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Identities involving some special numbers and polynomials on p-adic integral

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Abstract: In this paper, we get new identities involving Bernoulli, Daehee and Stirling numbers, and their representations by using p-adic integrals and combinatorial techniques.
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1 Introduction

Many interesting identities have been derived by using p-adic integrals for representations by Bernoulli, Stirling and Daehee numbers and polynomials. There are also studies involving degenerate types and q-representations of these numbers and polynomials (see [4,5,7–15,18,19]).

The q-calculus plays an important role in number theory, combinatorics and other branches of mathematics. The q-calculus was first examined by Euler [2], and the subject is still of current interest. Let p be an odd prime number. We use $\mathbb{Z}_p, \mathbb{Q}_p$, and $\mathbb{C}_p$ to denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|.|_p$ is normalized by $|p|_p = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{\frac{-1}{p-1}}$. The q-extension of number x, denoted by $[x]_q$, is

$$[x]_q = \frac{1-q^x}{1-q}.$$

It is clear that $\lim_{q\to 1}[x]_q = x$. Let f be a uniformly differentiable function on $\mathbb{Z}_p$. Then p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim in [12]:

$$\begin{aligned} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_q(x+p^N\mathbb{Z}_p) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x. \end{aligned} \tag{1}$$

Letting $q \to 0$ and $q \to -1$ in (1), the authors get

$$\int_{\mathbb{Z}_p} f(x+1)\, d\mu_0(x) - \int_{\mathbb{Z}_p} f(x)\, d\mu_0(x) = f'(0), \tag{2}$$

and

$$\int_{\mathbb{Z}_p} f(x+1)\, d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = 2f(0), \tag{3}$$

respectively. More generally, from (1),

$$q\int_{\mathbb{Z}_p} f(x+1)\, d\mu_q(x) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) + (q-1) f(0) + \frac{q-1}{\log q} f'(0), \tag{4}$$

where f' is the derivative of f with respect to x. The q-Bernoulli numbers, denoted by $B_{n,q}$, are defined with the help of $p-$adic integrals as follows:

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_q(x) = \sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{n!}. \tag{5}$$

From (4) and (5), the generating function of the q-Bernoulli numbers is

$$\frac{q-1+\frac{q-1}{\log q}t}{qe^t-1} = \sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{n!}. \tag{6}$$

The Carlitz's degenerate Bernoulli polynomials of order r, denoted by $\beta_n^{(r)}(x|\lambda)$, are defined via their generating function as follows:

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n^{(r)}(x|\lambda)\frac{t^n}{n!}. \tag{7}$$

The Daehee numbers, denoted by D_n, are defined via their generating function as follows:

$$\frac{\log(1+t)}{t}=\sum_{n=0}^{\infty}D_n\frac{t^n}{n!}. \tag{8}$$

It is clear that

$$D_0=1, D_1=-\frac{1}{2},\ldots,D_n=(-1)^n\frac{n!}{n+1}\ \ (n=1,2,\cdots). \tag{9}$$

The higher-order Daehee polynomials [6] are defined via their generating functions as follows:

$$\left(\frac{\log(1+t)}{t}\right)^r(1+t)^x=\sum_{n=0}^{\infty}D_n^{(r)}(x)\frac{t^n}{n!}.$$

Also, $D_n^{(r)}(x)$ can be given with the help of p-adic integral as follows [19]:

$$D_n^{(r)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+x_1+\cdots+x_r)_n d\mu_0(x_1)\cdots d\mu_0(x_r). \tag{10}$$

Specially, when $r=1$, $D_n^{(1)}(x)=D_n(x)$ are called the Daehee polynomials. J. W. Park generalized the q-Daehee polynomials, to what called the $\lambda-q-$Daehee polynomials, as follows [18]:

$$\frac{q-1+\frac{q-1}{\log q}\lambda\log(1+t)}{q(1+t)^{\lambda}-1}(1+t)^x=\sum_{n=0}^{\infty}D_{n,\lambda,q}(x)\frac{t^n}{n!}. \tag{11}$$

When $\lambda=1$ and $x=0$, $D_{n,1,q}(0)=D_{n,q}$ are called the q-Daehee numbers.
In [3], for every $\alpha\in\mathbb{R}^+$, the generalized harmonic numbers, denoted by $H_n(\alpha)$, are defined by

$$H_0(\alpha)=0,\ \ H_n(\alpha)=\sum_{k=1}^{n}\frac{1}{k\alpha^k}$$

for $n=1,2,\cdots$. When $\alpha=1$, the usual harmonic numbers are $H_n(1)=H_n$ for $n=0,1,2,\cdots$.
In [17], the authors showed that for $n\in\mathbb{Z}^+$ and $1<q$,

$$H_n\left(\frac{q}{q-1}\right)=\log q\frac{(1-q)^n}{q^n}\left(\frac{q^n}{(1-q)^n}-\frac{D_{n,q}}{n!}\right). \tag{12}$$

In [16], from the generalized harmonic numbers $H_n(\alpha)$, the authors defined the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$ by

$$H_n^r(\alpha)=\begin{cases}\sum_{k=1}^{n}H_k^{r-1}(\alpha) & \text{if } n,r\geq 1,\\ \frac{1}{n\alpha^n} & \text{if } r=0 \text{ and } n>0,\\ 0 & \text{if } r<0 \text{ or } n\leq 0,.\end{cases}$$

Specifically, when $r=1$, $H_n^1(\alpha)=H_n(\alpha)$. The generating function of these numbers is

$$\frac{-\log\left(1-\frac{t}{\alpha}\right)}{(1-t)^r}=\sum_{n=0}^{\infty}H_n^r(\alpha)t^n. \tag{13}$$

In [1], the authors defined the generalized harmonic numbers of rank r, denoted by $H(n,r,\alpha)$, for $n\geq 1$ and $r\geq 0$,

$$H(n,r,\alpha)=\sum_{1\leq n_0+n_1+\cdots+n_r\leq n}\frac{1}{n_0n_1\cdots n_r\alpha^{n_0+n_1+\cdots+n_r}}$$

and their generating function is

$$\frac{\left(-\log\left(1-\frac{t}{\alpha}\right)\right)^{r+1}}{1-t}=\sum_{n=0}^{\infty}H(n,r,\alpha)\,t^n. \tag{14}$$

The derangement numbers, denoted by d_n, are defined via their generating functions as follows:

$$\frac{e^{-t}}{1-t}=\sum_{n=0}^{\infty}d_n\frac{t^n}{n!}. \tag{15}$$

It is well known that Stirling numbers play an important role in combinatorial analysis. The Stirling numbers of the first kind $S(n,k)$ are defined by

$$x^{\underline{n}}=\sum_{k=0}^{n}S(n,k)x^k,$$

where for $n\geq 0$, $S(n,0)=\delta_{n0}$, δ_{ni} is the Kronecker delta and $x^{\underline{n}}$ stands for the falling factorial defined by $x^{\underline{n}}=x(x-1)\cdots(x-n+1)$. The generating function of these numbers is

$$\frac{\left(\log(1+t)\right)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{t^n}{n!}. \tag{16}$$

2 Identities are obtained by p-adic integral

In this section, we will give new identities involving Bernoulli, Daehee, Stirling numbers and their representations by using p-adic integral and combinatorial techniques.

Theorem 2.1. *For non-negative integer n and positive integer r, we have*

$$\begin{aligned}\frac{D_n(x)}{n!}&=\sum_{i=0}^{n}\sum_{k=0}^{i}(-1)^{n+i}\binom{r}{i-k}\binom{x}{k}\alpha^{n-k+1}H_{n-i+1}^{r}(\alpha)\\&=\sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^{i}\binom{r}{k-i}\frac{\beta_{n-k}^{(r)}(x|1)\,\alpha^{k+1}}{(n-k)!}H_{i+1}^{r}(\alpha).\end{aligned}$$

Proof. If we let $f(x)=\left(1-\frac{t}{\alpha}\right)^{x+y}$ in (2), it can be seen that

$$\int_{\mathbb{Z}_p}\left(1-\frac{t}{\alpha}\right)^{x+y}d\mu_0(y)=\frac{\log\left(1-\frac{t}{\alpha}\right)}{-\frac{t}{\alpha}}\left(1-\frac{t}{\alpha}\right)^{x}.$$

Using the binomial theorem, (7) and (13), we see that

$$\begin{aligned}\int_{\mathbb{Z}_p}\left(1-\frac{t}{\alpha}\right)^{x+y}d\mu_0\left(y\right)&=\alpha\frac{-\log\left(1-\frac{t}{\alpha}\right)}{\left(1-t\right)^r t}(1-t)^r\left(1-\frac{t}{\alpha}\right)^x\\&=\alpha\sum_{n=0}^{\infty}H_{n+1}^r\left(\alpha\right)t^n\left(1-t\right)^r\sum_{k=0}^{\infty}\left(-1\right)^k\beta_k^{(r)}\left(x|1\right)\frac{t^k}{\alpha^k k!}\\&=\alpha\sum_{n=0}^{\infty}H_{n+1}^r\left(\alpha\right)t^n\sum_{i=0}^{\infty}\left(-1\right)^i\binom{r}{i}t^i\sum_{k=0}^{\infty}\left(-1\right)^k\beta_k^{(r)}\left(x|1\right)\frac{t^k}{\alpha^k k!}\\&=\alpha\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}\left(-1\right)^{n-i}\binom{r}{k-i}H_{i+1}^r\left(\alpha\right)\frac{\beta_{n-k}^{(r)}\left(x|1\right)}{\alpha^{n-k}\left(n-k\right)!}t^n.\end{aligned}\tag{17}$$

Also, by the binomial theorem and (13), we have

$$\begin{aligned}\int_{\mathbb{Z}_p}\left(1-\frac{t}{\alpha}\right)^{x+y}d\mu_0\left(y\right)&=\alpha\frac{-\log\left(1-\frac{t}{\alpha}\right)}{\left(1-t\right)^r t}\left(1-t\right)^r\left(1-\frac{t}{\alpha}\right)^x\\&=\alpha\sum_{n=0}^{\infty}H_{n+1}^r\left(\alpha\right)t^n\sum_{i=0}^{\infty}\left(-1\right)^i\binom{r}{i}t^i\sum_{k=0}^{\infty}\frac{\left(-1\right)^k}{\alpha^k}\binom{x}{k}t^k\\&=\alpha\sum_{n=0}^{\infty}H_{n+1}^r\left(\alpha\right)t^n\sum_{i=0}^{\infty}\sum_{k=0}^{i}\frac{\left(-1\right)^i}{\alpha^k}\binom{r}{i-k}\binom{x}{k}t^i\\&=\alpha\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{k=0}^{i}\frac{\left(-1\right)^i}{\alpha^k}\binom{r}{i-k}\binom{x}{k}H_{n-i+1}^r\left(\alpha\right)t^n.\end{aligned}\tag{18}$$

On the other hand, the binomial theorem and (10) yield that

$$\begin{aligned}\int_{\mathbb{Z}_p}\left(1-\frac{t}{\alpha}\right)^{x+y}d\mu_0\left(y\right)&=\sum_{n=0}^{\infty}\left(-1\right)^n\frac{t^n}{\alpha^n}\int_{\mathbb{Z}_p}\binom{x+y}{n}d\mu_0\left(y\right)\\&=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n}{\alpha^n}\frac{D_n\left(x\right)}{n!}t^n.\end{aligned}\tag{19}$$

Thus, comparing the coefficients of t^n on right sides of (17)–(19), the proof is complete. □

Theorem 2.2. *For a non-negative integer* n *and* $q\in\mathbb{R}^+\setminus(0,1]$, *we have that*

$$\sum_{k=0}^{n}\frac{\left(-1\right)^k B_{k,q}}{k!}H\left(n,k-1,\alpha\right)$$
$$=\frac{\alpha^{n+1}\left(q-1\right)^{n+1}-q^{n+1}+\frac{1}{\log q}\left(q^{n+1}H_n\left(\frac{q}{q-1}\right)+\alpha^{n+1}\left(q-1\right)^{n+1}H_n\left(\alpha\right)\right)}{\alpha^n\left(q-1\right)^n\left(\alpha\left(q-1\right)-q\right)}.$$

Proof. Taking $f(x) = e^{x\log\left(1-\frac{t}{\alpha}\right)}$ in (4), by (6), we have

$$\int_{\mathbb{Z}_p} e^{x\log\left(1-\frac{t}{\alpha}\right)} d\mu_q(x) = \frac{q-1+\frac{q-1}{\log q}\log\left(1-\frac{t}{\alpha}\right)}{q\left(1-\frac{t}{\alpha}\right)-1} = \sum_{k=0}^{\infty} B_{k,q} \frac{(\log\left(1-\frac{t}{\alpha}\right))^k}{k!},$$

and by (14),

$$\begin{aligned}
\frac{1}{1-t}\int_{\mathbb{Z}_p} e^{x\log\left(1-\frac{t}{\alpha}\right)} d\mu_q(x) &= \sum_{k=0}^{\infty} \frac{B_{k,q}}{k!} \frac{\left(\log\left(1-\frac{t}{\alpha}\right)\right)^k}{1-t} \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{B_{k,q}}{k!} \sum_{n=0}^{\infty} H(n,k-1,\alpha)\, t^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} (-1)^k \frac{B_{k,q}}{k!} H(n,k-1,\alpha)\, t^n. \qquad (20)
\end{aligned}$$

So, (13) yields that

$$\begin{aligned}
&\frac{1}{1-t}\int_{\mathbb{Z}_p} e^{x\log\left(1-\frac{t}{\alpha}\right)} d\mu_q(x) \\
&= \frac{1}{1-t} \frac{q-1+\frac{q-1}{\log q}\log\left(1-\frac{t}{\alpha}\right)}{q\left(1-\frac{t}{\alpha}\right)-1} \\
&= \frac{1}{1-t}\sum_{n=0}^{\infty} \frac{q^n}{\alpha^n (q-1)^n} t^n \left(1+\frac{1}{\log q}\log\left(1-\frac{t}{\alpha}\right)\right) \\
&= \sum_{n=0}^{\infty} \frac{q^n}{\alpha^n (q-1)^n} t^n \left(\frac{1}{1-t}+\frac{1}{\log q}\frac{\log\left(1-\frac{t}{\alpha}\right)}{1-t}\right) \\
&= \sum_{n=0}^{\infty} \frac{q^n}{\alpha^n (q-1)^n} t^n \left(\sum_{k=0}^{\infty} t^k - \frac{1}{\log q}\sum_{i=0}^{\infty} H_i(\alpha)\, t^i\right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n}\left(\frac{q}{\alpha(q-1)}\right)^k - \frac{1}{\log q}\sum_{k=0}^{n}\left(\frac{q}{\alpha(q-1)}\right)^{n-k} H_k(\alpha)\right) t^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n}\left(\frac{q}{\alpha(q-1)}\right)^k - \frac{1}{\log q}\sum_{k=0}^{n}\left(\frac{q}{\alpha(q-1)}\right)^{n-k} \sum_{i=1}^{k}\frac{1}{i\alpha^i}\right) t^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n}\left(\frac{q}{\alpha(q-1)}\right)^k \right. \\
&\quad \left. - \frac{1}{\log q}\left(\frac{q}{\alpha(q-1)}\right)^{n} \sum_{i=1}^{n}\frac{1}{i\alpha^i}\sum_{k=i}^{n}\left(\frac{\alpha(q-1)}{q}\right)^k\right) t^n. \qquad (21)
\end{aligned}$$

From the power series identities in (20) and (21), finite geometric series and (12), we see that

$$
\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\frac{B_{k,q}}{k!}H(n,k-1,\alpha)\\
&\quad=\sum_{k=0}^{n}\left(\frac{q}{\alpha(q-1)}\right)^k-\frac{1}{\log q}\left(\frac{q}{\alpha(q-1)}\right)^n\sum_{i=0}^{n}\frac{1}{i\alpha^i}\sum_{k=i}^{n}\left(\frac{\alpha(q-1)}{q}\right)^k\\
&\quad=\frac{1-\left(\frac{q}{\alpha(q-1)}\right)^{n+1}}{1-\frac{q}{\alpha(q-1)}}-\frac{1}{\log q}\frac{1}{1-\left(\frac{\alpha(q-1)}{q}\right)}\left(\frac{q}{\alpha(q-1)}\right)^n\\
&\quad\quad\times\left(\sum_{i=1}^{n}\frac{1}{i}\left(\frac{q-1}{q}\right)^i+\left(\frac{\alpha(q-1)}{q}\right)^{n+1}\sum_{i=1}^{n}\frac{1}{i\alpha^i}\right)\\
&\quad=\frac{\alpha^{n+1}(q-1)^{n+1}-q^{n+1}+\frac{1}{\log q}\left(q^{n+1}H_n\left(\frac{q}{q-1}\right)+\alpha^{n+1}(q-1)^{n+1}H_n(\alpha)\right)}{\alpha^n(q-1)^n(\alpha(q-1)-q)},
\end{aligned}
$$

as claimed. □

Theorem 2.3. *For non-negative integers n, r, and $q\in\mathbb{R}^+\setminus\{1\}$, we have*

$$
\sum_{k=0}^{n}\sum_{i=0}^{n-k}(-1)^{k-i}\binom{r}{n-k-i}\frac{q^k}{\alpha^k(q-1)^k}H_i^r(\alpha)=\frac{\log q}{\alpha^n}\left(\frac{q^n}{(1-q)^n}-\frac{D_{n,q}}{n!}\right).
$$

Proof. Using $f(x)=\left(1-\frac{t}{\alpha}\right)^x$ in (4), observe that

$$
\int_{\mathbb{Z}_p}\left(1-\frac{t}{\alpha}\right)^x d\mu_q(x)=\frac{q-1}{q\left(1-\frac{t}{\alpha}\right)-1}\left(\frac{1}{\log q}\log\left(1-\frac{t}{\alpha}\right)+1\right). \tag{22}
$$

From (13) and the binomial theorem, we have

$$
\begin{aligned}
\int_{\mathbb{Z}_p}\left(1-\frac{t}{\alpha}\right)^x d\mu_q(x)&=\frac{q-1}{q\left(1-\frac{t}{\alpha}\right)-1}\left(\frac{1}{\log q}\frac{\log\left(1-\frac{t}{\alpha}\right)}{(1-t)^r}(1-t)^r+1\right)\\
&=\sum_{n=0}^{\infty}\frac{q^n}{\alpha^n(q-1)^n}t^n\left(-\frac{1}{\log q}\sum_{k=0}^{\infty}H_n^r(\alpha)t^n\sum_{i=0}^{\infty}(-1)^i\binom{r}{i}t^i+1\right)\\
&=\sum_{n=0}^{\infty}\frac{q^n}{\alpha^n(q-1)^n}t^n\left(-\frac{1}{\log q}\sum_{k=0}^{\infty}\sum_{i=0}^{k}H_i^r(\alpha)(-1)^{k-i}\binom{r}{k-i}t^k+1\right)\\
&=\sum_{n=0}^{\infty}\left(\frac{-1}{\log q}\sum_{k=0}^{n}\sum_{i=0}^{n-k}(-1)^{n-k-i}\binom{r}{n-k-i}\frac{q^k}{\alpha^k(q-1)^k}H_i^r(\alpha)\right.\\
&\qquad\left.+\frac{q^n}{\alpha^n(q-1)^n}\right)t^n.
\end{aligned}\tag{23}
$$

Also, by (8), (9), (12), and (22), we see that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \left(1-\frac{t}{\alpha}\right)^x d\mu_q(x) \\
&= \sum_{n=0}^{\infty} \frac{q^n}{\alpha^n (q-1)^n} t^n \left(\frac{1}{\log q} \sum_{k=1}^{\infty} (-1)^k D_{k-1} \frac{t^k}{\alpha^k (k-1)!} + 1 \right) \\
&= \frac{1}{\log q} \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} (-1)^{n-k} \frac{q^k}{\alpha^k (q-1)^k} \frac{D_{n-k-1}}{\alpha^{n-k} (n-k-1)!} t^n + \sum_{n=0}^{\infty} \frac{q^n}{\alpha^n (q-1)^n} t^n \\
&= \frac{1}{\log q} \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} (-1)^{k+1} \frac{q^{n-k-1}}{\alpha^n (q-1)^{n-k-1}} \frac{D_k}{k!} t^n + \sum_{n=0}^{\infty} \frac{q^n}{\alpha^n (q-1)^n} t^n \\
&= \frac{-1}{\log q} \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \frac{q^{n-k-1}}{\alpha^n (q-1)^{n-k-1}} \frac{1}{k+1} t^n + \sum_{n=0}^{\infty} \frac{q^n}{\alpha^n (q-1)^n} t^n \\
&= \frac{-1}{\log q} \sum_{n=0}^{\infty} \sum_{k=1}^{n} \frac{q^{n-k}}{\alpha^n (q-1)^{n-k}} \frac{1}{k} t^n + \sum_{n=0}^{\infty} \frac{q^n}{\alpha^n (q-1)^n} t^n \\
&= \sum_{n=0}^{\infty} \left(\frac{-1}{\log q} \frac{q^n}{\alpha^n (q-1)^n} H_n\left(\frac{q}{q-1}\right) + \frac{q^n}{\alpha^n (q-1)^n} \right) t^n \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{D_{n,q}}{\alpha^n n!} t^n. \qquad (24)
\end{aligned}$$

From (23) and (24), we get the desired result. □

Theorem 2.4. *For non-negative integers n and r, we have*

$$\sum_{i=0}^{n} \sum_{m=0}^{i} \sum_{k=0}^{n-i} \left((-1)^r H(m, r-1, \alpha) d_k \frac{\alpha^{m-n} + \alpha^{1+2i-m-n}}{\alpha+1} - (-1)^m \frac{S(m,r)}{m!} r! k! \alpha^{2i-2m-n} \right) \frac{(-1)^{k+i} S(n-i,k)}{(n-i)!} = 0.$$

Proof. Letting $f(x) = \left(\log\left(1-\frac{t}{\alpha}\right)\right)^{x+r}$ in (3), it is easily shown that

$$\int_{\mathbb{Z}_p} \left(\log\left(1-\frac{t}{\alpha}\right)\right)^{x+r} d\mu_{-1}(x) = \frac{2\left(\log\left(1-\frac{t}{\alpha}\right)\right)^r}{\log\left(1-\frac{t}{\alpha}\right)+1}. \qquad (25)$$

From (14)–(16), we have

$$\begin{aligned}
&\frac{1}{\alpha t - 1} \int_{\mathbb{Z}_p} \left(\log\left(1-\frac{t}{\alpha}\right)\right)^{x+r} d\mu_{-1}(x) \\
&= 2 \frac{1-\frac{t}{\alpha}}{\log\left(1-\frac{t}{\alpha}\right)+1} \frac{\left(\log\left(1-\frac{t}{\alpha}\right)\right)^r}{1-t} \frac{1-t}{1-\frac{t}{\alpha}} \frac{1}{\alpha t-1} \\
&= -2 \sum_{k=0}^{\infty} d_k \frac{\left(-\log\left(1-\frac{t}{\alpha}\right)\right)^k}{k!} \sum_{n=0}^{\infty} (-1)^r H(n, r-1, \alpha) t^n \sum_{m=0}^{\infty} \frac{\alpha^{-m} + \alpha^{1+m}}{\alpha+1} t^m
\end{aligned}$$

$$= -2\sum_{k=0}^{\infty}(-1)^k d_k \sum_{i=k}^{\infty}(-1)^i S(i,k)\frac{t^i}{\alpha^i i!}$$

$$\times \sum_{n=0}^{\infty}(-1)^r H(n,r-1,\alpha)\, t^n \sum_{m=0}^{\infty}\frac{\alpha^{-m}+\alpha^{1+m}}{\alpha+1}t^m$$

$$= -2\sum_{i=0}^{\infty}\sum_{k=0}^{i}(-1)^{k+i} d_k \frac{S(i,k)}{\alpha^i i!}t^i$$

$$\times \sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^r H(m,r-1,\alpha)\frac{\alpha^{-n+m}+\alpha^{1+n-m}}{\alpha+1}t^n$$

$$= -2\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{m=0}^{i}\sum_{k=0}^{n-i}(-1)^{k+r+n-i} d_k H(m,r-1,\alpha)\, S(n-i,k)$$

$$\times \frac{\alpha^{-i+m}+\alpha^{1+i-m}}{\alpha+1}\frac{t^n}{\alpha^{n-i}(n-i)!}. \tag{26}$$

On the other hand, (16) and (25) yield that

$$\frac{1}{1-\alpha t}\int_{\mathbb{Z}_p}\left(\log\left(1-\frac{t}{\alpha}\right)\right)^{x+r} d\mu_{-1}(x)$$

$$= -\,r!\frac{\left(\log\left(1-\frac{t}{\alpha}\right)\right)^r}{r!}\frac{1}{1-\alpha t}\int_{\mathbb{Z}_p}\left(\log\left(1-\frac{t}{\alpha}\right)\right)^{x} d\mu_{-1}(x)$$

$$= -\,2r!\sum_{n=0}^{\infty}(-1)^n S(n,r)\frac{t^n}{\alpha^n n!}\sum_{n=0}^{\infty}\alpha^n t^n\frac{1}{\log\left(1-\frac{t}{\alpha}\right)+1}$$

$$= -\,2r!\sum_{n=0}^{\infty}(-1)^n S(n,r)\frac{t^n}{\alpha^n n!}\sum_{n=0}^{\infty}\alpha^n t^n\sum_{k=0}^{\infty}k!(-1)^k\frac{\left(\log\left(1-\frac{t}{\alpha}\right)\right)^k}{k!}$$

$$= -\,2r!\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{m=0}^{i}\sum_{k=0}^{n-i}k!(-1)^k(-1)^{m+n-i}\frac{S(n-i,k)}{\alpha^{n-i}(n-i)!}\frac{S(m,r)}{\alpha^m m!}\alpha^{i-m}t^n. \tag{27}$$

By (26) and (27), the result is proved. □

We have the following corollary by taking $\alpha = 1$ in Theorem 2.4.

Corollary 2.1. *For non-negative integers n and r, we have*

$$\sum_{i=0}^{n}\sum_{m=0}^{i}\sum_{k=0}^{n-i}\left((-1)^r H(m,r-1)\, d_k-(-1)^m\frac{S(m,r)}{m!}r!k!\right)(-1)^{k+i}\frac{S(n-i,k)}{(n-i)!}=0.$$

For example, when $r = 1$ in Corollary 2.1, we write

$$\sum_{i=0}^{n}\sum_{m=0}^{i}\sum_{k=0}^{n-i}\left(H_m d_k+\frac{(-1)^m k!}{m}\right)\frac{(-1)^{k+i}S(n-i,k)}{(n-i)!}=0.$$

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References

[1] Duran, Ö., Ömür, N., & Koparal, S. (2020). On sums with generalized harmonic, hyperharmonic and special numbers. *Miskolc Mathematical Notes*, 21(2), 791–803.

[2] Euler, L. (1748). *Introductio in Analysin in Nitorum*. Apud Marcum-Michaelem Bousquet & Socios.

[3] Genčev, M. (2011). Binomial sums involving harmonic numbers. *Mathematica Slovaca*, 61(2), 215–226.

[4] Jang, L.-C., Kim, D. S., Kim, T. & Lee, H. (2020). p-adic integral on $\mathbb{Z}_p$ associated with degenerate Bernoulli polynomials of the second kind. *Advances in Difference Equations*, Article ID 278.

[5] Kim, T., Kim, D. S., Jang, L.-C., Lee, H. & Kim, H. (2022). Representations of degenerate Hermite polynomials. *Advances in Applied Mathematics*, 139, Article ID 102359.

[6] Kim, D. S., Kim, T., Lee, S. H., & Seo, J. J. (2014). Higher-order Daehee numbers and polynomials. *International Journal of Mathematical Analysis*, 8(6), 273–283.

[7] Kim, D. S., Kim, T., Kwon, J., Lee, S.-H., & Park, S. (2021). On λ-linear functionals arising from p-adic integrals on $\mathbb{Z}_p$. *Advances in Continuous and Discrete Models*, 2021, Article ID 479.

[8] Kim, T. (2007). A note on p-adic q-integral on $\mathbb{Z}_p$ associated with q-Euler numbers. *Advanced Studies in Contemporary Mathematics*, 15, 133–137.

[9] Kim, T. (2006). A note on some formulae for the q-Euler numbers and polynomials. *Proceedings of the Jangjeon Mathematical Society*, 9(2), 227–232.

[10] Kim, T. (2012). Lebesgue–Radon–Nikodym theorem with respect to fermionic p-adic invariant measure on $\mathbb{Z}_p$. *Russian Journal of Mathematical Physics*, 19(2), 193–196.

[11] Kim, T. (2016). On degenerate q-Bernoulli polynomials. *Bulletin of the Korean Mathematical Society*, 53(4), 1149–1156.

[12] Kim, T. (2002). q-Volkenborn integration. *Russian Journal of Mathematical Physics*, 9(3), 288–299.

[13] Kim, T. (2009). Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$. *Russian Journal of Mathematical Physics*, 16(4), 484–491.

[14] Lee, J. G., & Kwon, J. (2017). The modified degenerate q-Bernoulli polynomials arising from p-adic invariant integral on $\mathbb{Z}_p$. *Advances in Difference Equations*, 2017, Article ID 29.

[15] Ma, Y., Kim, T., Kim, D. S., & Lee, H. (2022). Study on q-analogues of Catalan–Daehee numbers and polynomials. *Filomat*, 36(5), 1499–1506.

[16] Ömür, N., & Bilgin, G. (2018). Some applications of the generalized hyperharmonic numbers of order $r,\ H_n^r(\alpha)$. *Advances and Applications in Mathematical Sciences*, 17(9), 617–627.

[17] Ömür, N., Südemen, K. N., & Koparal, S. *Some identities with special numbers* (submitted).

[18] Park, J. W. (2015). On the q-analogue of λ-Daehee polynomials. *Journal of Computational Analysis and Applications*, 19(6), 966–974.

[19] Yun, S. J., & Park, J.-W. (2020). On fully degenerate Daehee numbers and polynomials of the second kind. *Journal of Mathematics*, 2020, Article ID 7893498.