(2)} = GP_n^2$ ,
2.  $GP_{2n+1}^{(2)} = GP_n GP_{n+1}$ ,
3.  $GP_{2n+1}^{(2)} = 2GP_{2n}^{(2)} + GP_{2n-1}^{(2)}$ ,

4.  $GP_{3n}^{(2)} = GP_n^3$ ,
5.  $GP_{3n+1}^{(3)} = GP_n^2 GP_{n+1}$ ,
6.  $GP_{3n+1}^{(3)} = 2GP_{3n}^{(3)} + GP_{3n-1}^{(3)}$ ,
7.  $GP_{3n+2}^{(3)} = GP_n GP_{n+1}^2$ ,
8.  $GP_{4n}^{(4)} = GP_n^4$ ,
9.  $GP_{4n+1}^{(4)} = GP_n^3 GP_{n+1}$ ,
10.  $GP_{4n+1}^{(4)} = 2GP_{4n}^{(4)} + GP_{4n-1}^{(4)}$ ,
11.  $GP_{4n+2}^{(4)} = GP_n^2 GP_{n+1}^2$ ,
12.  $GP_{4n+3}^{(4)} = GP_n GP_{n+1}^3$ .

Various values for the generalized Gauss  $k$ -Pell numbers are given in Table 1.

$GP_n^{(k)}$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
$GP_0^{(k)}$	$i$	$-1$	$-i$	$1$	$i$	$-1$
$GP_1^{(k)}$	$1$	$i$	$-1$	$-i$	$1$	$i$
$GP_2^{(k)}$	$2 + i$	$1$	$i$	$-1$	$-i$	$1$
$GP_3^{(k)}$	$5 + 2i$	$2 + i$	$1$	$i$	$-1$	$-i$
$GP_4^{(k)}$	$12 + 5i$	$3 + 4i$	$2 + i$	$1$	$i$	$-1$
$GP_5^{(k)}$	$29 + 12i$	$8 + 9i$	$3 + 4i$	$2 + i$	$1$	$i$
$GP_6^{(k)}$	$70 + 29i$	$21 + 20i$	$2 + 11i$	$3 + 4i$	$2 + i$	$1$
$GP_7^{(k)}$	$169 + 70i$	$50 + 49i$	$7 + 26i$	$2 + 11i$	$3 + 4i$	$2 + i$
$GP_8^{(k)}$	$408 + 169i$	$119 + 120i$	$22 + 61i$	$-7 + 24i$	$2 + 11i$	$3 + 4i$
$GP_9^{(k)}$	$985 + 408i$	$288 + 289i$	$65 + 142i$	$-12 + 59i$	$-7 + 24i$	$2 + 11i$
$GP_{10}^{(k)}$	$2378 + 985i$	$697 + 696i$	$152 + 345i$	$-17 + 144i$	$-38 + 41i$	$-7 + 24i$

Table 1. The generalized Gauss  $k$ -Pell numbers  $GP_n^{(k)}$  for some  $k$  and  $n$ .

**Proposition 3.1.** For  $k \in \mathbb{Z}^+$  and  $n \in \mathbb{N}$ , we have  $GP_{kn}^{(k)} = GP_n^k$ .

*Proof.* By Eq. (13), we get  $GP_{kn}^{(k)} = GP_n^k GP_{n+1}^0 = GP_n^k$ . □

**Theorem 3.1.** For  $n, s \in \mathbb{N}$  such that  $n + s \geq 2$ , we have

$$GP_{n+s} GP_{n+s-2} - GP_{2(n+s-1)}^{(2)} = (-1)^{n+s-1} 2(1-i).$$

*Proof.* By Proposition 3.1 and Eq. (9), we get

$$\begin{aligned} GP_{n+s} GP_{n+s-2} - GP_{2(n+s-1)}^{(2)} &= GP_{n+s} GP_{n+s-2} - GP_{n+s-1}^2 \\ &= (-1)^{n+s-1} 2(1-i). \end{aligned}$$
□

**Theorem 3.2.** For  $k \in \mathbb{Z}^+$  and  $s \in \mathbb{N}$ , we have

$$GP_{s+1}^k - GP_s^k = GP_{(s+1)k}^{(k)} - GP_{sk}^{(k)}. \quad (14)$$

*Proof.* By Eq. (13) and Proposition 3.1, we get

$$\begin{aligned} GP_{(s+1)k}^{(k)} - GP_{sk}^{(k)} &= GP_s^{k-k} GP_{s+1}^k - GP_s^k \\ &= GP_{s+1}^k - GP_s^k. \end{aligned} \quad \square$$

**Theorem 3.3.** For  $k \in \mathbb{Z}^+$  and  $n \in \mathbb{N}$ , we have the relation

$$GP_{kn+1}^{(k)} = 2GP_{kn}^{(k)} + GP_{kn-1}^{(k)}.$$

*Proof.* By Eq. (1), Eq. (13) and Proposition 3.1, we obtain

$$\begin{aligned} 2GP_{kn}^{(k)} + GP_{kn-1}^{(k)} &= 2GP_n^k + GP_{n-1} GP_n^{k-1} \\ &= GP_n^{k-1} (2GP_n + GP_{n-1}) \\ &= GP_n^{k-1} GP_{n+1} \\ &= GP_{kn+1}^{(k)}. \end{aligned} \quad \square$$

**Theorem 3.4. (Cassini's Identity)** Let  $GP_n^{(k)}$  be the generalized Gauss  $k$ -Pell numbers. For integers  $n, k \geq 2$ , the following gives the Cassini's identity for  $GP_n^{(k)}$ :

$$GP_{kn+t}^{(k)} GP_{kn+t-2}^{(k)} - \left( GP_{kn+t-1}^{(k)} \right)^2 = \begin{cases} GP_n^{2k-2} (-1)^n 2(1-i), & t = 1, \\ 0, & t \neq 1. \end{cases}$$

*Proof.* By using Eq. (3), Eq. (13) and Proposition 3.1, we obtain the following equations. If  $t = 1$ ,

$$\begin{aligned} GP_{kn+1}^{(k)} GP_{kn-1}^{(k)} - \left( GP_{kn}^{(k)} \right)^2 &= (GP_n^{k-1} GP_{n+1}) (GP_{n-1} GP_n^{k-1}) - (GP_n^k)^2 \\ &= GP_n^{2k-2} (GP_{n+1} GP_{n-1} - GP_n^2) \\ &= GP_n^{2k-2} (-1)^n 2(1-i), \end{aligned}$$

and if  $t \neq 1$ ,

$$\begin{aligned} GP_{kn+t}^{(k)} GP_{kn+t-2}^{(k)} - \left( GP_{kn+t-1}^{(k)} \right)^2 &= (GP_n^{k-t} GP_{n+1}^t) (GP_n^{k-t+2} GP_{n+1}^{t-2}) - (GP_n^{k-t+1} GP_{n+1}^{t-1})^2 \\ &= GP_n^{2k-2t+2} (GP_{n+1}^{2t-2} - GP_{n+1}^{2t-2}) \\ &= 0. \end{aligned} \quad \square$$

For  $t = 0, 1, 2, \dots, k-1$ , we get

$$GP_{kn+t}^{(k)} = GP_n^{k-t} GP_{n+1}^t.$$

### 3.2 The generalized Gauss $k$ -Pell polynomials

**Definition 3.2.** Let  $m$  and  $r$  be unique numbers such that  $n = mk + r$  and  $0 \leq r < k$ , for  $n \in \mathbb{N}$  and  $m, k \in \mathbb{Z}^+$ . Then we define the generalized Gauss  $k$ -Pell numbers  $GP_n^{(k)}(x)$  by

$$GP_n^{(k)}(x) := \left[ \left( \frac{\alpha^m(x) - \beta^m(x)}{\alpha(x) - \beta(x)} \right) + i \left( \frac{\alpha(x)\beta^m(x) - \beta(x)\alpha^m(x)}{\alpha(x) - \beta(x)} \right) \right]^{k-r} \left[ \left( \frac{\alpha^{m+1}(x) - \beta^{m+1}(x)}{\alpha(x) - \beta(x)} \right) + i \left( \frac{\alpha(x)\beta^{m+1}(x) + \beta(x)\alpha^{m+1}(x)}{\alpha(x) - \beta(x)} \right) \right]^r, n = mk + r,$$

where  $\alpha(x) = x + \sqrt{1+x^2}$  and  $\beta(x) = x - \sqrt{1+x^2}$ .

Furthermore, using the matrix methods, we can derive the generalized Gauss  $k$ -Pell polynomials. Indeed, it is obvious that

$$GP_n^{k-1}(x) gp_n(x) = \begin{bmatrix} GP_{kn+1}^{(k)}(x) & GP_{kn}^{(k)}(x) \\ GP_{kn}^{(k)}(x) & GP_{kn-1}^{(k)}(x) \end{bmatrix},$$

where  $n > 0$  and

$$gp_n(x) = \begin{bmatrix} GP_{n+1}(x) & GP_n(x) \\ GP_n(x) & GP_{n-1}(x) \end{bmatrix}.$$

From Eq. (11) and Definition 3.2, we have the following relationship between the generalized Gauss  $k$ -Pell polynomials and the Gauss Pell polynomials:

$$GP_n^{(k)}(x) := (GP_m(x))^{k-r} (GP_{m+1}(x))^r, n = mk + r \quad (15)$$

If we take  $k = 1$  in Eq. (15), then we get that  $m = n$  and  $r = 0$ , so  $GQ_n^{(1)}(x) = GP_n(x)$ .

For  $k = 2, 3, 4, \dots, n$ , we obtain the following relationship between the generalized Gauss  $k$ -Pell polynomials and the Gauss Pell polynomials by Eq. (10) and Eq. (15):

1.  $GP_{2n}^{(2)}(x) = GP_n^2(x)$ ,
2.  $GP_{2n+1}^{(2)}(x) = GP_n(x) GP_{n+1}(x)$ ,
3.  $GP_{2n+1}^{(2)}(x) = 2x GP_{2n}^{(2)}(x) + GP_{2n-1}^{(2)}(x)$ ,
4.  $GP_{3n}^{(2)}(x) = GP_n^3(x)$ ,
5.  $GP_{3n+1}^{(3)}(x) = GP_n^2(x) GP_{n+1}(x)$ ,
6.  $GP_{3n+1}^{(3)}(x) = 2x GP_{3n}^{(3)}(x) + GP_{3n-1}^{(3)}(x)$ ,
7.  $GP_{3n+2}^{(3)}(x) = GP_n(x) GP_{n+1}^2(x)$ ,
8.  $GP_{4n}^{(4)}(x) = GP_n^4(x)$ ,
9.  $GP_{4n+1}^{(4)}(x) = GP_n^3(x) GP_{n+1}(x)$ ,
10.  $GP_{4n+1}^{(4)}(x) = 2x GP_{4n}^{(4)}(x) + GP_{4n-1}^{(4)}(x)$ ,
11.  $GP_{4n+2}^{(4)}(x) = GP_n^2(x) GP_{n+1}^2(x)$ ,
12.  $GP_{4n+3}^{(4)}(x) = GP_n(x) GP_{n+1}^3(x)$ .

Various values for the generalized Gauss  $k$ -Pell polynomials are given in Table 2.

$GP_n^{(k)}$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$GP_0^{(k)}(x)$	$i$	$-1$	$-i$	$1$
$GP_1^{(k)}(x)$	$1$	$i$	$-1$	$-i$
$GP_2^{(k)}(x)$	$2x + i$	$1$	$i$	$-1$
$GP_3^{(k)}(x)$	$4x^2 + 2ix + 1$	$2x + i$	$1$	$i$
$GP_4^{(k)}(x)$	$8x^3 + 4ix^2 + 4x + i$	$4x^2 + 4ix - 1$	$2x + i$	$1$
$GP_5^{(k)}(x)$	$16x^4 + 8ix^3 + 12x^2 + 4ix + 1$	$8x^3 + 8ix^2 + i$	$4x^2 + 4ix - 1$	$2x + i$
$GP_6^{(k)}(x)$	$32x^5 + 16ix^4 + 32x^3 + 12ix^2 + 6x + i$	$16x^4 + 16ix^3 + 4x^2 + 4ix + 1$	$8x^3 + 12ix^2 - 6x - i$	$4x^2 + 4ix - 1$
$GP_7^{(k)}(x)$	$64x^6 + 32ix^5 + 80x^4 + 32ix^3 + 24x^2 + 6ix + 1$	$32x^5 + 32ix^4 + 16x^3 + 16ix^2 + 2x + i$	$16x^4 + 24ix^3 - 8x^2 + 2ix - 1$	$8x^3 + 12ix^2 - 6x - i$
$GP_8^{(k)}(x)$	$128x^7 + 64ix^6 + 192x^5 + 80ix^4 + 80x^3 + 24ix^2 + 8x + i$	$64x^6 + 64ix^5 + 48x^4 + 48ix^3 + 8x^2 + 8ix - i$	$32x^5 + 48ix^4 - 8x^3 + 12ix^2 - 2x + i$	$16x^4 + 32ix^3 - 24x^2 - 8ix + 1$

Table 2. The generalized Gauss  $k$ -Pell polynomials  $GP_n^{(k)}$  for some  $k$  and  $n$ .

**Proposition 3.2.** For  $k \in \mathbb{Z}^+$  and  $n \in \mathbb{N}$ , we have  $GP_{kn}^{(k)}(x) = GP_n^k(x)$ .

*Proof.* By Eq. (15), we get  $GP_{kn}^{(k)}(x) = GP_n^k(x) GP_{n+1}^0(x) = GP_n^k(x)$ . □

**Theorem 3.5.** For  $n, s \in \mathbb{N}$  such that  $n + s \geq 2$ , we have

$$GP_{n+s}(x) GP_{n+s-2}(x) - GP_{2(n+s-1)}^{(2)}(x) = 2(-1)^{n+s-1}(1-xi).$$

*Proof.* By Proposition 3.2 and Eq. (12), we get

$$\begin{aligned} GP_{n+s}(x) GP_{n+s-2}(x) - GP_{2(n+s-1)}^{(2)}(x) &= GP_{n+s}(x) GP_{n+s-2}(x) - GP_{n+s-1}^2(x) \\ &= 2(-1)^{n+s-1}(1-xi). \end{aligned}$$
 □

**Theorem 3.6.** For  $k \in \mathbb{Z}^+$  and  $s \in \mathbb{N}$ , we have

$$GP_{s+1}^k(x) - GP_s^k(x) = GP_{(s+1)k}^{(k)}(x) - GP_{sk}^{(k)}(x). \quad (16)$$

*Proof.* By Eq. (15) and Proposition 3.2, we get

$$\begin{aligned} GP_{(s+1)k}^{(k)}(x) - GP_{sk}^{(k)}(x) &= GP_s^{k-k}(x) GP_{s+1}^k(x) - GP_s^k(x) \\ &= GP_{s+1}^k(x) - GP_s^k(x). \end{aligned}$$
 □



**Theorem 3.7.** For  $k \in \mathbb{Z}^+$  and  $n \in \mathbb{N}$ , we have the relation

$$GP_{kn+1}^{(k)}(x) = 2xGP_{kn}^{(k)}(x) + GP_{kn-1}^{(k)}(x).$$

*Proof.* By Eq. (10), Eq. (15) and Proposition 3.2, we obtain

$$\begin{aligned} 2xGP_{kn}^{(k)}(x) + GP_{kn-1}^{(k)}(x) &= 2xGP_n^k(x) + GP_{n-1}(x)GP_n^{k-1}(x) \\ &= GP_n^{k-1}(x)(2xGP_n(x) + GP_{n-1}(x)) \\ &= GP_n^{k-1}(x)GP_{n+1}(x) \\ &= GP_{kn+1}^{(k)}(x). \end{aligned} \quad \square$$

**Theorem 3.8. (Cassini's Identity)** Let  $GP_n^{(k)}(x)$  be the generalized Gauss  $k$ -Pell polynomials. For integers  $n, k \geq 2$ , the following gives the Cassini's identity for  $GP_n^{(k)}(x)$ :

$$GP_{kn+t}^{(k)}(x)GP_{kn+t-2}^{(k)}(x) - \left(GP_{kn+t-1}^{(k)}(x)\right)^2 = \begin{cases} GP_n^{2k-2}(x)2(-1)^n(1-xi), & t = 1, \\ 0, & t \neq 1. \end{cases}$$

*Proof.* By using Eq. (12), Eq. (15) and Proposition 3.2, we obtain the following equations.

If  $t = 1$ ,

$$\begin{aligned} GP_{kn+1}^{(k)}(x)GP_{kn-1}^{(k)}(x) - \left(GP_{kn}^{(k)}(x)\right)^2 &= (GP_n^{k-1}(x)GP_{n+1}(x))(GP_{n-1}(x)GP_n^{k-1}(x)) \\ &\quad - \left(GP_n^k(x)\right)^2 \\ &= GP_n^{2k-2}(x)(GP_{n+1}(x)GQ_{n-1}(x) - GP_n^2(x)) \\ &= GP_n^{2k-2}(x)2(-1)^n(1-xi), \end{aligned}$$

and if  $t \neq 1$ ,

$$\begin{aligned} GP_{kn+t}^{(k)}(x)GP_{kn+t-2}^{(k)}(x) - \left(GP_{kn+t-1}^{(k)}(x)\right)^2 &= (GP_n^{k-t}(x)GP_{n+1}^t(x))(GP_n^{k-t+2}(x)GP_{n+1}^{t-2}(x)) \\ &\quad - \left(GP_n^{k-t+1}(x)GP_{n+1}^{t-1}(x)\right)^2 \\ &= GP_n^{2k-2t+2}(x)(GP_{n+1}^{2t-2}(x) - GP_{n+1}^{2t-2}(x)) \\ &= 0. \end{aligned} \quad \square$$

For  $t = 0, 1, 2, \dots, k-1$ , we have the following relations:

$$GP_{kn+t}^{(k)}(x) = GP_n^{k-t}(x)GP_{n+1}^t(x).$$

## 4 Conclusions

Halıcı and Öz defined Gauss Pell numbers in [5] and Gauss Pell polynomials in [4]. We introduced a generalization of these numbers as the generalized Gauss  $k$ -Pell numbers and these polynomials as the generalized Gauss  $k$ -Pell polynomials. Some relations between the class of the generalized Gauss  $k$ -Pell numbers and the known Gauss Pell numbers are presented. Some relations between the class of the generalized Gauss  $k$ -Pell polynomials and the known Gauss Pell polynomials are presented. Then identities for these numbers and polynomials are proved.

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