Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2022, Volume 28, Number 3, 581–588
DOI: 10.7546/nntdm.2022.28.3.581-588

Eisenstein series of level 6 and level 10 with their applications to theta function identities of Ramanujan

A. I. Vijaya Shankar

Department of Studies in Mathematics, University of Mysore
Manasagangotri, Mysuru – 570 006, Karnataka, India
e-mail: vijayshankarai3@gmail.com

Received: 6 May 2022 **Revised:** 19 September 2022
Accepted: 21 September 2022 **Online First:** 28 September 2022

Abstract: S. Ramanujan recorded theta function identities of different levels in the unorganized pages of his second notebook and the lost notebook. In this paper, we prove level 6 and level 10 theta function identities by using Eisenstein series identities.
Keywords: Theta functions, Eisenstein series.
2020 Mathematics Subject Classification: 11F20, 11M36.

1 Introduction

Throughout the sequel, we use the following notation

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n),$$

where a and q are complex numbers with $|q| < 1$. For $|ab| < 1$, Ramanujan's general theta function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Further, Ramanujan [1, p. 36] considers the following three special cases of $f(a,b)$:

$$\varphi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (-q^2;q^2)_\infty},$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} = (q;q)_\infty.$$

After Ramanujan, we define

$$\chi(q) := (-q;q^2)_\infty \text{ and } f_n := f(-q^n),$$

where n is any positive integer. Let $P(q)$ denote Ramanujan's Eisenstein series of weight 2, defined by

$$P(q) := 1 - 24\sum_{k=1}^{\infty} \frac{kq^k}{1-q^k}.$$

For any positive integer n, set $P_n := P(q^n)$.

In the unorganized pages of his second notebook [10] and the lost notebook [11, p. 55], Ramanujan recorded theta function identities involving f_1, f_2, f_k and f_{2k}, for certain positive integer k. We call such identities as level $2k$ theta function identities. B. C. Berndt [2, pp. 204–208] proved level 6 and level 10 identities using Ramanujan's modular equations. Also S. Bhargava, K. R. Vasuki and K. R. Rajanna [3] have proved these identities by using Ramanujan's ${}_1\psi_1$ summation formula.

In a paper [7] S. Cooper and D. Ye deduced the below Eisenstein series identities: If

$$z = qf_1f_2f_7f_{14}, \quad v = q\left(\frac{f_2f_{14}}{f_1f_7}\right)^3 \text{ and } w = q\left(\frac{f_1f_{14}}{f_2f_7}\right)^4,$$

then

$$\begin{aligned}
zv &= \frac{1}{72}(-P_1 + P_2 + 7P_7 - 7P_{14}) - \frac{1}{3}z,\\
\frac{z}{v} &= \frac{1}{18}(P_1 - 4P_2 - 7P_7 + 28P_{14}) - \frac{8}{3}z,\\
zw &= \frac{1}{144}(5P_1 - 26P_2 + 91P_7 - 70P_{14}) + \frac{5}{6}z, \text{ and}\\
\frac{z}{w} &= \frac{1}{144}(-13P_1 + 10P_2 - 35P_7 + 182P_{14}) + \frac{5}{6}z.
\end{aligned}$$

Further they showed that

$$7z + 8zv + \frac{z}{v} - zw - \frac{z}{w} = 0.$$

This yields the following theta function identity of level 14:

$$\frac{1}{v} + 8v + 7 = w + \frac{1}{w},$$

which is equivalent to the modular equation recorded by Ramanujan [10, Ch. 19, Entry 19, ix]. Motivated by this, in this article we extend the above technique to obtain all level 6 and level 10 theta function identities. In Sections 2 and 4, we prove certain Eisenstein series identities which are required to prove Ramanujan's theta function identities of level 6 and level 10, respectively. In Sections 3 and 5, we prove level 6 and level 10 theta function identities, respectively.

2 Eisenstein series of level 6

In this section, we recall certain known Eisenstein series identities which are required to prove level 6 theta function identities. S. Cooper [6, Theorem 6.19, p. 378] has given a proof of the below six Eisenstein series identities (2.5)–(2.10) and it is interesting to see that these identities can be written as a linear combination of the form

$$aP_1 + bP_2 + cP_3 + dP_6,$$

where $6a+3b+2c+d = 0$. Among these identities, we prove (2.6)–(2.8). Then E. N. Bhuvan [4] set

$$l := \frac{\varphi^2(-q^3)}{\varphi^2(-q)}, \tag{2.1}$$

and then showed that

$$z = q\frac{d}{dq}\log(l) = 4q\psi^2(q)\psi^2(q^3), \tag{2.2}$$

$$\frac{\chi^9(-q)}{\chi^3(-q^3)} = \frac{8}{9l^2-1}, \tag{2.3}$$

$$\frac{\psi^4(q)}{q\psi^4(q^3)} = \frac{9l^2-1}{l^2-1}, \tag{2.4}$$

$$P_1 - 2P_2 - 9P_3 + 18P_6 = 8\frac{f_1^3 f_2^3}{f_3 f_6}, \tag{2.5}$$

and

$$P_1 - 4P_2 - 3P_3 + 12P_6 = 6\frac{f_1^4 f_3^4}{f_2^2 f_6^2}. \tag{2.6}$$

R. G. Veeresha [12, pp. 88–89] proved the below Eisenstein series identity by using W. N. Bailey's formula:

$$-P_1 + P_2 + 3P_3 - 3P_6 = 24q\frac{f_2^4 f_6^4}{f_1^2 f_3^2}. \tag{2.7}$$

Theorem 2.1. *The following identities hold:*

$$-P_1 + 2P_2 + P_3 - 2P_6 = 24q\frac{f_3^3 f_6^3}{f_1 f_2}, \tag{2.8}$$

$$P_1 - 10P_2 + 15P_3 - 6P_6 = -24q\frac{f_1^7 f_6^7}{f_2^5 f_3^5} \tag{2.9}$$

and

$$-5P_1 + 2P_2 - 3P_3 + 30P_6 = 24\frac{f_2^7 f_3^7}{f_1^5 f_6^5}. \tag{2.10}$$

Proof. From (2.3), we have

$$\frac{\chi^9(-q)}{\chi^3(-q^3)}=\frac{8}{9l^2-1}.$$

This implies

$$\frac{f_1^9f_6^3}{f_2^9f_3^3}=\frac{8}{9l^2-1}.$$

Differentiating logarithmically with respect to q, using (2.1), (2.3) and the definition of P_n, we obtain (2.8). From (2.1) and (2.3), we find that

$$l^2\frac{\chi^9(-q)}{\chi^3(-q^3)}=\frac{8l^2}{9l^2-1}.$$

This is

$$\frac{f_1f_3^5}{f_2^5f_6}=\frac{8l^2}{9l^2-1}.$$

Differentiating logarithmically with respect to q, using (2.1), (2.3) and the definition of P_n, we obtain (2.9). From (2.3) and (2.4), we have

$$q\frac{\chi^3(-q^3)\psi^4(q^3)}{\chi^9(-q)\psi^4(q)}=\frac{l^2-1}{8},$$

which implies

$$q\frac{f_2f_6^5}{f_1^5f_3}=\frac{l^2-1}{8}.$$

By logarithmic differentiation and using (2.1), (2.3) and (2.4), one can easily obtain (2.10). □

3 Theta function identities of level 6

In this section, we prove the below Ramanujan's theta function identities (3.1)–(3.3) of level 6 by applying the level 6 Eisenstein series identities proven in the previous section. For convenience, set $Z=q^{\frac{1}{2}}f_1f_2f_3f_6$.

Theorem 3.1. *Let*

$$X=\frac{f_1}{q^{\frac{1}{24}}f_2},\quad Y=\frac{f_3}{q^{\frac{1}{8}}f_6},\quad K=\frac{f_2}{q^{\frac{1}{24}}f_3},\quad L=\frac{f_1}{q^{\frac{5}{24}}f_6},\quad M=\frac{f_1}{q^{\frac{1}{12}}f_3}\ \textit{and}\ N=\frac{f_2}{q^{\frac{1}{6}}f_6}.$$

Then,

$$(XY)^3+\frac{8}{(XY)^3}=\left(\frac{Y}{X}\right)^6-\left(\frac{X}{Y}\right)^6,\tag{3.1}$$

$$(KL)^2-\frac{9}{(KL)^2}=\left(\frac{L}{K}\right)^3-8\left(\frac{K}{L}\right)^3\tag{3.2}$$

and

$$(MN)^2+\frac{9}{(MN)^2}=\left(\frac{N}{M}\right)^6+\left(\frac{M}{N}\right)^6.\tag{3.3}$$

Ramanujan recorded (3.2) and (3.3) in his second notebook [10, p. 327] and recorded a modular equation equivalent to (3.1) in his second notebook [10, p. 230].

Proof. From (2.6) and (2.7), we find that

$$-P_1 - 2P_2 + 3P_3 + 6P_6 = 6Z\left[(XY)^3 + \frac{8}{(XY)^3}\right].$$

Also from (2.9) and (2.10), we obtain that

$$-P_1 - 2P_2 + 3P_3 + 6P_6 = 6Z\left[\left(\frac{Y}{X}\right)^6 - \left(\frac{X}{Y}\right)^6\right].$$

Identity (3.1) follows from the above two identities. From (2.5) and (2.8), we find that

$$P_1 - 2P_2 - 3P_3 + 6P_6 = 2Z\left[(KL)^2 - \frac{9}{(KL)^2}\right].$$

Also from (2.6) and (2.7), we find that

$$P_1 - 2P_2 - 3P_3 + 6P_6 = 2Z\left[\left(\frac{L}{K}\right)^3 - 8\left(\frac{K}{L}\right)^3\right].$$

It is easy to see that from the above two identities, we obtain (3.2). From (2.5) and (2.8), we find that

$$-P_1 + 2P_2 - 3P_3 + 6P_6 = 4Z\left[(MN)^2 + \frac{9}{(MN)^2}\right].$$

Also from (2.9) and (2.10), we find that

$$-P_1 + 2P_2 - 3P_3 + 6P_6 = 4Z\left[\left(\frac{N}{M}\right)^6 + \left(\frac{M}{N}\right)^6\right].$$

From the above two identities, (3.3) holds. □

4 Eisenstein series of level 10

In this section, we obtained three new Eisenstein series identities by using the relations involving Ramanujan function k [10, p. 326] and is defined by

$$k = r(q)r^2(q^2),$$

where $r(q)$ is the Rogers–Ramanujan continued fraction and

$$r(q) = q^{\frac{1}{5}} \prod_{j=1}^{\infty} \frac{(1-q^{5j-4})(1-q^{5j-1})}{(1-q^{5j-3})(1-q^{5j-2})}.$$

Note that the below three Eisenstein series identities (4.6)–(4.8) can be obtained from [6, Theorem 10.12, p. 536] but we prove them by logarithmic differentiation. The identities (4.3)–(4.8) can be written as a linear combination of the form

$$aP_1 + bP_2 + cP_5 + dP_{10},$$

where $10a + 5b + 2c + d = 0$.

Of the identities (4.1) below, the first two are due to Ramanujan [11, p. 56] and the last two identities are due to C. Gugg [8].

$$\frac{1-k^2}{1-4k-k^2}=\frac{\varphi^2(-q^5)}{\varphi^2(-q)}, \qquad \frac{k}{1-k^2}=q\frac{\chi(-q)}{\chi^5(-q^5)},$$
$$\frac{1+k-k^2}{k}=\frac{\psi^2(q)}{q\psi^2(q^5)} \qquad \text{and} \qquad \sqrt{\alpha}=\frac{1+k^2}{k}=\frac{\psi(q)}{q^{1/2}\psi(q^5)}Y_0, \tag{4.1}$$

where

$$Y_0=\left(\frac{\psi(q)^2}{q\psi(q^5)^2}-2+5\frac{q\psi(q^5)^2}{\psi(q)^2}\right)^{\frac{1}{2}}.$$

S. Cooper [5], deduced the following identity which is a derivative of Ramanujan function k:

$$q\frac{d}{dq}\log(k)=\frac{f_1f_2^2f_5^3}{f_{10}^2}. \tag{4.2}$$

Further, S. Cooper [5], deduced the following Eisenstein series identities:

$$P_1-4P_2-5P_5+20P_{10}=12Y_0q^{\frac{1}{2}}f_1^2f_5^2, \tag{4.3}$$

$$-3P_1+2P_2-5P_5+30P_{10}=24Y_0q^{\frac{1}{2}}\frac{f_2^5f_5^3}{f_1^3f_{10}} \tag{4.4}$$

and

$$P_1-2P_2-25P_5+50P_{10}=24Y_0q^{\frac{1}{2}}\frac{f_1f_2^3f_{10}}{f_5}. \tag{4.5}$$

Theorem 4.1. *The following Eisenstein series identities hold:*

$$P_1-P_2-5P_5+5P_{10}=-24Y_0q^{\frac{3}{2}}\frac{f_2^4f_{10}^4}{f_1^2f_5^2}, \tag{4.6}$$

$$-P_1+6P_2-15P_5+10P_{10}=24Y_0q^{\frac{3}{2}}\frac{f_1^3f_{10}^5}{f_2f_5^3} \tag{4.7}$$

and

$$-P_1+2P_2+P_5-2P_{10}=24Y_0q^{\frac{3}{2}}\frac{f_2f_5f_{10}^3}{f_1}. \tag{4.8}$$

Proof. From (4.1), we have

$$\frac{\varphi^2(-q)}{\varphi^2(-q^5)}=\frac{1-4k-k^2}{1-k^2}.$$

This implies

$$\frac{f_1^4f_{10}^2}{f_2^2f_5^4}=\frac{1-4k-k^2}{1-k^2}.$$

Differentiating logarithmically with respect to q and using the definition of P_n and (4.1), (4.2), we arrive at (4.6). From (4.1), we see that

$$\frac{\psi^2(q)\chi(-q)}{\psi^2(q^5)\chi^5(-q^5)}=\frac{1+k-k^2}{1-k^2}.$$

This implies

$$\frac{f_2^3f_{10}}{f_1f_5^3}=\frac{1+k-k^2}{1-k^2}.$$

Differentiating logarithmically with respect to q and using the definition of P_n and (4.1), (4.2), one can easily deduce (4.7). From (4.1), we find that

$$\frac{\varphi^2(-q^5)\psi^2(q)\chi(-q)}{\varphi^2(-q)\psi^2(q^5)\chi^5(-q^5)} = \frac{1+k-k^2}{1-4k-k^2},$$

which implies

$$\frac{f_2^5 f_5}{f_1^5 f_{10}} = \frac{1+k-k^2}{1-4k-k^2}.$$

Differentiating logarithmically with respect to q and using the definition of P_n and (4.1), (4.2), we obtain (4.8). $\square$

5 Theta function identities of level 10

The following theorem gives the proof of Ramanujan's theta function identities of level 10 in which (5.1) and (5.3) are noted by Ramanujan [11, p. 55], and (5.2) is noted by Ramanujan in his second notebook [10, p. 327]. Now let us set $Z_0 = qf_2^2 f_{10}^2 Y_0$.

Theorem 5.1. *Let*

$$P = \frac{f_1}{q^{\frac{1}{24}} f_2},\ \ Q = \frac{f_5}{q^{\frac{5}{24}} f_{10}},\ \ A = \frac{f_2}{q^{\frac{1}{8}} f_5},\ \ B = \frac{f_1}{q^{\frac{3}{8}} f_{10}},\ \ C = \frac{f_1}{q^{\frac{1}{6}} f_5}, \textit{and}\ \ D = \frac{f_2}{q^{\frac{1}{3}} f_{10}}.$$

Then

$$(PQ)^2 + \frac{4}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3, \tag{5.1}$$

$$AB - \frac{5}{AB} = \left(\frac{B}{A}\right)^2 - 4\left(\frac{A}{B}\right)^2 \tag{5.2}$$

and

$$CD + \frac{5}{CD} = \left(\frac{D}{C}\right)^3 + \left(\frac{C}{D}\right)^3. \tag{5.3}$$

Proof. From (4.3) and (4.6), we find that

$$-P_1 - 2P_2 + 5P_5 + 10P_{10} = 12Z_0\left[(PQ)^2 + \frac{4}{(PQ)^2}\right].$$

Also, from (4.4) and (4.7), we find that

$$-P_1 - 2P_2 + 5P_5 + 10P_{10} = 12Z_0\left[\left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3\right].$$

From the above two identities, one can easily seen that (5.1) holds. From (4.5) and (4.8), we obtain that

$$P_1 - 2P_2 - 5P_5 + 10P_{10} = 4Z_0\left[AB - \frac{5}{AB}\right].$$

Also from (4.3) and (4.6), we find that

$$P_1 - 2P_2 - 5P_5 + 10P_{10} = 4Z_0\left[\left(\frac{B}{A}\right)^2 - 4\left(\frac{A}{B}\right)^2\right].$$

From the above two identities, it follows that (5.2) holds. From (4.5) and (4.8), we find that

$$-P_1 + 2P_2 - 5P_5 + 10P_{10} = 6Z_0\left[CD + \frac{5}{CD}\right].$$

Also from (4.4) and (4.7), we find that

$$-P_1 + 2P_2 - 5P_5 + 10P_{10} = 6Z_0 \left[\left(\frac{D}{C}\right)^3 + \left(\frac{C}{D}\right)^3 \right].$$

From the above two identities, it follows that (5.3) holds. □

Acknowledgements

The author would like to thank anonymous referee for their valuable comments.

The author is thankful to K. R. Vasuki, for his guidance during the preparation of article and University Grants Commission, New Delhi, India for supporting the research work by Grant No. 191620127010/(CSIR-UGC NET DEC.2019) under Joint CSIR-UGC JRF scheme.

References

[1] Berndt, B. C. (1991). *Ramanujan's Notebooks, Part III*. Springer, New York.

[2] Berndt, B. C. (1994). *Ramanujan's Notebooks, Part IV*. Springer, New York.

[3] Bhargava, S., Vasuki, K. R., & Rajanna, K. R. (2015). On some Ramanujan identities for the ratios of eta-functions. *Ukrainian Mathematical Journal*, 66, 1011–1028.

[4] Bhuvan, E. N. (2018). On some Eisenstein series identities associated with Borwein's cubic theta functions. *Indian Journal of Pure and Applied Mathematics*, 49, 689–703.

[5] Cooper, S. (2009). On Ramanujan's function $k = r(q)r^2(q^2)$. *Ramanujan Journal*, 20, 311–328.

[6] Cooper, S. (2017). *Ramanujan's Theta Functions*. Springer, Cham.

[7] Cooper, S., & Ye, D. (2016). Level 14 and 15 analogues of Ramanujan's Elliptic functions to alternative bases. *Transactions of the American Mathematical Society*, 368, 7883–7910.

[8] Gugg, C. (2009). Two modular equations for squares of the Rogers–Ramanujan functions with applications. *Ramanujan Journal*, 18, 183–207.

[9] Ramanujan, S., (1957). *Notebooks. Vol. 1*. Tata Institute of Fundamental Research, Bombay.

[10] Ramanujan, S., (1957). *Notebooks. Vol. 2*. Tata Institute of Fundamental Research, Bombay.

[11] Ramanujan, S., (1988). *The Lost Notebook and Other Unpublished Papers*. Narosa, New Delhi.

[12] Veeresha, R. G. (2015). *An elementary approach to Ramanujan's modular equations of degree 7 and its applications* [Doctoral dissertation, University of Mysore]. Shodhganga Repository. https://shodhganga.inflibnet.ac.in/handle/10603/108538