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On recurrence results from matrix transforms

Ömür Deveci[1] **and Anthony G. Shannon**[2]

[1] Department of Mathematics, Faculty of Science and Letters, Kafkas University
36100 Kars, Turkey
e-mail: odeveci36@hotmail.com

[2] Warrane College, The University of New South Wales
Kensington, NSW 2033, Australia
e-mail: t.shannon@warrane.unsw.edu.au

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Abstract: In this paper, the Laplace transform and various matrix operations are applied to the characteristic polynomial of the Fibonacci numbers. From this is generated some properties of the Jacobsthal numbers, including triangles where the row sums are known sequences. In turn these produce some new properties.
Keywords: Recurrence relations, Laplace transform, Fibonacci sequence, Jacobsthal numbers, Simson's formula.
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1 Introduction

Sburlati [4] used a recursive sequence defined as a repunit [2] by

$$k_n = \tfrac{1}{3}(4^n - 1).$$

It satisfies the second order homogenous linear recurrence relation

$$k_n = 5k_{n-1} - 4k_{n-2},\ n \geq 2,\ k_1 = 1,\ k_2 = 5,$$

which is a generalization of the well-known Fibonacci recurrence relation.

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2, F_1 = 1,\ \ F_2 = 1.$$

It is well known that the characteristic polynomial of the Fibonacci sequence is:

$$f(x) = x^2 - x - 1.$$

We now consider the Laplace transform of the polynomial $f(x)$. Since

$$L(f(x)) = F(s) = \frac{2}{s^3} - \frac{1}{s^2} - \frac{1}{s},$$

we have $s^3F(s) = 2 - s - s^2$. Thus, we can define the following recurrence sequence with respect to the polynomial $s^3F(s)$:

$$y_{n+2} = -y_{n+1} + 2y_n \tag{1.1}$$

with initial conditions $y_1 = 0$ and $y_2 = 1$. We now consider some consequences of this.

2 Some old and some new results

From (1.1), we can write the following companion matrix:

$$A = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}.$$

By an inductive argument, we may then write

$$(A)^n = \begin{bmatrix} y_{n+2} & -2y_{n+1} \\ y_{n+1} & -2y_n \end{bmatrix}$$

for $n \geq 1$. Since $\det A = -2$, we can write the associated Simson formula for the sequence $\{y_n\}$ as:

$$(y_{n+2})(-2y_n) + 2(y_{n+1})^2 = (-2)^n.$$

It is easy to see that

$$\{y_n\} = \{0, 1, -1, 3, -5, 11, -21, 43, -85, 171, -341, 683, -1365, 2731, -5461, \ldots\}$$

and so we get $y_3 = -k_1,\ y_5 = -k_2,\ y_7 = -k_3,\ y_9 = -k_4,\ y_{11} = -k_5, \ldots$. By mathematical induction on n, we obtain the relationships between the sequences $\{y_n\}$ and $\{k_n\}$ as follows:

$$y_{2n+1} = -k_n,\ n \geq 1.$$

The absolute values of the elements of the sequence $\{y_n\}$ yield the Jacobsthal sequences (A001045 in Sloane [7]), from which we can readily assemble a ‘Jacobsthal triangle’ (Figure 1).

This is different from, but related to results in Shapiro [6]. From them we can see that the Jacobsthal sequence its self is the union of $\{y_n\}$ and $\{k_n\}$ (A002450 and A007583). We can also see that the row sums of the triangle are the elements of the Jacobsthal sequence, the triangle of which contains sub-triangles, such as the one for A007583, the elements of which are similarly present in the row sums.

0
0 1
0 1 0
0 1 1 1
0 1 3 1 0
0 1 5 3 1 1
0 1 11 5 3 1 0
0 1 21 11 5 3 1 1
0 1 43 21 11 5 3 1 0
0 1 85 43 21 11 5 3 1 1

Figure 1. Jacobsthal triangle

1
1 1 1
1 5 3 1 1
1 21 11 5 3 1 1
1 85 43 21 11 5 3 1 1
1 341 171 85 43 21 11 5 3 1 1
1 1365 683 341 171 85 43 21 11 5 3 1 1
1 5461 2731 1365 683 341 171 85 43 21 11 5 3 1 1

Figure 2. Triangle for A007583

3 Concluding comments

This note continues the patterns of matrix generated recurrence relations pioneered by Leonard Carlitz and John Riordan [3] and illustrated in [5]. Thus, if we re-write the triangle in Figure 1, we obtain Figure 3.

0
0 1
0 –1 0
0 1 –1 1
0 –1 3 –1 0
0 1 –5 3 –1 1
0 –1 11 –5 3 –1 0
0 1 –21 11 –5 3 –1 1
0 –1 43 –21 11 –5 3 –1 0
0 1 85 43 –21 11 –5 3 –1 1

Figure 3. Triangle for $\{y_n\}$

Then, the triangles can be compared according to Barry's model [1]:

–										–								
–	1									–	1							
–1	–	–								1	–	–						
–	–	3	–							–	–	3	–					
–	–4	–	–	–1					= $(-1)^n$	–	4	–	–	1				
1	–	–	10	–	–					1	–	–	10	–	–			
–	–	–15	–	–	–6	–				–	–	15	–	–	6	–		
–	7	–	–	35	–	–	1			–	7	–	–	35	–	–	1	
–1	–	–	–56	–	–	–28	–	–		1	–	–	56	–	–	28	–	–

A *B*

Figure 4. The relation betwen trianles *A* and *B*

That is, the *n*-th row of the triangle *A* is the product of the *n*-th row of the triangle *B* with $(-1)^n$. From this we obtain the new result that if *n* is odd, then

$$4(y_n + y_{n+1}) - 2 = y_{n+2} + y_{n+3}.$$

And from this then we may write

$$4(-J_n + J_{n+1}) - 2 = -J_{n+2} + J_{n+3}.$$

Therefore, we obtain $J_{n+1} - 2J_n = 1$ when n is an odd integer, so that we can consider the Jacobsthal sequence as follows:

$$J_1 = 0, \quad J_2 = 1, \quad J_{n+2} = J_{n+1} + 2J_n.$$

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