On a new additive arithmetic function related to a fixed integer

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Abstract: The main purpose of this paper is to define a new additive arithmetic function related to a fixed integer \( k \geq 1 \) and to study some of its properties. This function is given by

\[ f_k (1) = 0 \quad \text{and} \quad f_k (n) = \sum_{\rho^\alpha \mid n} (k, \alpha) , \]

such that \((a, b)\) denotes the greatest common divisor of the integers \(a\) and \(b\).

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1 Introduction

For all integers \( a, b \geq 1 \), we denote by \( \gcd(a, b) = (a, b) \) the largest common divisor of the integers \(a\) and \(b\). Let

\[ n = \prod_{i=1}^{r} p_i^{\alpha_i} \]

be the prime factorization of the positive integer \(n > 1\). In [1] Atanassov defined and studied the following function:
\[ \text{mult}(n) = \prod_{i=1}^{r} p_i, \quad \text{mult}(1) = 1, \]

and in [5], Andrei V. Shubin defined the following two additive arithmetic functions

\[
\Omega (k, n) = \sum_{p^\alpha \parallel n, \alpha \leq k} (k, \alpha) \quad \text{and} \quad \omega (k, n) = \sum_{p^\alpha \parallel n, \alpha > k} 1. \tag{1}
\]

The two functions \( \Omega(k, n) \) and \( \omega(k, n) \) are generalizations of the well-known functions \( \Omega(n) \) and \( \omega(n) \) respectively the number of prime divisors and the number of distinct prime divisors of \( n \). The definition and study of the properties of new arithmetic functions is a topic of interest to many researchers (see for example, [2–4]).

In this paper, a new additive arithmetic function will be defined and some of its basic properties are investigated.

## 2 Main results

Let \( k \) be a positive integer. Then we define \( f_k \) to be the arithmetic function such that \( f_k(1) = 0 \) and

\[ f_k(n) = \sum_{p^\alpha \parallel n} (k, \alpha). \]

We note that the function \( f_k(n) \) is equal to the function \( \omega(n) \) or to the function \( \Omega(n) \) for some particular cases of the integer \( k \).

Indeed, let \( n = \prod_{i=1}^{r} p_i^{\alpha_i} \). If \( k = 1 \), then \( (k, \alpha_i) = 1 \) for all \( (1 \leq i \leq r) \). Thus

\[ f_1(n) = \sum_{p_i \parallel n} (1, \alpha_i) = \sum_{p_i \parallel n} 1 = \omega(n), \quad \text{for all } n, \]

and if \( \alpha_i \geq 2 \) for all \( 1 \leq i \leq r \), then

\[ f_1(n) = \sum_{p_i \parallel n, \alpha_i > 1} 1 = \omega(1, n). \]

If \( k = \text{lcm}(\alpha_1, \alpha_2, \ldots, \alpha_r) \), then \( (k, \alpha_i) = \alpha_i \) for all \( (1 \leq i \leq r) \). Thus

\[ f_k(n) = \sum_{p_i \parallel n} (k, \alpha_i) = \sum_{p_i \parallel n} \alpha_i = \Omega(n), \]

and it can also be noticed that \( \alpha_i \leq k \), then

\[ f_k(n) = \sum_{p_i \parallel n, \alpha_i \leq k} \alpha_i = \Omega(k, n). \]
Let \( m \) be a positive integer such that \( m = \prod_{j=1}^{s} q_j^{\beta_j} \), its canonical decomposition. If \((m, n) = 1\) (i.e., \(q_j \neq p_i\) for all \(1 \leq i \leq r\) and \(1 \leq j \leq s\)), then for all \(k \in \mathbb{Z}_{\geq 1}\)

\[
f_k(nm) = \sum_{p_i|n} (k, \alpha_i) + \sum_{q_j|n} (k, \beta_j) = f_k(n) + f_k(m).
\]

On the other hand, if \(p_1, p_2\), and \(p_3\) are different primes, then for all \(k \in \mathbb{Z}_{\geq 1}\):

\[
f_k(p_1p_2^{k+1}p_3) = (k, 1) + (k, k + 1) + (k, 1) = 3,
\]

while

\[
f_k(p_1p_2^2) + f_k(p_2p_3) = (k, 1) + (k, k) + (k, 1) + (k, 1) = k + 3.
\]

Therefore, it can be shown that the function \(f_k\) is additive but not completely additive.

We know that for all \(\alpha_i\) \((1 \leq i \leq r)\) and for all \(k \in \mathbb{Z}_{\geq 1}\), we have

\[
1 \leq (k, \alpha_i) \leq \alpha_i,
\]

which implies that

\[
\sum_{p_i|n} 1 \leq \sum_{p_i|n} (k, \alpha_i) \leq \sum_{p_i|n} \alpha_i,
\]

i.e.,

\[
\omega(n) \leq f_k(n) \leq \Omega(n) \quad \text{for all } n > 1.
\] (2)

**Theorem 2.1.** For any integer \(k \in \mathbb{Z}_{\geq 1}\), \(f_k(n) = \omega(n)\) if and only if \(n\) is a square-free positive integer.

**Proof.** Clearly, if \(n\) is a square-free positive integer, we have \(n = \prod_{i=1}^{r} p_i^{\alpha_i}\), i.e., \(\alpha_1 = \alpha_2 = \cdots = \alpha_r = 1\). Then for \(k \in \mathbb{Z}_{\geq 1}\), it comes that

\[
f_k(n) = \sum_{p_i|n} (k, 1) = \sum_{p_i|n} 1 = \omega(n).
\]

Conversely, if \(n\) is a positive integer such that

\[
\forall k \in \mathbb{Z}_{\geq 1}, \quad f_k(n) = \omega(n),
\]

then \((k, \alpha_i) = 1\) for all \(k \in \mathbb{Z}_{\geq 1}\), and that is true if \(\alpha_i = 1\) for all \(1 \leq i \leq r\), i.e., only if \(n\) is a square-free number.

For a fixed integer \(k \in \mathbb{Z}_{\geq 1}\) there is an infinity of positive integers \(n\) where \(f_k(n) = \omega(n)\). For example if \(k\) an odd integer, then all \(n = \prod_{i=1}^{r} p_i^{\alpha_i}\) with \(\alpha_i\) are even for all \(1 \leq i \leq r\) the property is true.

**Corollary 2.1.** For any integer \(k \in \mathbb{Z}_{\geq 1}\), and for every integer \(n > 1\)

\[
f_k(\text{mult}(n)) = \omega(n).
\]
Theorem 2.2. For any integer $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, such that $\ell = \text{lcm} (\alpha_1, \alpha_2, \ldots, \alpha_r)$, the function $f_k(n)$ of the variable $k$ is $\ell$-periodic. In other words,

$$f_{k+\ell}(n) = f_k(n), \text{ for all } k \in \mathbb{Z}_{\geq 1}.$$

Proof. Let $\ell = \text{lcm} (\alpha_1, \alpha_2, \ldots, \alpha_r)$. So for all $\alpha_i (1 \leq i \leq r)$ there exists $\lambda_i$, such that $\ell = \lambda_i \alpha_i$. It follows that

$$(\alpha_i, k + \ell) = (\alpha_i, k + \lambda_i \alpha_i) = (\alpha_i, k) \quad (1 \leq i \leq r),$$

by this last property we get

$$f_{k+\ell}(n) = \sum_{p_i \mid n} (k + \ell, \alpha_i) = \sum_{p_i \mid n} (k, \alpha_i) = f_k(n). \quad \square$$

Theorem 2.3. For any integer $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, such that $k = \text{gcd} (\alpha_1, \alpha_2, \ldots, \alpha_r)$, we have

$$\frac{f_k(n)}{k} = \omega(n).$$

Proof. Firstly, as we have $k = \text{gcd} (\alpha_1, \alpha_2, \ldots, \alpha_r)$ there exist $r$ positive integers $(\alpha_1', \alpha_2', \ldots, \alpha_r')$ such that $\alpha_i = k \alpha_i' \quad (1 \leq i \leq r)$. This shows that

$$(k, \alpha_i) = k \quad (1 \leq i \leq r),$$

from which, we have for every integer $n > 1$,

$$f_k(n) = \sum_{p_i \mid n} (k, \alpha_i) = \sum_{p_i \mid n} k = k \omega(n). \quad \square$$

Theorem 2.4. Let $k_1$ and $k_2$ be positive integers such that $k_1$ is a multiple of $k_2$. For any integer $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ such that $\left( \frac{k_1}{k_2}, \alpha_i \right) = 1 \quad (1 \leq i \leq r)$. Then

$$f_{k_1}(n) = f_{k_2}(n).$$

Proof. Since $k_1$ is a multiple of $k_2$, then $k_1 = dk_2$ where $d \geq 1$. If $(d, \alpha_i) = 1$ for all $1 \leq i \leq r$, it comes that

$$(k_1, \alpha_i) = (dk_2, \alpha_i) = (k_2, \alpha_i).$$

Thus

$$f_{k_1}(n) = \sum_{p_i \mid n} (k_1, \alpha_i) = \sum_{p_i \mid n} (k_2, \alpha_i) = f_{k_2}(n). \quad \square$$

Theorem 2.5. Let $k \geq 1$ be an integer. If $k$ is odd, then for every even perfect number $n$, we have

$$f_k(n) = 2,$$

and if $k$ is even, then for every odd perfect number $n$ (if exists), there exists an integer $m$ such that

$$f_k(n) - f_{k/2}(m) = 1.$$
Proof. We know that every even perfect number \( n \) has the form \( 2^{p-1} (2^p - 1) \) where \( (2^p - 1) \) is a Mersenne prime (therefore, \( p \) is prime). So for a prime number \( p \) such that \( n = 2^{p-1} (2^p - 1) \) is perfect, we have

\[
\begin{align*}
f_k (n) &= f_k (2^{p-1} (2^p - 1)) \\
&= f_k (2^{p-1}) + f_k (2^p - 1) \\
&= (k, p - 1) + 1.
\end{align*}
\]

The result comes directly if \( k \) is odd.

If \( n \) is an odd perfect number, then

\[ n = p^{4Q+1}m^2, \]

where \( p \) is a prime number such that \( p \equiv 1 \pmod{4} \) and does not divide \( Q \). Then

\[
\begin{align*}
f_k (n) &= f_k (p^{4Q+1}m^2) \\
&= f_k (p^{4Q+1}) + f_k (m^2) \\
&= (4Q + 1, k) + \sum_{p^\alpha||m} (k, 2\alpha).
\end{align*}
\]

So, if \( k \) even, then

\[
\begin{align*}
f_k (n) &= 1 + 2 \sum_{p^\alpha||m} (k/2, \alpha) \\
&= 1 + 2f_{k/2} (m).
\end{align*}
\]

3 Conclusion

In this paper we have defined a new additive arithmetic function related to a fixed integer and studied some of its properties.

We know that

\[
\omega (n) \leq \frac{\log \tau (n)}{\log 2} \leq \Omega (n) \quad \text{for all } n \geq 1,
\]

such that \( \tau (n) \) is the number of divisors of \( n \). So, according to (2) it is important to ask ourselves what is the amplitude of the difference

\[
\left| f_k (n) - \frac{\log \tau (n)}{\log 2} \right|.
\]

This is what can be taken care of later.

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References


