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On a new additive arithmetic function related to a fixed integer

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Abstract: The main purpose of this paper is to define a new additive arithmetic function related to a fixed integer $k \geq 1$ and to study some of its properties. This function is given by

$$f_k(1) = 0 \text{ and } f_k(n) = \sum_{p^\alpha \| n} (k, \alpha),$$

such that (a, b) denotes the greatest common divisor of the integers a and b.
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1 Introduction

For all integers $a,\ b \geq 1$, we denote by $\gcd(a, b) = (a, b)$ the largest common divisor of the integers a and b. Let

$$n = \prod_{i=1}^{r} p_i^{\alpha_i}$$

be the prime factorization of the positive integer $n > 1$. In [1] Atanassov defined and studied the following function:

$$\underline{mult}\,(n) = \prod_{i=1}^{r} p_i, \quad \underline{mult}\,(1) = 1,$$

and in [5], Andrei V. Shubin defined the following two additive arithmetic functions

$$\Omega\,(k,n) = \sum_{\substack{p^\alpha \| n \\ \alpha \le k}} \alpha \text{ and } \omega\,(k,n) = \sum_{\substack{p^\alpha \| n \\ \alpha > k}} 1. \tag{1}$$

The two functions $\Omega(k,n)$ and $\omega(k,n)$ are generalizations of the well-known functions $\Omega(n)$ and $\omega(n)$ are respectively the number of prime divisors and the number of distinct prime divisors of n. The definition and study of the properties of new arithmetic functions is a topic of interest to many researchers (see for example, [2–4]).

In this paper, a new additive arithmetic function will be defined and some of its basic properties are investigated.

2 Main results

Let k be a positive integer. Then we define f_k to be the arithmetic function such that $f_k\,(1) = 0$ and

$$f_k\,(n) = \sum_{p^\alpha \| n} (k,\alpha)\,.$$

We note that the function $f_k\,(n)$ is equal to the function $\omega\,(n)$ or to the function $\Omega\,(n)$ for some particular cases of the integer k.

Indeed, let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$. If $k = 1$, then $(k,\alpha_i) = 1$ for all $(1 \le i \le r)$. Thus

$$f_1\,(n) = \sum_{p_i | n} (1,\alpha_i) = \sum_{p_i | n} 1 = \omega(n), \text{ for all } n,$$

and if $\alpha_i \ge 2$ for all $1 \le i \le r$, then

$$f_1\,(n) = \sum_{\substack{p_i | n \\ \alpha_i > 1}} 1 = \omega\,(1,n)\,.$$

If $k = \operatorname{lcm}(\alpha_1, \alpha_2, \ldots, \alpha_r)$, then $(k,\alpha_i) = \alpha_i$ for all $(1 \le i \le r)$. Thus

$$f_k\,(n) = \sum_{p_i | n} (k,\alpha_i) = \sum_{p_i | n} \alpha_i = \Omega\,(n)\,,$$

and it can also be noticed that $\alpha_i \le k$, then

$$f_k\,(n) = \sum_{\substack{p_i | n \\ \alpha_i \le k}} \alpha_i = \Omega\,(k,n)\,.$$

Let m be a positive integer such that $m = \prod_{j=1}^{s} q_j^{\beta_j}$, its canonical decomposition. If $(m, n) = 1$ (i.e., $q_j \neq p_i$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$), then for all $k \in \mathbb{Z}_{\geq 1}$

$$f_k(nm) = \sum_{p_i|n} (k, \alpha_i) + \sum_{q_j|n} (k, \beta_j) = f_k(n) + f_k(m).$$

On the other hand, if p_1, p_2 and p_3 are different primes, then for all $k \in \mathbb{Z}_{\geq 1}$:

$$f_k\left(p_1 p_2^{k+1} p_3\right) = (k, 1) + (k, k+1) + (k, 1) = 3,$$

while

$$f_k\left(p_1 p_2^k\right) + f_k(p_2 p_3) = (k, 1) + (k, k) + (k, 1) + (k, 1) = k + 3.$$

Therefore, it can be shown that the function f_k is additive but not completely additive.

We know that for all α_i $(1 \leq i \leq r)$ and for all $k \in \mathbb{Z}_{\geq 1}$, we have

$$1 \leq (k, \alpha_i) \leq \alpha_i,$$

which implies that

$$\sum_{p_i|n} 1 \leq \sum_{p_i|n} (k, \alpha_i) \leq \sum_{p_i|n} \alpha_i,$$

i.e.,

$$\omega(n) \leq f_k(n) \leq \Omega(n) \text{ for all } n > 1. \tag{2}$$

Theorem 2.1. *For any integer $k \in \mathbb{Z}_{\geq 1}$, $f_k(n) = \omega(n)$ if and only if n is a square-free positive integer.*

Proof. Clearly, if n is a square-free positive integer, we have $n = \prod_{i=1}^{r} p_i^1$. i.e., $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 1$. Then for $k \in \mathbb{Z}_{\geq 1}$, it comes that

$$f_k(n) = \sum_{p_i|n} (k, 1) = \sum_{p_i|n} 1 = \omega(n).$$

Conversely, if n is a positive integer such that

$$\forall k \in \mathbb{Z}_{\geq 1}, \quad f_k(n) = \omega(n),$$

then $(k, \alpha_i) = 1$ for all $k \in \mathbb{Z}_{\geq 1}$, and that is true if $\alpha_i = 1$ for all $1 \leq i \leq r$, i.e., only if n is a square-free number. □

For a fixed integer $k \in \mathbb{Z}_{\geq 1}$ there is an infinity of positive integers n where $f_k(n) = \omega(n)$. For example if k an odd integer, then all $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ with α_i are even for all $1 \leq i \leq r$ the property is true.

Corollary 2.1. *For any integer $k \in \mathbb{Z}_{\geq 1}$, and for every integer $n > 1$*

$$f_k(\underline{mult}(n)) = \omega(n).$$

Theorem 2.2. *For any integer* $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, *such that* $\ell = \operatorname{lcm}(\alpha_1, \alpha_2, \ldots, \alpha_r)$, *the function* $f_k(n)$ *of the variable* k *is* ℓ*-periodic. In other words,*

$$f_{k+\ell}(n) = f_k(n), \textit{ for all } k \in \mathbb{Z}_{\geq 1}.$$

Proof. Let $\ell = \operatorname{lcm}(\alpha_1, \alpha_2, \ldots, \alpha_r)$. So for all α_i $(1 \leq i \leq r)$ there exists λ_i, such that $\ell = \lambda_i \alpha_i$. It follows that

$$(\alpha_i, k + \ell) = (\alpha_i, k + \lambda_i \alpha_i) = (\alpha_i, k) \quad (1 \leq i \leq r),$$

by this last property we get

$$f_{k+\ell}(n) = \sum_{p_i | n} (k + \ell, \alpha_i) = \sum_{p_i | n} (k, \alpha_i) = f_k(n). \qquad \square$$

Theorem 2.3. *For any integer* $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, *such that* $k = \gcd(\alpha_1, \alpha_2, \ldots, \alpha_r)$, *we have*

$$\frac{f_k(n)}{k} = \omega(n).$$

Proof. Firstly, as we have $k = \gcd(\alpha_1, \alpha_2, \ldots, \alpha_r)$ there exist r positive integers $\left(\alpha_1', \alpha_2', \ldots, \alpha_r'\right)$ such that $\alpha_i = k\alpha_i'$ $\ (1 \leq i \leq r)$. This shows that

$$(k, \alpha_i) = k \quad (1 \leq i \leq r),$$

from which, we have for every integer $n > 1$,

$$f_k(n) = \sum_{p_i | n} (k, \alpha_i) = \sum_{p_i | n} k = k\omega(n). \qquad \square$$

Theorem 2.4. *Let* k_1 *and* k_2 *be positive integers such that* k_1 *is a multiple of* k_2. *For any integer* $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ *such that* $\left(\frac{k_1}{k_2}, \alpha_i\right) = 1$ $(1 \leq i \leq r)$. *Then*

$$f_{k_1}(n) = f_{k_2}(n).$$

Proof. Since k_1 is a multiple of k_2, then $k_1 = dk_2$ where $d \geq 1$. If $(d, \alpha_i) = 1$ for all $1 \leq i \leq r$, it comes that

$$(k_1, \alpha_i) = (dk_2, \alpha_i) = (k_2, \alpha_i).$$

Thus

$$f_{k_1}(n) = \sum_{p_i | n} (k_1, \alpha_i) = \sum_{p_i | n} (k_2, \alpha_i) = f_{k_2}(n). \qquad \square$$

Theorem 2.5. *Let* $k \geq 1$ *be an integer. If* k *is odd, then for every even perfect number* n, *we have*

$$f_k(n) = 2,$$

and if k *is even, then for every odd perfect number* n *(if exists), there exists an integer* m *such that*

$$f_k(n) - f_{k/2}(m) = 1.$$

Proof. We know that every even perfect number n has the form $2^{p-1}(2^p-1)$ where (2^p-1) is a Mersenne prime (therefore, p is prime). So for a prime number p such that $n=2^{p-1}(2^p-1)$ is perfect, we have

$$\begin{aligned} f_k(n) &= f_k\left(2^{p-1}(2^p-1)\right) \\ &= f_k\left(2^{p-1}\right)+f_k(2^p-1) \\ &= (k,p-1)+1. \end{aligned}$$

The result comes directly if k is odd.

If n is an odd perfect number, then

$$n=p^{4Q+1}m^2,$$

where p is a prime number such that $p\equiv 1 \,(\mathrm{mod}\ 4)$ and does not divide Q. Then

$$\begin{aligned} f_k(n) &= f_k\left(p^{4Q+1}m^2\right) \\ &= f_k\left(p^{4Q+1}\right)+f_k\left(m^2\right) \\ &= (4Q+1,k)+\sum_{p^\alpha\|m}(k,2\alpha). \end{aligned}$$

So, if k even, then

$$\begin{aligned} f_k(n) &= 1+2\sum_{p^\alpha\|m}(k/2,\alpha) \\ &= 1+2f_{k/2}(m). \end{aligned}$$

□

3 Conclusion

In this paper we have defined a new additive arithmetic function related to a fixed integer and studied some of its properties.

We know that

$$\omega(n)\leq\frac{\log\tau(n)}{\log 2}\leq\Omega(n)\ \text{ for all } n\geq 1, \tag{3}$$

such that $\tau(n)$ is the number of divisors of n. So, according to (2) it is important to ask ourselves what is the amplitude of the difference

$$\left|f_k(n)-\frac{\log\tau(n)}{\log 2}\right|.$$

This is what can be taken care of later.

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